# BOUNDS ON MOMENTS OF WEIGHTED SUMS OF FINITE RIESZ PRODUCTS

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Abstract. We establish matching lower and upper bounds for moments of weighted sums of finite Riesz products based on lacunary integer sequences with large enough ratios. Our bounds essentially show that those moments behave as though the products were functions with disjoint supports. Constants depend only on the order of the moments.

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## 1. introduction

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ be the one dimensional torus and $m$ be the normalized Haar measure on $\mathbb{T}$. Let $(n_j)_{j\geq 1}$ be an increasing sequence of positive integers. Riesz products are defined on $\mathbb{T}$ by

$$R_0 \equiv 1 \quad \text{and} \quad R_N(t) := \prod_{j=1}^{N}(1+\cos(n_j t)) \quad \text{for } N = 1, 2, \ldots. \tag{1}$$

To simplify the notation we also put

$$X_0 \equiv 1 \quad \text{and} \quad X_j(t) := 1 + \cos(n_j t), \quad j = 1, 2, \ldots.$$

It was Frigyes Riesz who first realized the usefulness of these objects treated as probability measures. Suppose $n_{j+1}/n_j \geq 2$ for $j \geq 1$. Then the numbers $\sum_{j=1}^{N} \varepsilon_j n_j$ are all nonzero for nonzero vectors $(\varepsilon_j)_{j=1}^{N} \in \{-1, 0, 1\}^N$, due to the fact that for every $l$, $\sum_{k=1}^{l} n_k < n_{l+1}$. In particular, the zero mode of $R_N$ has Fourier weight 1 and thus $R_N$ are densities of probability measures $\mu_N$. The weak-$*$ limit of $(\mu_N)$ is a singular measure which admits a number of remarkable Fourier-analytic properties. The reader is referred for instance to [10] for more information on properties of Riesz products and general trigonometric polynomials as well as to the short survey [5] of some applications of Riesz products.

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In this article we shall study the sum $\sum_{k=0}^N v_k R_k$ where $v_k$ are vectors in a normed space $(E, \|\cdot\|)$. By the triangle inequality, we trivially have

$$\int_{\mathbb{T}} \left\| \sum_{k=0}^N v_k R_k \right\| \mathrm{d}m \leq \sum_{k=0}^N \|v_k\| . \tag{2}$$

Wojciechowski asked for the validity of the reverse bound up to some universal constant (personal communication) and studied this problem in the following probabilistic context. Suppose we replace the functions $X_1, X_2, \ldots$ appearing in the definition of the Riesz products with a sequence of independent random variables $\bar{X}_1, \bar{X}_2, \ldots$ (defined on some probability space $(\Omega, \mathbb{P})$), each having the same distribution as $1 + \cos(Y)$, where $Y$ is uniform on $[0, 2\pi]$. We then take $\bar{R}_N = \prod_{k=1}^N \bar{X}_k$ and of course $\bar{R}_0 \equiv 1$. Note that the functions $X_j$ defined on the probability space $(\mathbb{T}, m)$ have the same distribution as the random variables $\bar{X}_j$. Even though the $X_j$ are not independent, we shall see that they behave, in many ways, like independent random variables. Capturing this phenomenon in a quantitative way is one of the main difficulties in our investigation.

In [9], Wojciechowski showed the existence of universal constants $c$ and $C$ as well as real numbers $a_1, a_2, \ldots$ such that for every $n$, $|\sum_{i=0}^k a_i| \leq C$ for all $k \leq n$ and $\mathbb{E}|\sum_{i=0}^n a_i \bar{R}_i| \geq cn$. This result was used in [4] to show the continuity of Fourier multipliers on the homogeneous Sobolev space $\dot{W}_1^1(\mathbb{R}^d)$, generalizing a result of Bonami and Poornima from [1]. Another application of Wojciechowski's result appeared in [3] where the authors gave an alternative proof of the lack of a priori estimates for certain differential operators, first established by Ornstein in [8].

The reverse of (2) for $\bar{R}_k$ was proved by the first named author in [6] for general random variables. Namely, for any sequence $\bar{X}_1, \bar{X}_2, \ldots$ of i.i.d. non-negative random variables with mean one and such that $\mathbb{P}(\bar{X}_1 = 1) < 1$, we have

$$\mathbb{E} \left\| \sum_{k=0}^N v_k \bar{R}_k \right\| \geq c_{\bar{X}_1} \sum_{k=0}^N \|v_k\| , \tag{3}$$

for any vectors $v_i$ in an arbitrary normed space $(E, \|\cdot\|)$, with a constant $c_{\bar{X}_1}$ depending only on the distribution of $\bar{X}_1$ (see Theorem 4 in [6]; see also Theorem 3 therein for non identically distributed sequences $(\bar{X}_i)$). This clearly implies Wojciechowski's result with $a_i = (-1)^i$ (here $E = \mathbb{R}$). According to a theorem of Y. Meyer (see [7]), under a stronger divergence of the sequence of modes, namely when $\sum_{k=1}^\infty \frac{n_k}{n_{k+1}} < \infty$, for any real numbers $a_i$, we have

$$\int_{\mathbb{T}} \left| \sum_{k=0}^N a_k R_k \right| \geq c_S \mathbb{E} \left| \sum_{k=0}^N a_k \bar{R}_k \right|$$

for a positive constant $c_S$ which depends only on the $n_k$. In [6], this principle was combined with (3) to show the reverse of (2) in the real case and under the above restrictive condition on the modes $n_i$.

Later the results of [6] have been generalized by Damek et al. in [2], where it was shown that for any $p > 0$ and under the same assumptions on the i.i.d. sequence $(\bar{X}_i)$, we have

$$\frac{1}{C_{p,\bar{X}_1}} \sum_{k=0}^{N} \|v_k\|^p \ \mathbb{E}\bar{R}_k^p \ \le \ \mathbb{E} \left\| \sum_{k=0}^{N} v_k \bar{R}_k \right\|^p \ \le \ C_{p,\bar{X}_1} \sum_{k=0}^{N} \|v_k\|^p \ \mathbb{E}\bar{R}_k^p \qquad N \ge 1,$$

with a constant $C_{p,\bar{X}_1}$ depending only on $p$ and the distribution of $\bar{X}_1$.

The aim of this article is to prove the following theorem.

**Theorem 1.** *For every $p \ge 1$ there are positive constants $d_p, c_p, C_p$ depending only on $p$, such that for any integers $n_j$ satisfying $n_{j+1}/n_j \ge d_p$, $j = 1, 2, \dots$ and for any vectors $v_0, v_1, \dots$ in a normed space $(E, \|\cdot\|)$, we have*

$$c_p \sum_{k=0}^{N} \|v_k\|^p \int_{\mathbb{T}} R_k^p \mathrm{d}m \ \le \ \int_{\mathbb{T}} \left\| \sum_{k=0}^{N} v_k R_k \right\|^p \mathrm{d}m \ \le \ C_p \sum_{k=0}^{N} \|v_k\|^p \int_{\mathbb{T}} R_k^p \mathrm{d}m,$$

*for any $N \ge 1$, where $R_k$ are defined via* (1).

