On $\mathcal{Z}_p$-norms of random vectors

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Abstract

To any n-dimensional random vector X we may associate its L_p-centroid body $\mathcal{Z}_p(X)$ and the corresponding norm. We formulate a conjecture concerning the bound on the $\mathcal{Z}_p(X)$-norm of X and show that it holds under some additional symmetry assumptions. We also relate our conjecture with estimates of covering numbers and Sudakov-type minorization bounds.

1 Introduction. Formulation of the Problem.

Let $p \geq 2$ and $X = (X_1, \ldots, X_n)$ be a random vector in $\mathbb{R}^n$ such that $\mathbb{E}|X|^p < \infty$. We define the following two norms on $\mathbb{R}^n$:

$$\|t\|_{\mathcal{M}_p(X)} := (\mathbb{E}|\langle t, X\rangle|^p)^{1/p} \quad \text{and} \quad \|t\|_{\mathcal{Z}_p(X)} := \sup\{|\langle t, s\rangle| \colon \ \|s\|_{\mathcal{M}_p(X)} \leq 1\}.$$

By $\mathcal{M}_p(X)$ and $\mathcal{Z}_p(X)$ we will also denote unit balls in these norms, i.e.

$$\mathcal{M}_p(X) := \{t \in \mathbb{R}^n \colon \ \|t\|_{\mathcal{M}_p(X)} \leq 1\} \text{ and } \mathcal{Z}_p(X) := \{t \in \mathbb{R}^n \colon \ \|t\|_{\mathcal{Z}_p(X)} \leq 1\}.$$

The set $\mathcal{Z}_p(X)$ is called the L_p-centroid body of X (or rather of the distribution of X). It was introduced (under a different normalization) for uniform distributions on convex bodies in [9]. Investigation of L_p-centroid bodies played a crucial role in the Paouris proof of large deviations bounds for Euclidean norms of log-concave vectors [10]. Such bodies also appears in questions related to the optimal concentration of log-concave vectors [7].

Let us introduce a bit of useful notation. We set $|t| := \|t\|_2 = \sqrt{\langle t, t\rangle}$ and $B_2^n = \{t \in \mathbb{R}^n \colon |t| \leq 1\}$. By $\|Y\|_p = (\mathbb{E}|Y|^p)^{1/p}$ we denote the L_p-norm of a random variable Y. Letter C denotes universal constants (that may differ at each occurence), we write $f \sim g$ if $\frac{1}{C} f \leq g \leq Cf$.

Let us begin with a simple case, when a random vector X is rotationally invariant. Then $X = RU$, where U has a uniform distribution on S^{n-1} and $R = |X|$ is a nonnegative random variable, independent of U. We have for any vector $t \in \mathbb{R}^n$ and $p \geq 2$,

$$\|\langle t, U\rangle\|_p = |t|\|U_1\|_p \sim \sqrt{\frac{p}{n+p}}|t|,$$

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where U_1 is the first coordinate of U. Therefore

$$\|t\|_{\mathcal{M}_p(X)} = \|U_1\|_p \|R\|_p |t| \quad \text{and} \quad \|t\|_{\mathcal{Z}_p(X)} = \|U_1\|_p^{-1} \|R\|_p^{-1} |t|.$$

So

$$\left(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^p\right)^{1/p} = \|U_1\|_p^{-1} \|R\|_p^{-1} (\mathbb{E}|X|^p)^{1/p} = \|U_1\|_p^{-1} \sim \sqrt{\frac{n+p}{p}}. \tag{1}$$

This motivates the following problem.

Problem 1. Is it true that for (at least a large class of) centered n-dimensional random vectors X,

$$\left(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^2\right)^{1/2} \leq C\sqrt{\frac{n+p}{p}} \quad \text{ for } p \geq 2,$$

or maybe even

$$\left(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^p\right)^{1/p} \leq C\sqrt{\frac{n+p}{p}} \quad \text{ for } p \geq 2?$$

Notice that the problem is linearly-invariant, since

$$\|AX\|_{\mathcal{Z}_p(AX)} = \|X\|_{\mathcal{Z}_p(X)} \quad \text{ for any } A \in \mathrm{GL}(n). \tag{2}$$

For any centered random vector X with nondegenerate covariance matrix, random vector $Y = \mathrm{Cov}(X)^{-1/2} X$ is isotropic (i.e. centered with identity covariance matrix). We have $\mathcal{M}_2(Y) = \mathcal{Z}_2(Y) = B_2^n$, hence

$$\mathbb{E}\|X\|_{\mathcal{Z}_2(X)}^2 = \mathbb{E}\|Y\|_{\mathcal{Z}_2(Y)}^2 = \mathbb{E}|Y|^2 = n.$$

Next remark shows that the answer to our problem is positive in the case $p \geq n$.

