

# On the best constant in the Khintchine-Kahane inequality \*

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## Abstract

We prove that if  $r_i$  is the Rademacher system of functions then

$$\left( \int \left\| \sum_{i=1}^n x_i r_i(t) \right\|^2 dt \right)^{1/2} \leq \sqrt{2} \int \left\| \sum_{i=1}^n x_i r_i(t) \right\| dt$$

for any sequence of vectors  $x_i$  in any linear normed space  $F$ .

**Introduction:** The classical result of Khintchine [3] states that for each  $p, q > 0$  there exists a constant  $c_{p,q}$  such that for any real numbers  $x_1, \dots, x_n$

$$\left( \int \left| \sum_{i=1}^n x_i r_i(t) \right|^p dt \right)^{1/p} \leq c_{p,q} \left( \int \left| \sum_{i=1}^n x_i r_i(t) \right|^q dt \right)^{1/q} \quad (1)$$

The smallest constant  $c_{p,q}$  we will denote by  $C_{p,q}^R$  and will call the best one. Obviously for  $p \leq q$   $C_{p,q}^R = 1$ , but it took some efforts to calculate the other best constants. The especially interesting case  $p = 2, q = 1$  was first solved by S.J. Szarek [4], who proved  $C_{2,1}^R = \sqrt{2}$ . A simpler proof was given by U. Haagerup [1] who also found  $C_{p,2}^R$  and  $C_{2,p}^R$  for each  $p > 0$ . A simple and elementary proof that  $C_{2,1}^R = \sqrt{2}$  was also presented by B. Tomaszewski [6].

J.-P. Kahane [2] generalized the result of Khintchine to Banach space valued sequences  $x_1, \dots, x_n$  replacing in (1) the absolute value by the norm. Let  $C_{p,q}$  denote the smallest constant in the vector-valued inequality. It is of interest if the constants are the same in the vector and the real case. As far as we know the best result for  $p=2$  and  $q=1$  known up to now was obtained by B. Tomaszewski [5], who proved that  $C_{2,1} \leq \sqrt{3}$ . In this paper we show that  $C_{2,1} = \sqrt{2}$ , and we think that our proof is simpler than the ones known for real numbers.

**Notation:** For  $\sigma = (\sigma_1, \dots, \sigma_n) \in \{0, 1\}^n$ ,  $\alpha = (\alpha_1, \dots, \alpha_n) \in R^n$   $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ ,  $\eta = (\eta_1, \dots, \eta_n) \in \{-1, 1\}^n$  and  $x_1, \dots, x_n \in F$  let us define

$$|\sigma| = \sum_{i=1}^n \sigma_i$$

$$\alpha^\sigma = \prod_{i=1}^n \alpha_i^{\sigma_i} \quad (\text{where } x^0 = 1 \text{ for any } x \in R)$$

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$$\begin{aligned}
-\varepsilon &= (-\varepsilon_1, \dots, -\varepsilon_n) \\
X_\varepsilon &= \left\| \sum_{i=1}^n \varepsilon_i x_i \right\| \\
d(\varepsilon, \eta) &= \text{card}\{i : \varepsilon_i \neq \eta_i\}
\end{aligned}$$

By  $E(\cdot)$  we will denote the mean value of  $(\cdot)$ .

**Results:** We will prove the following theorem:

**Theorem 1** *Let  $S = \sum_{i=1}^n \varepsilon_i x_i$ , where  $\varepsilon_i$  are independent Bernoulli random variables and  $x_i$  are vectors of a normed linear space  $F$ . Then*

$$(E\|S\|^2)^{1/2} \leq \sqrt{2}E\|S\|$$

The constant  $\sqrt{2}$  is the best possible.

**Proof:**

Differentiating in  $t$  both sides of the equality

$$t^2 \prod_{i=1}^n (1 + t^{-1} \alpha_i) = \sum_{\sigma \in \{0,1\}^n} t^{2-|\sigma|} \alpha^\sigma$$

and evaluating them at  $t=1$  we get

$$2 \prod_{i=1}^n (1 + \alpha_i) - \sum_{j=1}^n \alpha_j \prod_{i=1, i \neq j}^n (1 + \alpha_i) = \sum_{\sigma \in \{0,1\}^n} (2 - |\sigma|) \alpha^\sigma$$