The lower bound in the case $p = 1$ answers the original question of Wojciechowski. Let us also note that for $p > 1$, both the upper and the lower bound are non-trivial. Theorem 1 was proved in [2] in the real case ($E = \mathbb{R}$) under the condition $\sum_{k=1}^{\infty} \frac{n_k}{n_{k+1}} < \infty$ mentioned earlier (again by combining the independent case with the decoupling inequality of Meyer).

The values of the constants $d_p, c_p$ and $C_p$ that can be obtained from our proofs are not optimal. In particular, we have $\lim_{p\to 1^+} d_p = \infty$ and $\lim_{p \to 1^+} c_p = 0$, which is inconsistent with the case $p = 1$. Due to these blow-ups as $p \to 1^+$, our proof in the case $p = 1$ is slightly different from the proof for $p > 1$. We restate the result for $p = 1$ with numerical values of the constants (for explicit bounds on the constants for $p > 1$, see Remark 28).

**Theorem 2.** *There exist constants $d_1 < 1.2 \cdot 10^9$ and $c_1 > 5.8 \cdot 10^{-6}$ such that for any positive integers $n_j$ satisfying $n_{j+1}/n_j \ge d_1$ and for any vectors $v_0, v_1, \dots$ in a normed space $(E, \|\cdot\|)$, we have*

$$\int_{\mathbb{T}} \Big\| \sum_{j=0}^{N} v_j R_j \Big\| \mathrm{d}m \ge c_1 \sum_{j=0}^{N} \|v_j\|$$

*for $R_k$ defined in* (1).

We conclude with two questions: 1) Can the constant $d_p$ in Theorem 1 be chosen so that it does not depend on $p$ (is universal)? 2) Does Theorem 2 hold with $d_1 = 2$?

The article is organized as follows. In Section 2 we provide two lemmas concerning lower bounds for the $p$th norm of a sum of two functions. In Section 3 we give some auxiliary lemmas concerning factorization of integrals under the presence of highly oscillating factors. Section 4 is devoted to the proof of Theorem 2. Section 5 deals with the lower estimate for $p > 1$. Finally, in Section 6 we give a proof of the upper bound for $p > 1$.

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## 2. Auxiliary general bounds

The following simple result will lie in the heart of an induction procedure in the case $p = 1$.

**Lemma 3.** *Let $\mu$ be a measure on $X$ and let $f, g\colon X \to E$ be measurable functions. Suppose $A \subset X$ is a measurable set such that*

$$\int_A \|f\| \mathrm{d}\mu \leq \frac{1}{6} \int_X \|f\| \mathrm{d}\mu \quad \textit{and} \quad \int_{X \setminus A} \|g\| \mathrm{d}\mu \leq \frac{1}{6} \int_X \|f\| \mathrm{d}\mu.$$

*Then*

$$\int_X \|f+g\| \mathrm{d}\mu \geq \frac{1}{3} \int_X \|f\| \mathrm{d}\mu + \int_X \|g\| \mathrm{d}\mu.$$

*Proof.* We have

$$\begin{aligned}\int_X \|f+g\| \mathrm{d}\mu &\geq \int_A (\|f\| + \|g\| - 2\|f\|) \mathrm{d}\mu + \int_{X \setminus A} (\|f\| + \|g\| - 2\|g\|) \mathrm{d}\mu \\ &= \int_X \|f\| \mathrm{d}\mu + \int_X \|g\| \mathrm{d}\mu - 2 \int_A \|f\| \mathrm{d}\mu - 2 \int_{X \setminus A} \|g\| \mathrm{d}\mu \\ &\geq \frac{1}{3} \int_X \|f\| \mathrm{d}\mu + \int_X \|g\| \mathrm{d}\mu.\end{aligned}$$

□

The next result will play a role of Lemma 3 for $p > 1$. It is basically [2, Lemma 9] .

**Lemma 4.** *Let $\mu$ be a measure on $X$ and let $f, g\colon X \to E$ be measurable functions. Suppose that for some $p \geq 1$ and $\gamma > 0$, we have*

$$\int_X \|g\|^{p-1} \|f\| \mathrm{d}\mu \leq \gamma \int_X \|f\|^p \mathrm{d}\mu.$$

*Then,*

$$\int_X \|f+g\|^p \mathrm{d}\mu \geq \left( \frac{1}{3^p} - 2p\gamma \right) \int_X \|f\| \mathrm{d}\mu + \int_X \|g\|^p \mathrm{d}\mu.$$

*Proof.* For any real numbers $a, b$ we have $|a+b|^p \geq |a|^p - p|a|^{p-1}|b|$. If, additionally, $|a| \leq \frac{1}{3}|b|$, then $|a+b| \geq |b| - |a| \geq |a| + \frac{1}{3}|b|$ and thus $|a+b|^p \geq |a|^p + \frac{1}{3^p}|b|^p$. Taking $a = \|g\|$, $b = -\|f\|$ and using the inequality $\|f+g\| \geq |\|f\| - \|g\||$, we obtain

$$\begin{aligned}\int_X \|f+g\|^p \mathrm{d}\mu &= \int_X \|f+g\|^p \mathbb{1}_{\{\|g\| \leq \frac{1}{3}\|f\|\}} \mathrm{d}\mu + \int_X \|f+g\|^p \mathbb{1}_{\{\|g\| > \frac{1}{3}\|f\|\}} \mathrm{d}\mu \\ &\geq \int_X \|g\|^p \mathbb{1}_{\{\|g\| \leq \frac{1}{3}\|f\|\}} \mathrm{d}\mu + \frac{1}{3^p} \int_X \|f\|^p \mathbb{1}_{\{\|g\| \leq \frac{1}{3}\|f\|\}} \mathrm{d}\mu \\ &\quad + \int_X \|g\|^p \mathbb{1}_{\{\|g\| > \frac{1}{3}\|f\|\}} \mathrm{d}\mu - p \int_X \|g\|^{p-1} \|f\| \mathbb{1}_{\{\|g\| > \frac{1}{3}\|f\|\}} \mathrm{d}\mu \\ &= \int_X \|g\|^p \mathrm{d}\mu + \frac{1}{3^p} \int_X \|f\|^p (1 - \mathbb{1}_{\{\|g\| > \frac{1}{3}\|f\|\}}) \mathrm{d}\mu - p \int_X \|g\|^{p-1} \|f\| \mathbb{1}_{\{\|g\| > \frac{1}{3}\|f\|\}} \mathrm{d}\mu.\end{aligned}$$

Note that

$$\int_X \left(\frac{1}{3^p}\|f\|^p + p\|g\|^{p-1}\|f\|\right) \mathbb{1}_{\{\|g\|>\frac{1}{3}\|f\|\}}\mathrm{d}\mu \le \left(\frac{1}{3}+p\right)\int_X \|g\|^{p-1}\|f\|\mathrm{d}\mu \le 2p\gamma \int_X \|f\|^p\mathrm{d}\mu.$$

Therefore,

$$\int_X \|f+g\|^p\mathrm{d}\mu \ge \int_X \|g\|^p\mathrm{d}\mu + \frac{1}{3^p}\int_X \|f\|^p\mathrm{d}\mu - 2p\gamma\int_X \|f\|^p\mathrm{d}\mu.$$

□

## 3. Exact and approximate factorization of integrals

Our next lemma concerns exact algebraic factorization for integrals of products of trigonometric polynomials.