Remark 1. For $p \geq n$ and any n-dimensional random vector X we have $(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^p)^{1/p} \leq 10$.

Proof. Let S be a $1/2$-net in the unit ball of $\mathcal{M}_p(X)$ such that $|S| \leq 5^n$ (such net exists by the volume-based argument, cf. [1, Corollary 4.1.15]). Then

$$\begin{aligned}(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^p)^{1/p} &\leq 2\left(\mathbb{E}\sup_{t\in S} |\langle t, X\rangle|^p\right)^{1/p} \leq 2\left(\mathbb{E}\sum_{t\in S} |\langle t, X\rangle|^p\right)^{1/p} \\ &\leq 2|S|^{1/p} \sup_{t\in S} (\mathbb{E}\langle t, X\rangle|^p)^{1/p} \leq 2\cdot 5^{n/p}.\end{aligned}$$

□

L_p-centroid bodies play an important role in the study of vectors uniformly distributed on convex bodies and a more general class of log-concave vectors. A random vector with a nondenerate covariance matrix is called log-concave if its density has the form e^{-h}, where $h\colon \mathbb{R}^n \to (-\infty,\infty]$ is convex. If X is centered and log-concave then

$$\|\langle t,X\rangle\|_p \le \lambda\frac{p}{q}\|\langle t,X\rangle\|_q \quad \text{for } p\ge q\ge 2, \tag{3}$$

where $\lambda=2$ ($\lambda=1$ if X is symmetric and log-concave and $\lambda=3$ for arbitrary log-concave vectors). One of open problems for log-concave vectors [7] states that for such vectors, arbitrary norm $\|\ \|$ and $q\ge 1$,

$$(\mathbb{E}\|X\|^q)^{1/q} \le C\left(\mathbb{E}\|X\| + \sup_{\|t\|_*\le 1}\|\langle t,X\rangle\|_q\right).$$

In particular one may expect that for log-concave vectors

$$(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^q)^{1/q} \le C\left(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)} + \sup_{t\in\mathcal{M}_p(X)}\|\langle t,X\rangle\|_q\right) \le C\left(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)} + \frac{\max\{p,q\}}{p}\right).$$

As a result it is natural to state the following variant of Problem 1.

Problem 2. Let X be a centered log-concave n-dimensional random vector. Is it true that

$$(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^q)^{1/q} \le C\sqrt{\frac{n}{p}} \quad \text{for } 2\le p\le n,\ 1\le q\le \sqrt{pn}.$$

In Section 2 we show that Problems 1 and 2 have affirmative solutions in the class of unconditional vectors. In Section 3 we relate our problems to estimates of covering numbers. We also show that the first estimate in Problem 1 holds if the random vector X satisfies the Sudakov-type minorization bound.

2 Bounds for unconditional random vectors

In this section we consider the class of *unconditional* random vectors in $\mathbb{R}^n$, i.e. vectors X having the same distribution as $(\varepsilon_1|X_1|,\varepsilon_2|X_2|,\ldots,\varepsilon_n|X_n|)$, where (ε_i) is a sequence of independent symmetric ± 1 random variables (Rademacher sequence), independent of X.

Our first result shows that formula (1) may be extended to the unconditional case for p even. We use the standard notation – for a multiindex $\alpha=(\alpha_1,\ldots,\alpha_n)$, $x\in\mathbb{R}^n$ and $m=\sum\alpha_i$, $x^\alpha := \prod_i x_i^{\alpha_i}$ and $\binom{m}{\alpha} := m!/(\prod_i \alpha_i!)$.

Proposition 2. *We have for any $k = 1, 2, \ldots$ and any n-dimensional unconditional random vector X such that $\mathbb{E}|X|^{2k} < \infty$,*

$$\Big(\mathbb{E}\|X\|^{2k}_{\mathcal{Z}_{2k}(X)}\Big)^{1/(2k)} \le c_{2k} := \left(\sum_{\|\alpha\|_1=k} \frac{\binom{k}{\alpha}^2}{\binom{2k}{2\alpha}}\right)^{1/(2k)} \sim \sqrt{\frac{n+k}{k}},$$

where the summation runs over all multiindices $\alpha = (\alpha_1, \ldots, \alpha_n)$ with nonnegative integer coefficients such that $\|\alpha\|_1 = \sum_{i=1}^n \alpha_i = k$.