Hence we obtain

$$\begin{aligned}
\sum_{\varepsilon, \eta \in \{-1,1\}^n} (2 \prod_{i=1}^n (1 + \varepsilon_i \eta_i) - \sum_{j=1}^n \varepsilon_j \eta_j \prod_{i=1, i \neq j}^n (1 + \varepsilon_i \eta_i)) X_\varepsilon X_\eta &= \sum_{\varepsilon, \eta \in \{-1,1\}^n} \sum_{\sigma \in \{0,1\}^n} (2 - |\sigma|) \varepsilon^\sigma \eta^\sigma X_\varepsilon X_\eta \\
&= \sum_{\sigma \in \{0,1\}^n} (2 - |\sigma|) \left( \sum_{\varepsilon \in \{-1,1\}^n} \varepsilon^\sigma X_\varepsilon \right)^2 \leq 2 \left( \sum_{\varepsilon \in \{-1,1\}^n} X_\varepsilon \right)^2
\end{aligned}$$

The last inequality holds because  $X_\varepsilon = X_{-\varepsilon}$  for each  $\varepsilon$ , so that obviously

$$\sum_{\varepsilon \in \{-1,1\}^n} \varepsilon^\sigma X_\varepsilon = 0$$

for each  $\sigma$  with  $|\sigma| = 1$

Since the left-hand side of the above inequality is equal to

$$2^{n+1} \sum_{\varepsilon \in \{-1,1\}^n} X_\varepsilon^2 - n 2^{n-1} \sum_{\varepsilon \in \{-1,1\}^n} X_\varepsilon^2 + 2^{n-1} \sum_{\varepsilon, \eta \in \{-1,1\}^n, d(\varepsilon, \eta)=1} X_\varepsilon X_\eta$$

we arrive at

$$2^n \sum_{\varepsilon \in \{-1,1\}^n} X_\varepsilon^2 + 2^{n-1} \sum_{\varepsilon \in \{-1,1\}^n} X_\varepsilon \left( \sum_{\eta \in \{-1,1\}^n, d(\varepsilon, \eta)=1} X_\eta - (n-2) X_\varepsilon \right) \leq 2 \left( \sum_{\varepsilon \in \{-1,1\}^n} X_\varepsilon \right)^2 \quad (2)$$

By the triangle inequality for each fixed  $\varepsilon$  we get

$$(n-2)X_\varepsilon \leq \sum_{\eta \in \{-1,1\}^n, d(\varepsilon, \eta)=1} X_\eta$$

So inequality (2) yields

$$2^n \sum_{\varepsilon \in \{-1,1\}^n} X_\varepsilon^2 \leq 2 \left( \sum_{\varepsilon \in \{-1,1\}^n} X_\varepsilon \right)^2$$

Dividing by  $2^{2n}$  we get

$$E\|S\|^2 \leq 2(E\|S\|)^2$$

To see that the constant  $\sqrt{2}$  is the best possible it suffices to take  $n = 2$ ,  $x_1 = x_2 \neq 0$ .

**Remark 1:** If we replace in the above proof  $X_\varepsilon$  by  $X_\varepsilon^p$  and use the inequality

$$X_\varepsilon \leq \frac{1}{n-2} \sum_{\eta \in \{-1,1\}^n, d(\varepsilon, \eta)=1} X_\eta \leq \frac{n}{n-2} \left( \frac{1}{n} \sum_{\eta \in \{-1,1\}^n, d(\varepsilon, \eta)=1} X_\eta^p \right)^{1/p}$$

we will obtain that for  $p \in [1, 2)$

$$(E\|S\|^{2p})^{1/2p} \leq (1-p/2)^{-1/2p} (E\|S\|^p)^{1/p}$$

but we do not think that the above constants are optimal for  $p > 1$ .

**Remark 2:** Since for each bounded real random variable  $X$  the function  $f(r) = r \ln E|X|^{1/r}$  is convex Theorem 1 yields that for each  $q \in (0, 1], p \in (0, 2], q \leq p$  the following inequality holds:

$$(E\|S\|^p)^{1/p} \leq 2^{1/q-1/p} (E\|S\|^q)^{1/q}$$

and the constants  $2^{1/q-1/p}$  are optimal.

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## References

- [1] U. Haagerup *The best constants in the Khintchine inequality*, Studia Math. 70 (1982), 231-283
- [2] J.-P. Kahane *Sur les sommes vectorielles  $\sum \pm u_n$* , Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris, 259 (1964), 2577-2580

- [3] A. Khintchine *Über dyadische Brüche*”, Math.Z. 18 (1923), 109-116
- [4] S.J. Szarek *On the best constants in the Khinchine inequality*, Studia Math. 58 (1976), 197-208
- [5] B. Tomaszewski *Two remarks on the Khinchine-Kahane inequality*, Colloq.Math. 46 (1982) no.2, 283-288
- [6] B. Tomaszewski *A simple and elementary proof of the Khintchine inequality with the best constant* Bull.Sci.Math.(2) 111 (1987), 103-109

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