**Lemma 5.** *Suppose that $g_1,\ldots,g_{N-1}$ are trigonometric polynomials of degree at most $d$, $g_N$ is an arbitrary continuous function on $\mathbb{T}$ and $n_{j+1}/n_j \ge d+1$ for $j\ge 1$. Then*

$$\int_{\mathbb{T}} \prod_{j=1}^N g_j(n_jt)\mathrm{d}m = \prod_{j=1}^N \int_{\mathbb{T}} g_j(n_jt)\mathrm{d}m.$$

*Proof.* Since trigonometric polynomials are dense in the space $C(\mathbb{T})$ of continuous functions with the sup norm, we may assume that $g_N$ is also a trigonometric polynomial. Let $g_j(t) = \sum_{l=-d_j}^{d_j} a_{j,l}e^{ilt}$, where $d_j = d$ for $j\le N-1$. Observe that an integer of the form $\sum_{j=1}^N l_jn_j$, $l_j\in\mathbb{Z}$, $|l_j|\le d$ for $j\le N-1$ is zero if and only if all $l_j$ are zero. Hence

$$\prod_{j=1}^N g_j(n_jt) = \prod_{j=1}^N a_{j,0} + \sum_s b_se^{im_st},$$

where $m_s$ are nonzero integers. Therefore

$$\int_{\mathbb{T}} \prod_{j=1}^N g_j(n_jt)\mathrm{d}m = \prod_{j=1}^N a_{j,0} = \prod_{j=1}^N \int_{\mathbb{T}} g_j(n_jt)\mathrm{d}m.$$

□

Even if the exact factorization does not hold, one can establish approximate factorization in the presence of a highly oscillating factor. This idea is quantified in the following lemma.

**Lemma 6.** *Suppose that $f$ is a Lipschitz function on $\mathbb{T}$ and $g$ is an integrable function on $\mathbb{T}$. Then for any integer $n\ge 1$, we have*

$$\Big|\int_{\mathbb{T}} f(t)g(nt)\mathrm{d}m - \int_{\mathbb{T}} f\mathrm{d}m\int_{\mathbb{T}} g(nt)\mathrm{d}m\Big| \le \frac{2\pi}{n}\int_{\mathbb{T}} |f'(t)|\mathrm{d}m\int_{\mathbb{T}} |g(nt)|\mathrm{d}m.$$

*Proof.* Let $I_k = [\frac{k}{n}2\pi, \frac{k+1}{n}2\pi]$ for $k = 0, 1, \ldots, n-1$. Observe that for any $k$, $\int_{\mathbb{T}} g(nt)\mathrm{d}m = \frac{1}{|I_k|}\int_{I_k} g(nt)\mathrm{d}t$, hence

$$\begin{aligned}\Big|\int_{I_k} f(t)\Big(g(nt) - \int_{\mathbb{T}} g(ns)\mathrm{d}m(s)\Big)\mathrm{d}t\Big| &= \frac{1}{|I_k|}\Big|\int_{I_k\times I_k}(f(t)-f(s))g(nt)\mathrm{d}t\mathrm{d}s\Big| \\ &\le \sup_{t,s\in I_k}|f(t)-f(s)|\int_{I_k}|g(nt)|\mathrm{d}t \le \int_{I_k}|f'(u)|\mathrm{d}u\int_{I_k}|g(nt)|\mathrm{d}t \\ &= \frac{2\pi}{n}\int_{I_k}|f'(u)|\mathrm{d}u\int_{\mathbb{T}}|g(nt)|\mathrm{d}m.\end{aligned}$$

Summing the above estimate over $0 \le k \le n-1$ yields the lemma. □

In the context of trigonometric polynomials, in the above lemma we can pass from the bound in terms of $f'$ to the bound in terms of the original factor $f$. Namely, we have the following lemma.

**Lemma 7.** *Suppose that $f$ is a vector-valued trigonometric polynomial of order at most $d$. Then*

$$\int_{\mathbb{T}}\|f'\|^p\mathrm{d}m \le d^p\int_{\mathbb{T}}\|f\|^p\mathrm{d}m. \tag{4}$$

*Moreover, for any integrable (complex valued) function $h$ on $\mathbb{T}$, we have*

$$\left|\int_{\mathbb{T}}\|f(t)\|^p h(nt)\mathrm{d}m - \int_{\mathbb{T}}\|f\|^p\mathrm{d}m\int_{\mathbb{T}}h(nt)\mathrm{d}m\right| \le 2\pi\frac{pd}{n}\int_{\mathbb{T}}\|f\|^p\mathrm{d}m\int_{\mathbb{T}}|h(nt)|\mathrm{d}m. \tag{5}$$

*Proof.* Formula (3.11) in [10, Chapter X] gives $f'(t) = \sum_{k=1}^{2d} b_k f(t+t_k)$, where $\sum_{k=1}^{2d}|b_k| = d$ and $t_k = \frac{1}{d}(k-\frac{1}{2})\pi$. Thus $\|f'\|_p \le \sum_{k=1}^{2d}|b_k|\|f\|_p = d\|f\|_p$ and (4) follows.

To show (5), take $g = \|f\|^p$. Then $|g'| \le p\|f\|^{p-1}\|f'\|$ ($g$ is in fact almost everywhere differentiable) and

$$\int_{\mathbb{T}}|g'|\mathrm{d}m \le p\left(\int_{\mathbb{T}}\|f\|^p\mathrm{d}m\right)^{(p-1)/p}\left(\int_{\mathbb{T}}\|f'\|^p\mathrm{d}m\right)^{1/p} \le pd\int_{\mathbb{T}}\|f\|^p\mathrm{d}m,$$

by Hölder's inequality and estimate (4). Thus Lemma 6 yields (5). □

## 4. Lower bound for $p = 1$

As in the independent case established in [6], the proof is based on a more general lower estimate, which is shown by induction. For technical reasons (that enable the induction procedure) we need to consider a larger class of measures on the torus. Namely, for $k, l \ge 0$ by $\mathcal{F}_{k,l}$ we denote the class of all measures on $\mathbb{T}$ with densities of the form $\frac{\mathrm{d}\mu}{\mathrm{d}m} = \prod_{j=1}^{l} g_j(n_j t)$, where $g_1, \ldots, g_l$ are nonnegative trigonometric polynomials of degree at most $k$. Observe that $\mathcal{F}_{0,l}$ consists only of positive multiples of the measure $m$, $\mathcal{F}_{k,0}$ consists only of $m$ and $\mathcal{F}_{k,l} \subset \mathcal{F}_{k',l'}$ for $k' \ge k$ and $l' \ge l$.