Proof. Observe first that

$$\mathbb{E}|\langle t, X\rangle|^{2k} = \mathbb{E}\left|\sum_{i=1}^n t_i\varepsilon_i X_i\right|^{2k} = \sum_{\|\alpha\|_1=k} \binom{2k}{2\alpha} t^{2\alpha}\mathbb{E}X^{2\alpha}.$$

For any $t, s \in \mathbb{R}^n$ we have

$$|\langle t, s\rangle|^k = \sum_{\|\alpha\|_1=k} \binom{k}{\alpha} t^\alpha s^\alpha.$$

So by the Cauchy-Schwarz inequality,

$$\|s\|^k_{\mathcal{Z}_{2k}(X)} = \sup\{|\langle t, s\rangle|^k \colon \mathbb{E}|\langle t, X\rangle|^{2k} \le 1\} \le \left(\sum_{\|\alpha\|_1=k} \frac{\binom{k}{\alpha}^2}{\binom{2k}{2\alpha}} \frac{s^{2\alpha}}{\mathbb{E}X^{2\alpha}}\right)^{1/2}.$$

To see that $c_{2k} \sim \sqrt{(n+k)/k}$ observe that

$$\frac{\binom{k}{\alpha}^2}{\binom{2k}{2\alpha}} = \binom{2k}{k}^{-1} \prod_{i=1}^n \binom{2\alpha_i}{\alpha_i}.$$

Therefore, since $1 \le \binom{2l}{l} \le 2^{2l}$, we get

$$4^{-k}\binom{n+k-1}{k} \le c_{2k}^{2k} \le 4^k \binom{n+k-1}{k}.$$

$\square$

Corollary 3. *Let X be an unconditional n-dimensional random vector. Then*

$$\Big(\mathbb{E}\|X\|^{2k}_{\mathcal{Z}_p(X)}\Big)^{1/2k} \le C\sqrt{\frac{n+p}{p}} \quad \text{for any positive integer } k \le \frac{p}{2}.$$

Proof. By the monotonicity of L_{2k}-norms we may and will assume that $k = \lfloor p/2 \rfloor$. Then by Proposition 2,

$$\left(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^{2k}\right)^{1/2k} \leq \left(\mathbb{E}\|X\|_{\mathcal{Z}_{2k}(X)}^{2k}\right)^{1/2k} \leq C\sqrt{\frac{n+k}{k}} \leq C\sqrt{\frac{n+p}{p}}.$$

□

In the unconditional log-concave case we may bound higher moments of $\|X\|_{\mathcal{Z}_p(X)}$.

Theorem 4. *Let X be an unconditional log-concave n-dimensional random vector. Then for $p, q \geq 2$,*

$$(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^q)^{1/q} \leq C\left(\sqrt{\frac{n+p}{p}} + \sup_{t\in\mathcal{M}_p(X)} \|\langle t, X\rangle\|_q\right) \leq C\left(\sqrt{\frac{n+p}{p}} + \frac{q}{p}\right).$$

In order to show this result we will need the following lemma.

Lemma 5. *Let $2 \leq p \leq n$, X be an unconditional random vector in $\mathbb{R}^n$ such that $\mathbb{E}|X|^p < \infty$ and $\mathbb{E}|X_i| = 1$. Then*

$$\|s\|_{\mathcal{Z}_p(X)} \leq \sup_{I\subset[n],|I|\leq p} \sup_{\|t\|_{\mathcal{M}_p(X)}\leq 1} \left|\sum_{i\in I} t_i s_i\right| + C_1 \sup_{\|t\|_{\mathcal{M}_p(X)}\leq 1, \|t\|_2\leq p^{-1/2}} \left|\sum_{i=1}^n t_i s_i\right|. \tag{4}$$

Proof. We have by the unconditionality of X and Jensen's inequality,

$$\|t\|_{\mathcal{M}_p(X)} = \left\|\sum_{i=1}^n t_i\varepsilon_i|X_i|\right\|_p \geq \left\|\sum_{i=1}^n t_i\varepsilon_i\mathbb{E}|X_i|\right\|_p.$$

By the result of Hitczenko [5], for numbers $a_1, \ldots, a_n$,

$$\left\|\sum_{i=1}^n a_i\varepsilon_i\right\|_p \sim \sum_{i\leq p} a_i^* + \sqrt{p}\left(\sum_{i>p} |a_i^*|^2\right)^{1/2}, \tag{5}$$

where $(a_i^*)_{i\leq n}$ denotes the nonincreasing rearrangement of $(|a_i|)_{i\leq n}$. Thus

$$\sqrt{p}\left(\sum_{i>p} |t_i^*|^2\right)^{1/2} \leq C_1\|t\|_{\mathcal{M}_p(X)}$$

and (4) easily follows. □

Proof of Theorem 4. The last bound in the assertion follows by (3). It is easy to see that (increasing q if necessary) it is enough to consider the case $q \geq \sqrt{np}$.

If $q \geq n$ then the similar argument as in the proof of Remark 1 shows that

$$\Big(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^q\Big)^{1/q} \leq 2 \cdot 5^{n/q} \sup_{t \in \mathcal{M}_p(X)} \|\langle t, X\rangle\|_q \leq 10 \sup_{t \in \mathcal{M}_p(X)} \|\langle t, X\rangle\|_q.$$

Finally, consider the remaining case $\sqrt{pn} \leq q \leq n$. By (2) we may assume that $\mathbb{E}|X_i| = 1$ for all i. By the log-concavity $\|\langle t, X\rangle\|_{q_1} \leq C\frac{q_1}{q_2}\|\langle t, X\rangle\|_{q_2}$ for $q_1 \geq q_2 \geq 1$, in particular $\sigma_i := \|X_i\|_2 \leq C$.

Let $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be i.i.d. symmetric exponential random variables with variance 1. By [6, Theorem 3.1] we have

$$\begin{aligned}
&\left\| \sup_{\|t\|_{\mathcal{M}_p(X)} \leq 1, \|t\|_2 \leq p^{-1/2}} \left|\sum_{i=1}^n t_i X_i\right| \right\|_q \\
&\qquad \leq C\left(\left\| \sup_{\|t\|_{\mathcal{M}_p(X)} \leq 1, \|t\|_2 \leq p^{-1/2}} \left|\sum_{i=1}^n t_i \sigma_i \mathcal{E}_i\right| \right\|_1 + \sup_{\|t\|_{\mathcal{M}_p(X)} \leq 1, \|t\|_2 \leq p^{-1/2}} \|\langle t, X\rangle\|_q\right).
\end{aligned}$$

We have

$$\sup_{\|t\|_{\mathcal{M}_p(X)} \leq 1, \|t\|_2 \leq p^{-1/2}} \|\langle t, X\rangle\|_q \leq \sup_{\|t\|_{\mathcal{M}_p(X)} \leq 1} \|\langle t, X\rangle\|_q$$

and

$$\left\| \sup_{\|t\|_{\mathcal{M}_p(X)} \leq 1, \|t\|_2 \leq p^{-1/2}} \left|\sum_{i=1}^n t_i \sigma_i \mathcal{E}_i\right| \right\|_1 \leq \frac{1}{\sqrt{p}} \left\| \sqrt{\sum_{i=1}^n \sigma_i^2 \mathcal{E}_i^2} \right\|_1 \leq \frac{1}{\sqrt{p}} \sqrt{\sum_{i=1}^n \sigma_i^2} \leq C\sqrt{\frac{n}{p}}.$$

Thus

$$\left\| \sup_{\|t\|_{\mathcal{M}_p(X)} \leq 1, \|t\|_2 \leq p^{-1/2}} \left|\sum_{i=1}^n t_i X_i\right| \right\|_q \leq C\left(\sqrt{\frac{n}{p}} + \sup_{\|t\|_{\mathcal{M}_p(X)} \leq 1} \|\langle t, X\rangle\|_q\right).$$

Let for each $I \subset [n]$, $P_I X = (X_i)_{i \in I}$ and S_I be a $1/2$-net in $\mathcal{M}_p(P_I X)$ of cardinality at most $5^{|I|}$. We have

$$\begin{aligned}
\left\|\sup_{I\subset[n],|I|\le p}\ \sup_{\|t\|_{\mathcal{M}_p(X)}\le 1}\left|\sum_{i\in I} t_iX_i\right|\right\|_q &\le 2\left\|\sup_{I\subset[n],|I|\le p}\ \sup_{t\in S_I}\left|\sum_{i\in I} t_iX_i\right|\right\|_q \\
&\le 2\left(\sum_{I\subset[n],|I|\le p}\ \sum_{t\in S_I}\mathbb{E}\left|\sum_{i\in I} t_iX_i\right|^q\right)^{1/q} \\
&\le 2\cdot 5^{p/q}|\{I\subset[n],|I|\le p\}|^{1/q}\sup_I\sup_{t\in S_I}\left\|\sum_{i\in I} t_iX_i\right\|_q \\
&\le 10\left(\frac{en}{p}\right)^{p/q}\sup_{t\in\mathcal{M}_p(X)}\left\|\sum_{i\in I} t_iX_i\right\|_q \\
&\le C\sup_{t\in\mathcal{M}_p(X)}\left\|\sum_{i\in I} t_iX_i\right\|_q,
\end{aligned}$$

where the last estimate follows from $q\ge\sqrt{np}$.

Hence the assertion follows by Lemma 5. □

Corollary 6. *Let X be an unconditional log-concave n-dimensional random vector and $2\le p\le n$. Then*

$$\frac{1}{C}\sqrt{\frac{n}{p}}\le\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}\le\left(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^{\sqrt{np}}\right)^{1/\sqrt{np}}\le C\sqrt{\frac{n}{p}} \tag{6}$$

and

$$\mathbb{P}\left(\|X\|_{\mathcal{Z}_p(X)}\ge\frac{1}{C}\sqrt{\frac{n}{p}}\right)\ge\frac{1}{C},\quad \mathbb{P}\left(\|X\|_{\mathcal{Z}_p(X)}\ge Ct\sqrt{\frac{n}{p}}\right)\le e^{-t\sqrt{np}}\ \textit{for}\ t\ge 1.$$

Proof. The upper bound in (6) easily follows by Theorem 4. In fact we have for $t\ge 1$,

$$\left(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^{t\sqrt{np}}\right)^{1/(t\sqrt{np})}\le Ct\sqrt{\frac{n}{p}},$$

hence the Chebyshev inequality yields the upper tail bound for $\|X\|_{\mathcal{Z}_p(X)}$.