**Proposition 8.** *There exist positive constants $C_0 \le 92\pi$ and $\lambda_0 \le \frac{47}{48}$ with the following property. If $k \ge 2, N \ge l \ge 0$, $n_{j+1}/n_j \ge C_0(k+1)+1$ for $j \ge 1$, then for any $\mu \in \mathcal{F}_{k,l}$ and any vectors $v_0, v_1, \dots, v_N$ in a normed space $(E, \|\cdot\|)$, we have*

$$\int_{\mathbb{T}} \Big\| \sum_{j=0}^{N} v_j R_j \Big\| \mathrm{d}\mu \ge \alpha \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l} v_j R_j \Big\| \mathrm{d}\mu + \sum_{j=l+1}^{N} (\beta - c_{j-l}) \|v_j\| \int_{\mathbb{T}} R_j \mathrm{d}\mu,$$

*where*

$$\alpha = \frac{1}{12} 4^{-k} \binom{2k}{k}, \quad \beta = \frac{\alpha}{2}, \quad \gamma = \frac{432\alpha}{(1-\lambda_0)(k+1)} \quad \text{and} \quad c_j = \gamma \sum_{i=0}^{j-1} \lambda_0^i, \ j \ge 1.$$

Let us see how this proposition implies Theorem 2.

*Proof of Theorem 2.* Let $k$ be the smallest integer such that $(1-\lambda_0)^2(k+1) \ge 1728$ and put $C_1 = C_0(k+1)+1$. We use the notation of Proposition 8. Observe that

$$c_m \le \frac{\gamma}{1-\lambda_0} = \frac{432\alpha}{(1-\lambda_0)^2(k+1)} \le \frac{\alpha}{4} = \frac{\beta}{2}.$$

We apply Proposition 8 with $\mu = m$, $l = 0$ and get

$$\int_{\mathbb{T}} \Big\| \sum_{j=0}^{N} v_j R_j \Big\| \mathrm{d}m \ge \alpha \|v_0\| + \sum_{j=1}^{N} (\beta - c_j) \|v_j\| \ge \frac{\alpha}{4} \sum_{j=0}^{N} \|v_j\|.$$

Since $\lambda_0 \le \frac{47}{48}$ and $C_0 \le 92\pi$, then $k+1 \le 1728 \cdot 48^2$, $C_1 \le 92\pi \cdot 1728 \cdot 48^2 + 1 \le 1.2 \cdot 10^9$, by Stirling's formula, $\alpha \sim \frac{1}{12\sqrt{\pi k}}$ and it can be checked that $\frac{\alpha}{4} \ge 5.8 \cdot 10^{-6}$. □

We now formulate several preparatory facts needed in the proof of Proposition 8. The following is an easy corollary of Lemma 7.

**Corollary 9.** *Let $\mu \in \mathcal{F}_{k,l}$ and let $v_0, v_1, \dots, v_l$ be vectors in a normed space $(E, \|\cdot\|)$. Suppose $g$ is a nonnegative integrable function on $\mathbb{T}$. If $\varepsilon > 0$, $l \ge 0$ and $n_{j+1}/n_j \ge \frac{2}{\varepsilon}\pi(k+1)+1$ for $j \ge 1$, then*

$$\begin{aligned}(1-\varepsilon) \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l} v_j R_j \Big\| \mathrm{d}\mu \int_{\mathbb{T}} g(n_{l+1}t) \mathrm{d}m \ &\le \ \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l} v_j R_j(t) \Big\| g(n_{l+1}t) \mathrm{d}\mu \\ &\le (1+\varepsilon) \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l} v_j R_j \Big\| \mathrm{d}\mu \int_{\mathbb{T}} g(n_{l+1}t) \mathrm{d}m.\end{aligned}$$

*Proof.* Let $\delta = \frac{2}{\varepsilon}\pi(k+1)+1$ and let $h := \sum_{j=0}^{l} v_j R_j \frac{d\mu}{dm}$. By the assumption $n_{j+1}/n_j \ge \delta$, so $h$ is a (vector valued) trigonometric polynomial of order

$$d \le (k+1) \sum_{j=1}^{l} n_j \le (k+1) \sum_{i=1}^{\infty} \delta^{-i} n_{l+1} = \frac{\varepsilon}{2\pi} n_{l+1}.$$

The assertion follows by Lemma 7 with $n = n_{l+1}$ and $p = 1$. □

**Lemma 10.** *Suppose that $n_{j+1}/n_j \geq k+2$ for $j \geq 1$. Then for any $\mu \in \mathcal{F}_{k,l}$ and any $v_0, \ldots, v_{l+1}$ in a normed space $(E, \|\cdot\|)$, we have*

$$\int_{\mathbb{T}} \Big\| \sum_{j=0}^{l+1} v_j R_j \Big\| \mathrm{d}\mu \geq \frac{1}{2} \|v_{l+1}\| \int_{\mathbb{T}} R_{l+1} \mathrm{d}\mu.$$

*Proof.* We have

$$\begin{aligned}\int_{\mathbb{T}} \Big\| \sum_{j=0}^{l+1} v_j R_j \Big\| \mathrm{d}\mu &\geq \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l+1} v_j R_j(t) \cos(n_{l+1}t) \Big\| \mathrm{d}\mu \\ &\geq \Big\| \int_{\mathbb{T}} \sum_{j=0}^{l+1} v_j R_j(t) \cos(n_{l+1}t) \mathrm{d}\mu \Big\|.\end{aligned}$$

Lemma 5 yields $\int_{\mathbb{T}} R_j(t)\cos(n_{l+1}t)\mathrm{d}\mu = 0$ for $j \leq l$ and

$$\begin{aligned}\int_{\mathbb{T}} R_{l+1}(t) \cos(n_{l+1}t) \mathrm{d}\mu &= \int_{\mathbb{T}} R_l \mathrm{d}\mu \int_{\mathbb{T}} (1 + \cos(n_{l+1}t)) \cos(n_{l+1}t) \mathrm{d}m(t) \\ &= \frac{1}{2} \int_{\mathbb{T}} R_l \mathrm{d}\mu = \frac{1}{2} \int_{\mathbb{T}} R_{l+1} \mathrm{d}\mu.\end{aligned}$$

□

The next lemma presents a simple upper estimate for $\sqrt{R_1}$.