To establish lower bounds we may assume that X is additionally isotropic. Then by the result of Bobkov and Nazarov [3] we have $\|\langle t,X\rangle\|_p\le C(\sqrt{p}\|t\|_2+p\|t\|_\infty)$. This easily gives

$$\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}\ge\frac{1}{C}\sqrt{\frac{n}{p}}\mathbb{E}X^*_{\lceil n/2\rceil}\ge\frac{1}{C}\sqrt{\frac{n}{p}},$$

where the last inequality follows by Lemma 7 below.

By the Paley-Zygmund inequality we get

$$\mathbb{P}\left(\|X\|_{\mathcal{Z}_p(X)} \geq \frac{1}{C}\sqrt{\frac{n}{p}}\right) \geq \mathbb{P}\left(\|X\|_{\mathcal{Z}_p(X)} \geq \frac{1}{2}\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}\right) \geq \frac{(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)})^2}{4\mathbb{E}\|X\|^2_{\mathcal{Z}_p(X)}} \geq c.$$

□

Lemma 7. *Let X by a symmetric isotropic n-dimensional log-concave vector. Then* $\mathbb{E}X^*_{\lceil n/2\rceil} \geq \frac{1}{C}$.

Proof. Let $a_i > 0$ be such that $\mathbb{P}(X_i \geq a_i) = 3/8$. Then by the log-concavity of X_i, $\mathbb{P}(|X_i| \geq ta_i) = 2\mathbb{P}(X_i \geq ta_i) \leq (3/4)^t$ for $t \geq 1$ and integration by parts yields $\|X_i\|_2 \leq Ca_i$. Thus $a_i \geq c_1$ for a constant $c_1 > 0$.

Let $S = \sum_{i=1}^n I_{\{|X_i|\geq c_1\}}$. Then $\mathbb{E}S = \sum_{i=1}^n \mathbb{P}(|X_i| \geq c_1) \geq 3n/4$. On the other hand $\mathbb{E}S \leq \frac{n}{2} + n\mathbb{P}(X^*_{\lceil n/2\rceil} \geq c_1)$, so

$$\mathbb{E}X^*_{\lceil n/2\rceil} \geq c_1\mathbb{P}(X^*_{\lceil n/2\rceil} \geq c_1) \geq c_1/4.$$

□

The next example shows that the tail and moment bounds in Corollary 6 are optimal.

Example. Let $X = (X_1, \ldots, X_n)$ be an isotropic random vector with i.i.d. symmetric exponential coordinates (i.e. X has the density $2^{n/2}\exp(-\sqrt{2}\|x\|_1)$). Then $(\mathbb{E}|X_i|^p)^{1/p} \leq p/2$, so $\frac{2}{p}e_i \in \mathcal{M}_p(X)$ and

$$\mathbb{P}\left(\|X\|_{\mathcal{Z}_p(X)} \geq t\sqrt{n/p}\right) \geq \mathbb{P}(|X_i| \geq t\sqrt{np}/2) \geq e^{-t\sqrt{np}/\sqrt{2}}$$

and for $q = s\sqrt{np}$, $s \geq 1$,

$$\left(\mathbb{E}\|X\|^q_{\mathcal{Z}_p(X)}\right)^{1/q} \geq \frac{2}{p}\|X_i\|_q \geq cq/p = cs\sqrt{n/p}.$$

3 General case – approach via entropy numbers

In this section we propose a method of deriving estimates for $\mathcal{Z}_p$-norms via entropy estimates for $\mathcal{M}_p$-balls and Euclidean distance. We use a standard notation – for sets $T, S \subset \mathbb{R}^n$, by $N(T, S)$ we denote the minimal number of translates of S that are enough to cover T. If S is the ε-ball with respect to some translation-invariant metric d then $N(T, S)$ is also denoted as $N(T, d, \varepsilon)$ and is called the metric entropy of T with respect to d.

We are mainly interested in log-concave vectors or random vectors which satisfy moment estimates

$$\|\langle t, X\rangle\|_p \leq \lambda\frac{p}{q}\|\langle t, X\rangle\|_q \quad \text{for } p \geq q \geq 2. \tag{7}$$

Let us start with a simple bound.

Proposition 8. *Suppose that X is isotropic in $\mathbb{R}^n$ and* (7) *holds. Then for any $p \geq 2$ and $\varepsilon > 0$ we have*

$$\left(\mathbb{E}\|X\|^2_{\mathcal{Z}_p(X)}\right)^{1/2} \leq \varepsilon\sqrt{n} + \frac{e\lambda}{p}\max\left\{p, \log N(\mathcal{M}_p(X), \varepsilon B_2^n)\right\}.$$

Proof. Let $N = N(\mathcal{M}_p(X), \varepsilon B_2^n)$ and choose $t_1, \ldots, t_N \in \mathcal{M}_p(X)$ such that $\mathcal{M}_p(X) \subset \bigcup_{i=1}^N (t_i + \varepsilon B_2^n)$. Then

$$\|x\|_{\mathcal{Z}_p(X)} \leq \varepsilon|x| + \sup_{i\leq N}\langle t_i, x\rangle.$$

Let $r = \max\{p, \log N\}$. We have

$$\begin{aligned}\left(\mathbb{E}\sup_{i\leq N}|\langle t_i, X\rangle|^2\right)^{1/2} &\leq \left(\mathbb{E}\sup_{i\leq N}|\langle t_i, X\rangle|^r\right)^{1/r} \leq \left(\sum_{i=1}^N \mathbb{E}|\langle t_i, X\rangle|^r\right)^{1/r} \\ &\leq N^{1/r}\sup_i \|\langle t_i, X\rangle\|_r \leq e\lambda\frac{r}{p}\sup_i\|\langle t_i, X\rangle\|_p \leq e\lambda\frac{r}{p}.\end{aligned}$$

□

Remark 9. The Paouris inequality [10] states that for isotropic log-concave vectors and $q \geq 2$, $(\mathbb{E}|X|^q)^{1/q} \leq C(\sqrt{n} + q)$, so for such vectors and $q \geq 2$,

$$\left(\mathbb{E}\|X\|^q_{\mathcal{Z}_p(X)}\right)^{1/q} \leq C\varepsilon(\sqrt{n} + q) + \frac{2e}{p}\max\{p, q, \log N(\mathcal{M}_p(X), \varepsilon B_2^n)\}.$$

Unfortunately, the known estimates for entropy numbers of $\mathcal{M}_p$-balls are rather weak.

Theorem 10 ([4, Proposition 9.2.8])**.** *Assume that X is isotropic log-concave and $2 \leq p \leq \sqrt{n}$. Then*

$$\log N\left(\mathcal{M}_p(X), \frac{t}{\sqrt{p}}B_2^n\right) \leq C\frac{n\log^2 p\log t}{t} \qquad \text{for } 1 \leq t \leq \min\left\{\sqrt{p}, \frac{1}{C}\frac{n\log p}{p^2}\right\}.$$

Corollary 11. *Let X be isotropic log-concave, then*

$$\left(\mathbb{E}\|X\|^p_{\mathcal{Z}_p(X)}\right)^{1/p} \leq C\left(\frac{n}{p}\right)^{3/4}\log p\sqrt{\log n} \quad \text{for } 2 \leq p \leq \frac{1}{C}n^{3/7}\log^{-2/7} n.$$

Proof. We apply Theorem 10 with $t = (n/p)^{1/4}\log p\log^{1/2} n$ and Proposition 8 with $\varepsilon = tp^{-1/2}$. □

Remark 12. Suppose that X is centered and the following stronger bound than (7) (satisfied for example for Gaussian vectors) holds

$$\|\langle t, X\rangle\|_p \leq \lambda\sqrt{\frac{p}{q}}\|\langle t, X\rangle\|_q \qquad \text{for } p \geq q \geq 2. \tag{8}$$

Then for any $2 \le p \le n$,

$$\frac{1}{\lambda}\sqrt{\frac{2n}{p}} \le \Big(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^2\Big)^{1/2} \le \Big(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^n\Big)^{1/n} \le 10\lambda\sqrt{\frac{n}{p}}.$$

Proof. Without loss of generality we may assume that X is isotropic. We have

$$\|\langle t,X\rangle\|_p \le \lambda\sqrt{p/2}\|\langle t,X\rangle\|_2 = \lambda\sqrt{p/2}|t|,$$

so $\mathcal{M}_p(X) \supset \lambda^{-1}\sqrt{2/p}B_2^n$ and

$$\Big(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^2\Big)^{1/2} \ge \frac{1}{\lambda}\sqrt{\frac{2}{p}}\left(\mathbb{E}|X|^2\right)^{1/2} = \frac{1}{\lambda}\sqrt{\frac{2n}{p}}.$$

On the other hand let S be a $1/2$-net in $\mathcal{M}_p(X)$ of cardinality at most 5^n. Then

$$\begin{aligned}\Big(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^n\Big)^{1/n} &\le 2\left(\mathbb{E}\sup_{t\in S}|\langle t,X\rangle|^n\right)^{1/n} \le 2\left(\sum_{t\in S}\mathbb{E}|\langle t,X\rangle|^n\right)^{1/n} \\ &\le 2|S|^{1/n}\sup_{t\in S}\|\langle t,X\rangle\|_n \le 10\lambda\sqrt{\frac{n}{p}}\sup_{t\in S}\|\langle t,X\rangle\|_p \le 10\lambda\sqrt{\frac{n}{p}}.\end{aligned}$$