**Lemma 11.** *For any $x \in [-1,1]$, we have $\sqrt{1+x} \leq 1 + \frac{1}{2}x - \frac{1}{12}x^2$. In particular,*

$$\sqrt{1+\cos x} \leq 1 + \frac{1}{2}\cos x - \frac{1}{12}\cos^2 x = \frac{23}{24} + \frac{1}{4}(e^{ix} + e^{-ix}) - \frac{1}{48}(e^{2ix} + e^{-2ix}).$$

*Proof.* We have

$$\Big(1 + \frac{1}{2}x - \frac{1}{12}x^2\Big)^2 - (1+x) = \frac{1}{12}x^2(1-x) + \frac{1}{144}x^4.$$

□

*Remark* 12. Even though we will not use this observation, let us point out that the above bound can be used to quantify singular behaviour of Riesz products. Assume $n_{k+1}/n_k \geq 3$ for $k \geq 1$. Then for $N \geq 0$,

$$m\left(R_N \geq \left(\frac{23}{24}\right)^N\right) \leq \left(\frac{23}{24}\right)^{N/2}.$$

Indeed, using Lemmas 5 and 11 we get

$$\int_{\mathbb{T}} \sqrt{R_N} \mathrm{d}m \leq \int_{\mathbb{T}} \prod_{k=1}^{N} \left(\frac{23}{24} + \frac{1}{4}(e^{in_k t} + e^{-in_k t}) - \frac{1}{48}(e^{2in_k t} + e^{-2in_k t})\right) \mathrm{d}m = \left(\frac{23}{24}\right)^N.$$

So,

$$m\left(R_N \geq \left(\frac{23}{24}\right)^N\right) = m\left(\sqrt{R_N} \geq \left(\frac{23}{24}\right)^{N/2}\right) \leq \left(\frac{23}{24}\right)^{-N/2} \int_{\mathbb{T}} \sqrt{R_N}\mathrm{d}m \leq \left(\frac{23}{24}\right)^{N/2}.$$

To formulate our next proposition, we set for $l = 1, 2, \ldots$,

$$R_{l,l-1} \equiv 1 \quad \text{and} \quad R_{l,k}(t) := \prod_{j=l}^{k}(1+\cos(n_j t)) \quad \text{for } k \geq l.$$

**Proposition 13.** *There exist $C_0 \leq 92\pi$ and $\lambda_0 \leq \frac{47}{48}$ such that the following holds. If $k \geq 2$, $l \geq 0$, $n_{j+1}/n_j \geq C_0(k+1)+1$, then for any $\mu \in \mathcal{F}_{k,l+1}$, $v_0, v_1, \ldots, v_l$ in a normed space $(E, \|\cdot\|)$ and $r \geq l+1$, we have*

$$\int_{\mathbb{T}} \Big\|\sum_{j=0}^{l} v_j R_j(t)\Big\| \sqrt{R_{l+2,r}(t)}\mathrm{d}\mu \leq \lambda_0^{r-l-1} \int_{\mathbb{T}} \Big\|\sum_{j=0}^{l} v_j R_j(t)\Big\|\mathrm{d}\mu.$$

*Proof.* Let

$$g(x) := \frac{23}{24} + \frac{1}{4}(e^{ix}+e^{-ix}) - \frac{1}{48}(e^{2ix}+e^{-2ix}).$$

By Lemma 11 we have

$$\int_{\mathbb{T}} \Big\|\sum_{j=0}^{l} v_j R_j(t)\Big\| \sqrt{R_{l+2,r}(t)}\mathrm{d}\mu \leq \int_{\mathbb{T}} \Big\|\sum_{j=0}^{l} v_j R_j(t)\Big\| \prod_{j=l+2}^{r} g(n_j t)\mathrm{d}\mu,$$

where we adopt the convention that $\prod_{j=l+2}^{l+1} g(n_j t) \equiv 1$. We will show by induction on $r$ that if $n_{j+1}/n_j \geq 92\pi(k+1)+1$, then

$$\int_{\mathbb{T}} \Big\|\sum_{j=0}^{l} v_j R_j(t)\Big\| \prod_{j=l+2}^{r} g(n_j t)\mathrm{d}\mu \leq \left(\frac{47}{48}\right)^{r-l-1} \int_{\mathbb{T}} \|\sum_{j=0}^{l} v_j R_j(t)\|\mathrm{d}\mu.$$

For $r = l+1$ this is obvious, so it is enough to show that if the bound holds for $r \geq l+1$, then it is also satisfied for $r+1$. Let $\tilde{\mu}$ be the measure with the density $\prod_{j=l+2}^{r} g(n_j t)\frac{\mathrm{d}\mu}{\mathrm{d}m}$. Then $\tilde{\mu} \in \mathcal{F}_{k,r}$ (here we have used the assumption $k \geq 2$). Hence Corollary 9 used with

$\varepsilon = \varepsilon_0 = \frac{1}{46}$, and $r, \tilde{\mu}$ instead of $l$, $\mu$ yields

$$\begin{aligned}\int_{\mathbb{T}} \Big\| \sum_{j=0}^{l} v_j R_j(t) \Big\| \prod_{j=l+2}^{r+1} g(n_j t) \mathrm{d}\mu \\ &\le (1+\varepsilon_0) \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l} v_j R_j(t) \Big\| \prod_{j=l+2}^{r} g(n_j t) \mathrm{d}\mu \int_{\mathbb{T}} g(n_{r+1} t) \mathrm{d}m \\ &= \frac{47}{48} \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l} v_j R_j(t) \Big\| \prod_{j=l+2}^{r} g(n_j t) \mathrm{d}\mu \\ &\le \Big(\frac{47}{48}\Big)^{(r+1)-l-1} \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l} v_j R_j(t) \Big\| \mathrm{d}\mu.\end{aligned}$$

□

**Corollary 14.** *If $k \ge 2$, $n_{j+1}/n_j \ge C_0(k+1)+1$ for $j \ge 1$, then for any $\mu \in \mathcal{F}_{k,l+1}$, $N \ge l+1$ and $v_0, v_1, \ldots, v_N$ in a normed space $(E, \|\cdot\|)$, we have*

$$\begin{aligned}\int_{\mathbb{T}} \Big\| \sum_{j=0}^{l} v_j R_j(t) \Big\| \Big\| \sum_{j=l+1}^{N} v_j R_{l+2,j}(t) \Big\|^{1/2} \mathrm{d}\mu \\ &\le \sum_{j=l+1}^{N} \lambda_0^{j-l-1} \|v_j\|^{1/2} \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l} v_j R_j(t) \Big\| \mathrm{d}\mu. \end{aligned} \tag{6}$$

*Moreover for any $u > 0$,*

$$\int_{A_u} \Big\| \sum_{j=0}^{l} v_j R_j(t) \Big\| \mathrm{d}\mu \le \frac{1}{\sqrt{u}} \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l} v_j R_j(t) \Big\| \mathrm{d}\mu, \tag{7}$$

*where*

$$A_u := \Big\{ t \in \mathbb{T} \colon \ \Big\| \sum_{j=l+1}^{N} v_j R_{l+2,j}(t) \Big\| \ge \frac{u}{1-\lambda_0} \sum_{j=l+1}^{N} \lambda_0^{j-l-1} \|v_j\| \Big\}.$$