□

Recall that the Sudakov minoration principle [11] states that if G is an isotropic Gaussian vector in $\mathbb{R}^n$ then for any bounded $T \subset \mathbb{R}^n$ and $\varepsilon > 0$,

$$\mathbb{E}\sup_{t\in T}\langle t,G\rangle \ge \frac{1}{C}\varepsilon\sqrt{\log N(T,\varepsilon B_2^n)}.$$

So we can say that a random vector X in $\mathbb{R}^n$ satisfies *the L_2-Sudakov minoration with a constant C_X* if for any bounded $T \subset \mathbb{R}^n$ and $\varepsilon > 0$,

$$\mathbb{E}\sup_{t\in T}\langle t,X\rangle \ge \frac{1}{C_X}\varepsilon\sqrt{\log N(T,\varepsilon B_2^n)}.$$

Example. Any unconditional n-dimensional random vector satisfies the L_2-Sudakov minoration with constant $C\sqrt{\log(n+1)}/(\min_{i\le n}\mathbb{E}|X_i|)$.

Indeed, we have by the unconditionality, Jensen's inequality and the contraction principle,

$$\mathbb{E}\sup_{t\in T}\sum_{i=1}^n t_iX_i = \mathbb{E}\sup_{t\in T}\sum_{i=1}^n t_i\varepsilon_i|X_i| \ge \mathbb{E}\sup_{t\in T}\sum_{i=1}^n t_i\varepsilon_i\mathbb{E}|X_i| \ge \min_{i\le n}\mathbb{E}|X_i|\mathbb{E}\sup_{t\in T}\sum_{i=1}^n t_i\varepsilon_i.$$

On the other hand, the classical Sudakov minoration and the contraction principle yields

$$\frac{1}{C}\varepsilon\sqrt{\log N(T,\varepsilon B_2^n)} \le \mathbb{E}\sup_{t\in T}\sum_{i=1}^n t_i g_i \le \mathbb{E}\max_{i\le n}|g_i|\mathbb{E}\sup_{t\in T}\sum_{i=1}^n t_i\varepsilon_i$$
$$\le C\sqrt{\log(n+1)}\mathbb{E}\sup_{t\in T}\sum_{i=1}^n t_i\varepsilon_i.$$

However the L_2-Sudakov minoration constant may be quite large in the isotropic case even for unconditional vectors if we do not assume that L_1 and L_2 norms of X_i are comparable. Indeed, let $\mathbb{P}(X=\pm n^{1/2}e_i)=\frac{1}{2n}$ for $i=1,\ldots,n$, where $e_1,\ldots,e_n$ is the canonical basis of $\mathbb{R}^n$. Then X is isotropic and unconditional. Let $T=\{t\in\mathbb{R}^n\colon\ \|t\|_\infty\le n^{-1/2}\}$. Then

$$\mathbb{E}\sup_{t\in T}|\langle t,X\rangle|\le 1.$$

However, by the volume-based estimate,

$$N(T,\varepsilon B_2^n)\ge\frac{\mathrm{vol}(T)}{\mathrm{vol}(\varepsilon B_2^n)}\ge\left(\frac{1}{C\varepsilon}\right)^n,$$

hence

$$\sup_{\varepsilon>0}\varepsilon\sqrt{\log N((T,\varepsilon B_2^n)}\ge\frac{1}{C}\sqrt{n}.$$

Thus the L_2-Sudakov constant $C_X\ge\sqrt{n}/C$ in this case.

Next proposition shows that random vectors with uniformly log-convex density satisfy the L_2-Sudakov minoration.

Proposition 13. *Suppose that a symmetric random vector X in $\mathbb{R}^n$ has the density of the form e^h such that $\mathrm{Hess}(h)\ge-\alpha\mathrm{Id}$ for some $\alpha>0$. Then X satisfies the L_2-Sudakov minoration with constant $C_X\le C\sqrt{\alpha}$.*

Proof. We will follow the method of the proof of the (dual) classical Sudakov inequality (cf. (3.15) and its proof in [8]).