*Proof.* We have

$$\Big\| \sum_{j=l+1}^{N} v_j R_{l+2,j}(t) \Big\|^{1/2} \le \sum_{j=l+1}^{N} \| v_j R_{l+2,j}(t) \|^{1/2}$$

and (6) follows by Proposition 13. By the Cauchy-Schwarz inequality

$$\Big(\sum_{j=l+1}^{N}\lambda_0^{j-l-1}\|v_j\|^{1/2}\Big)^2 \le \sum_{j=l+1}^{N}\lambda_0^{j-l-1}\sum_{j=l+1}^{N}\lambda_0^{j-l-1}\|v_j\| \le \frac{1}{1-\lambda_0}\sum_{j=l+1}^{N}\lambda_0^{j-l-1}\|v_j\|,$$

hence $A_u \subset B_u$, where

$$B_u := \Big\{t\in\mathbb{T}\colon \ \Big\|\sum_{j=l+1}^{N} v_j R_{l+2,j}(t)\Big\|^{1/2} \ge \sqrt{u}\sum_{j=l+1}^{N}\lambda_0^{j-l-1}\|v_j\|^{1/2}\Big\}.$$

Therefore

$$\int_{A_u}\Big\|\sum_{j=0}^{l} v_j R_j(t)\Big\|\mathrm{d}\mu \le \int_{B_u}\Big\|\sum_{j=0}^{l} v_j R_j(t)\Big\|\frac{\|\sum_{j=l+1}^{N} v_j R_{l+2,j}(t)\|^{1/2}}{\sqrt{u}\sum_{j=l+1}^{N}\lambda_0^{j-l-1}\|v_j\|^{1/2}}\mathrm{d}\mu$$

and (7) follows by (6). □

Before we finally give a proof of Proposition 8, we need two more lemmas.

**Lemma 15.** *For any integers $n, k \ge 1$, we have*

$$\int_{\mathbb{T}}(1-\cos(nt))^k\mathrm{d}m = 2^{-k}\binom{2k}{k}$$

*and*

$$\int_{\mathbb{T}}(1-\cos(nt))^k(1+\cos(nt))\mathrm{d}m = 2^{-k}\binom{2k}{k}\frac{1}{k+1}.$$

*Proof.* We have

$$\int_{\mathbb{T}}(1-\cos(nt))^k\mathrm{d}m = \int_{\mathbb{T}}(2\sin^2(nt/2))^k\mathrm{d}m = 2^k\int_{\mathbb{T}}\sin^{2k}t\mathrm{d}m = 2^{-k}\int_{\mathbb{T}}(-1)^k(e^{it}-e^{-it})^{2k}\mathrm{d}m = 2^{-k}\binom{2k}{k},$$

where the last equality follows from $\int_{\mathbb{T}} e^{ilt}\mathrm{d}m = \delta_{0,l}$. Similar calculations show that

$$\int_{\mathbb{T}}(1-\cos(nt))^k(1+\cos(nt))\mathrm{d}m = 2^{k+1}\int_{\mathbb{T}}\sin^{2k}t\cos^2 t\mathrm{d}m$$
$$= 2^{-k-1}\int_{\mathbb{T}}(-1)^k(e^{it}-e^{-it})^{2k}(2+e^{2it}+e^{-2it})\mathrm{d}m$$
$$= 2^{-k-1}\Big[2\binom{2k}{k}-\binom{2k}{k-1}-\binom{2k}{k+1}\Big] = 2^{-k}\binom{2k}{k}\frac{1}{k+1}.$$

□

**Lemma 16.** *Let $k, l \geq 0$, $\varphi_k(t) = \left(\frac{1-\cos(t)}{2}\right)^k$, $\mu \in \mathcal{F}_{k,l}$ and $n_{j+1}/n_j \geq k+1$ for $j \geq 1$. Then*
$$\int_{\mathbb{T}} (1+\cos(n_{l+1}t))\varphi_k(n_{l+1}t)\mathrm{d}\mu = \frac{1}{k+1}\int_{\mathbb{T}} \varphi_k(n_{l+1}t)\mathrm{d}\mu.$$

*Proof.* Let $\mathrm{d}\mu = \prod_{j=1}^{l} g_j(n_j t)\mathrm{d}m$. Lemma 5 yields
$$\begin{aligned}\int_{\mathbb{T}} &(1+\cos(n_{l+1}t))\varphi_k(n_{l+1}t)\mathrm{d}\mu \\ &= \Big(\prod_{j=1}^{l}\int_{\mathbb{T}} g_j(n_j t)\mathrm{d}m\Big)\int_{\mathbb{T}}(1+\cos(n_{l+1}t))\varphi_k(n_{l+1}t)\mathrm{d}m \\ &= \mu(\mathbb{T})2^{-k}\int_{\mathbb{T}}(1+\cos(n_{l+1}t))(1-\cos(n_{l+1}t))^k\mathrm{d}m.\end{aligned}$$
By the same token, Lemma 5 also implies that
$$\int_{\mathbb{T}} \varphi_k(n_{l+1}t)\mathrm{d}\mu = \mu(\mathbb{T})2^{-k}\int_{\mathbb{T}}(1-\cos(n_{l+1}t))^k\mathrm{d}m.$$
The assertion easily follows by Lemma 15. □

We are now ready to give a proof of Proposition 8.

*Proof of Proposition 8.* We proceed by induction on $N-l$. If $N-l=0$, the assertion is obvious, since $\alpha \leq 1$. To show the induction step we may assume that $l$ is fixed and we increased $N$. We consider two cases.

**Case 1.** $\alpha\int_{\mathbb{T}}\left\|\sum_{j=0}^{l} v_j R_j\right\|\mathrm{d}\mu \leq \gamma\sum_{j=l+1}^{N+1}\lambda_0^{j-l-1}\|v_j\|\int_{\mathbb{T}} R_j\mathrm{d}\mu$.

By the induction assumption (applied to $N+1$ and $l+1$), we have
$$\begin{aligned}\int_{\mathbb{T}}\Big\|\sum_{j=0}^{N+1} v_j R_j\Big\|\mathrm{d}\mu &\geq \alpha\int_{\mathbb{T}}\Big\|\sum_{j=0}^{l+1} v_j R_j\Big\|\mathrm{d}\mu + \sum_{j=l+2}^{N+1}(\beta - c_{j-l-1})\|v_j\|\int_{\mathbb{T}} R_j\mathrm{d}\mu \\ &\geq \beta\|v_{l+1}\|\int_{\mathbb{T}} R_{l+1}\mathrm{d}\mu + \sum_{j=l+2}^{N+1}(\beta - c_{j-l-1})\|v_j\|\int_{\mathbb{T}} R_j\mathrm{d}\mu \\ &\geq \alpha\int_{\mathbb{T}}\Big\|\sum_{j=0}^{l} v_j R_j\Big\|\mathrm{d}\mu - \gamma\sum_{j=l+1}^{N+1}\lambda_0^{j-l-1}\|v_j\|\int_{\mathbb{T}} R_j\mathrm{d}\mu \\ &\quad + \beta\|v_{l+1}\|\int_{\mathbb{T}} R_{l+1}\mathrm{d}\mu + \sum_{j=l+2}^{N+1}(\beta - c_{j-l-1})\|v_j\|\int_{\mathbb{T}} R_j\mathrm{d}\mu \\ &= \alpha\int_{\mathbb{T}}\Big\|\sum_{j=0}^{l} v_j R_j\Big\|\mathrm{d}\mu + \sum_{j=l+1}^{N+1}(\beta - c_{j-l})\|v_j\|\int_{\mathbb{T}} R_j\mathrm{d}\mu,\end{aligned}$$

where the second inequality follows by Lemma 10.