Let T be a bounded symmetric set and

$$A:=\mathbb{E}\sup_{t\in T}|\langle t,X\rangle|.$$

By the duality of entropy numbers [2] we need to show that $\log^{1/2}N(\varepsilon^{-1}B_2^n,T^o)\le C\varepsilon^{-1}\alpha^{1/2}A$ for $\varepsilon>0$ or equivalently that

$$N(\delta B_2^n,6AT^o)\le\exp(C\alpha\delta^2)\quad\text{ for }\delta>0. \tag{9}$$

To this end let $N = N(\delta B_2^n, 6AT^o)$. If $N = 1$ there is nothing to show, so assume that $N \geq 2$. Then we may choose $t_1, \ldots, t_N \in \delta B_2^n$ such that the balls $t_i + 3AT^0$ are disjoint. Let $\mu = \mu_X$ be the distribution of X. By the Chebyshev inequality,

$$\mu(3AT^0) = 1 - \mathbb{P}\left(\sup_{t \in T} |\langle t, X\rangle| > 3A\right) \geq \frac{2}{3}.$$

Observe also that for any symmetric set K and $t \in \mathbb{R}^n$,

$$\begin{aligned}\mu(t+K) &= \int_K e^{h(x-t)}dx = \int_K e^{h(x+t)}dx = \int_K \frac{1}{2}(e^{h(x-t)} + e^{h(x+t)})dx \\ &\geq \int_K e^{(h(x-t)+h(x+t))/2}dx.\end{aligned}$$

By Taylor's expansion we have for some $\theta \in [0,1]$,

$$\frac{h(x-t)+h(x+t)}{2} = h(x) + \frac{1}{4}(\langle \mathrm{Hess} h(x+\theta t)t, t\rangle + \langle \mathrm{Hess} h(x-\theta t)t, t\rangle) \geq h(x) - \frac{1}{2}\alpha|t|^2.$$

Thus

$$\mu(t+K) \geq \int_K e^{h(x) - \alpha|t|^2/2} = e^{-\alpha|t|^2/2}\mu(K)$$

and

$$1 \geq \sum_{i=1}^N \mu(t_i + 3AT^0) \geq \sum_{i=1}^N e^{-\alpha|t_i|^2/2}\mu(3AT^0) \geq \frac{2N}{3}e^{-\alpha\delta^2/2} \geq N^{1/3}e^{-\alpha\delta^2/2}$$

and (9) easily follows.

□

Proposition 14. *Suppose that X satisfies the L_2-Sudakov minoration with constant C_X. Then for any $p \geq 2$*

$$N\left(\mathcal{M}_p(X), \frac{eC_X}{\sqrt{p}}B_2^n\right) \leq e^p.$$

In particular if X is isotropic we have for $2 \leq p \leq n$,

$$\left(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^2\right)^{1/2} \leq e\left(C_X\sqrt{\frac{n}{p}} + 1\right).$$

Proof. Suppose that $N = N(\mathcal{M}_p(X), eC_X p^{-1/2}B_2^n) \geq e^p$. We can choose $t_1, \ldots, t_N \in \mathcal{M}_p(X)$, such that $\|t_i - t_j\|_2 \geq eC_X p^{-1/2}$. We have

$$\mathbb{E}\sup_{i \geq N}\langle t_i, X\rangle \geq \frac{1}{C_X}eC_X p^{-1/2}\sqrt{\log N} > e.$$

However on the other hand,

$$\mathbb{E}\sup_{i\geq N}\langle t_i, X\rangle \leq \left(\mathbb{E}\sup_{i\geq N}|\langle t_i, X\rangle|^p\right)^{1/p} \leq \left(\sum_{i\geq N}\mathbb{E}|\langle t_i, X\rangle|^p\right)^{1/p} \leq N^{1/p}\max_i\|\langle t_i, X\rangle\|_p \leq e.$$

To show the second estimate we proceed in a similar way as in the proof of Proposition 8. We choose $T\subset\mathcal{M}_p(X)$ such that $|T|\leq e^p$ and $\mathcal{M}_p(X)\subset T+eC_Xp^{-1/2}B_2^n$. We have

$$\|X\|_{\mathcal{Z}_p(X)} \leq eC_Xp^{-1/2}|X| + \sup_{t\in T}|\langle t, X\rangle|$$

so that

$$\left(\mathbb{E}\|X\|^2_{\mathcal{Z}_p(X)}\right)^{1/2} \leq eC_Xp^{-1/2}(\mathbb{E}|X|^2)^{1/2} + \left(\mathbb{E}\sup_{t\in T}|\langle t, X\rangle|^2\right)^{1/2}.$$

Vector X is isotropic, so $\mathbb{E}|X|^2 = n$ and since $T\subset\mathcal{M}_p(X)$ and $p\geq 2$ we get

$$\begin{aligned}\left(\mathbb{E}\sup_{t\in T}|\langle t, X\rangle|^2\right)^{1/2} &\leq \left(\mathbb{E}\sup_{t\in T}|\langle t, X\rangle|^p\right)^{1/p} \leq \left(\sum_{t\in T}\mathbb{E}|\langle t, X\rangle|^p\right)^{1/p}\\ &\leq |T|^{1/p}\max_{t\in T}\||\langle t, X\rangle\|_p \leq e.\end{aligned}$$

□

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