**Case 2.** $\alpha \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l} v_j R_j \Big\| \mathrm{d}\mu > \gamma \sum_{j=l+1}^{N+1} \lambda_0^{j-l-1} \|v_j\| \int_{\mathbb{T}} R_j \mathrm{d}\mu$.

Let $\varphi_k$ be as in Lemma 16 and set $p_k = \int_{\mathbb{T}} \varphi_k dm$,

$$\mathrm{d}\mu_1 := (1 - \varphi_k(n_{l+1}t))\mathrm{d}\mu \quad \text{and} \quad \mathrm{d}\mu_2 := \varphi_k(n_{l+1}t)\mathrm{d}\mu.$$

The induction assumption applied to $l+1$ and $N+1$ with measure $\mu_1 \in \mathcal{F}_{k,l+1}$ yields

$$\int_{\mathbb{T}} \Big\| \sum_{j=0}^{N+1} v_j R_j \Big\| \mathrm{d}\mu_1 \geq \alpha \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l+1} v_j R_j \Big\| \mathrm{d}\mu_1 + \sum_{j=l+2}^{N+1} (\beta - c_{j-l-1}) \|v_j\| \int_{\mathbb{T}} R_j \mathrm{d}\mu_1.$$

By Lemma 10, we have

$$\begin{aligned} \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l+1} v_j R_j \Big\| \mathrm{d}\mu_1 &= \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l+1} v_j R_j \Big\| \mathrm{d}\mu - \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l+1} v_j R_j \Big\| \mathrm{d}\mu_2 \\ &\geq \frac{1}{2} \|v_{l+1}\| \int_{\mathbb{T}} R_{l+1} \mathrm{d}\mu - \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l+1} v_j R_j \Big\| \mathrm{d}\mu_2, \end{aligned}$$

hence

$$\begin{aligned} \int_{\mathbb{T}} \Big\| \sum_{j=0}^{N+1} v_j R_j \Big\| \mathrm{d}\mu_1 \geq & \beta \|v_{l+1}\| \int_{\mathbb{T}} R_{l+1} d\mu - \alpha \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l+1} v_j R_j \Big\| \mathrm{d}\mu_2 \\ & + \sum_{j=l+2}^{N+1} (\beta - c_{j-l-1}) \|v_j\| \int_{\mathbb{T}} R_j \mathrm{d}\mu_1. \end{aligned} \tag{8}$$

Define

$$f = \sum_{j=0}^{l} v_j R_j \quad \text{and} \quad g = \sum_{j=l+1}^{N+1} v_j R_j = R_{l+1} \sum_{j=l+1}^{N+1} v_j R_{l+2,j}.$$

Corollary 9 with $\varepsilon = 1/2$ and the assumptions of Case 2 yield

$$\int_{\mathbb{T}} \|f\| \mathrm{d}\mu_2 \geq \frac{1}{2} \int_{\mathbb{T}} \|f\| \mathrm{d}\mu \int_{\mathbb{T}} \varphi_k(n_{l+1}t) \mathrm{d}m \geq \frac{\gamma}{2\alpha} p_k \sum_{j=l+1}^{N+1} \lambda_0^{j-l-1} \|v_j\| \int_{\mathbb{T}} R_j \mathrm{d}\mu.$$

Observe that for $j \geq l+1$ by Lemma 5, $\int_{\mathbb{T}} R_j \mathrm{d}\mu = \int_{\mathbb{T}} R_l \mathrm{d}\mu = \frac{1}{p_k} \int_{\mathbb{T}} R_l \mathrm{d}\mu_2$. Hence, recalling the definition of $\gamma$, we get

$$\int_{\mathbb{T}} \|f\| \mathrm{d}\mu_2 \geq \frac{216}{(1-\lambda_0)(k+1)} \sum_{j=l+1}^{N+1} \lambda_0^{j-l-1} \|v_j\| \int_{\mathbb{T}} R_l \mathrm{d}\mu_2. \tag{9}$$

Let

$$A = \Big\{ t \in \mathbb{T} \colon \Big\| \sum_{j=l+1}^{N+1} v_j R_{l+2,j}(t) \Big\| \geq \frac{36}{1-\lambda_0} \sum_{j=l+1}^{N+1} \lambda_0^{j-l-1} \|v_j\| \Big\}.$$

Then (7) with $u = 36$ implies (note that $\mu_2 \in \mathcal{F}_{k,l+1}$)

$$\int_A \|f\| \mathrm{d}\mu_2 \leq \frac{1}{6} \int_{\mathbb{T}} \|f\| \mathrm{d}\mu_2.$$

We also have

$$\int_{\mathbb{T}\setminus A} \|g\| \mathrm{d}\mu_2 \leq \frac{36}{1-\lambda_0} \sum_{j=l+1}^{N+1} \lambda_0^{j-l-1} \|v_j\| \int_{\mathbb{T}} R_{l+1} \mathrm{d}\mu_2.$$

Lemmas 5 and 16 give

$$\begin{aligned}\int_{\mathbb{T}} R_{l+1} \mathrm{d}\mu_2 &= \int_{\mathbb{T}} R_l \mathrm{d}\mu \int_{\mathbb{T}} (1+\cos(n_{l+1}t))\varphi_k(n_{l+1}t)\mathrm{d}m \\ &= \frac{1}{k+1} \int_{\mathbb{T}} R_l \mathrm{d}\mu \int_{\mathbb{T}} \varphi_k(n_{l+1}t)\mathrm{d}m = \frac{1}{k+1} \int_{\mathbb{T}} R_l \mathrm{d}\mu_2.\end{aligned}$$

Hence

$$\int_{\mathbb{T}\setminus A} \|g\| \mathrm{d}\mu_2 \leq \frac{36}{(1-\lambda_0)(k+1)} \sum_{j=l+1}^{N+1} \lambda_0^{j-l-1} \|v_j\| \int_{\mathbb{T}} R_l \mathrm{d}\mu_2 \leq \frac{1}{6} \int_{\mathbb{T}} \|f\| \mathrm{d}\mu_2,$$

where the last inequality follows by (9).

Therefore the assumptions of Lemma 3 (with $\mu_2$ instead of $\mu$) are satisfied and

$$\int_{\mathbb{T}} \|f+g\| \mathrm{d}\mu_2 \geq \frac{1}{3} \int_{\mathbb{T}} \|f\| \mathrm{d}\mu_2 + \int_{\mathbb{T}} \|g\| \mathrm{d}\mu_2.$$

Corollary 9 with $\varepsilon = 1/2$ gives

$$\int_{\mathbb{T}} \|f\| \mathrm{d}\mu_2 \geq \frac{1}{2} \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l} v_j R_j \Big\| \mathrm{d}\mu \int_{\mathbb{T}} \varphi_k(n_{l+1}t)\mathrm{d}m = 6\alpha \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l} v_j R_j \Big\| \mathrm{d}\mu,$$

where the equality follows by Lemma 15.

The induction assumption applied to $l+1, N+1$, measure $\mu_2 \in \mathcal{F}_{k,l+1}$ and $v_0 = v_1 = \ldots = v_l = 0$ yields

$$\int_{\mathbb{T}} \|g\| \mathrm{d}\mu_2 \geq \alpha \int_{\mathbb{T}} \|v_{l+1} R_{l+1}\| \mathrm{d}\mu_2 + \sum_{j=l+2}^{N+1} (\beta - c_{j-l-1}) \|v_j\| \int_{\mathbb{T}} R_j \mathrm{d}\mu_2.$$

Hence

$$\begin{aligned}\int_{\mathbb{T}} \Big\| \sum_{j=0}^{N+1} v_j R_j \Big\| \mathrm{d}\mu_2 = & \int_{\mathbb{T}} \|f+g\| \mathrm{d}\mu_2 \\ & \geq 2\alpha \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l} v_j R_j \Big\| \mathrm{d}\mu + \alpha \int_{\mathbb{T}} \|v_{l+1} R_{l+1}\| \mathrm{d}\mu_2 \\ & + \sum_{j=l+2}^{N+1} (\beta - c_{j-l-1}) \|v_j\| \int_{\mathbb{T}} R_j \mathrm{d}\mu_2. \end{aligned} \tag{10}$$

Since $\int_{\mathbb{T}} \| \sum_{j=0}^{l} v_j R_j \| \mathrm{d}\mu \geq \int_{\mathbb{T}} \| \sum_{j=0}^{l} v_j R_j \| \mathrm{d}\mu_2$, we get

$$\int_{\mathbb{T}} \Big\| \sum_{j=0}^{l} v_j R_j \Big\| \mathrm{d}\mu + \int_{\mathbb{T}} \|v_{l+1} R_{l+1}\| \mathrm{d}\mu_2 \geq \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l+1} v_j R_j \Big\| \mathrm{d}\mu_2.$$

Thus adding (8) and (10) we obtain

$$\begin{aligned}\int_{\mathbb{T}} \Big\| \sum_{j=0}^{N+1} v_j R_j \Big\| \mathrm{d}\mu \geq & \alpha \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l} v_j R_j \Big\| \mathrm{d}\mu + \beta \|v_{l+1}\| \int_{\mathbb{T}} R_{l+1} \mathrm{d}\mu \\ & + \sum_{j=l+2}^{N+1} (\beta - c_{j-l-1}) \|v_j\| \int_{\mathbb{T}} R_j \mathrm{d}\mu \\ \geq & \alpha \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l} v_j R_j \Big\| \mathrm{d}\mu + \sum_{j=l+1}^{N+1} (\beta - c_{j-l}) \|v_j\| \int_{\mathbb{T}} R_j \mathrm{d}\mu. \end{aligned}$$

□

## 5. Lower bound for $p > 1$

In this section by $C_i(p)$ and $c_i(p)$ we will denote positive and finite constants, depending only on $p$. For $f \in L^p(\mathbb{T}, m)$ we will write $\|f\|_p$ for its $L_p$-norm.

We start by establishing facts needed to derive an analogue of Lemma 10. The next lemma is a rather standard application of Bernstein polynomials, but we prove it for completeness.

**Lemma 17.** *Let $p \geq 1$ and $f_p(t) = (1 - \frac{1}{2}t^p)^{1/p}$, $t \in [0,1]$. For any $\varepsilon > 0$, there exists a polynomial $w_{\varepsilon,p}$ of degree at most $\lceil 4\varepsilon^{-2} \rceil$ such that*

$$f_p(t) \leq w_{\varepsilon,p}(t) \leq (1+\varepsilon) f_p(t) \quad \text{for } t \in [0,1].$$

*Proof.* We have $|f_p'(t)| = \frac{1}{2}t^{p-1}(1 - \frac{1}{2}t^p)^{1/p-1} \leq 2^{-1/p} \leq 1$, so $f_p$ is 1-Lipschitz. Let $S_{n,t}$ have the binomial distribution with parameters $n$ and $t$ and define $\tilde{w}_{n,p}(t) := \mathbb{E} f_p(\frac{1}{n} S_{n,t})$.

Then $\tilde{w}_{n,p}$ is a polynomial of degree at most $n$ and

$$|\tilde{w}_{n,p}(t)-f_p(t)| \le \mathbb{E}\left|f_p\left(\frac{1}{n}S_{n,t}\right)-f_p(t)\right| \le \mathbb{E}\left|\frac{1}{n}S_{n,t}-t\right| \le \frac{1}{n}\left(\mathbb{E}|S_{n,t}-nt|^2\right)^{1/2}$$
$$=\frac{1}{n}\sqrt{nt(1-t)} \le \frac{1}{2\sqrt{n}}.$$

Define $w_{\varepsilon,p}=\tilde{w}_{n,p}+\frac{1}{2\sqrt{n}}$, where $n=\lceil 4\varepsilon^{-2}\rceil$. Observe that

$$f_p(t) \le w_{\varepsilon,p}(t) \le f_p(t)+\frac{1}{\sqrt{n}} \le f_p(t)+\frac{\varepsilon}{2} \le (1+\varepsilon)f_p(t).$$

□

As in the previous section $\varphi_k(t)=(\frac{1-\cos t}{2})^k$. The definition of $\mathcal{F}^p_{k,l}$ for $p>1$ is more technical than the definition of $\mathcal{F}_{k,l}$. For $k,l\ge 1$, we say that a function $g$ on $\mathbb{T}$ belongs to $\mathcal{F}^p_{k,l}$ if it has the form

$$g(t):=\prod_{j=1}^{l} h_j(n_jt), \qquad \text{where } h_j \in \left\{1, \frac{1}{2}\varphi_k^p, 1-\frac{1}{2}\varphi_k^p\right\} \text{ for } j=1,\ldots,l.$$

We also set $\mathcal{F}^p_{k,0}:=\{1\}$. With a slight abuse of notation we will say that a measure $\mu$ on $\mathbb{T}$ belongs to $\mathcal{F}^p_{k,l}$ if it has the form $\mathrm{d}\mu = g\mathrm{d}m$ for some $g\in\mathcal{F}^p_{k,l}$.

**Lemma 18.** *Suppose that $n_{j+1}/n_j \ge 8$ for all $j\ge 1$ and let $k\ge 1$, $l\ge 0$. Then for any $g\in\mathcal{F}^p_{k,l}$, there exists a trigonometric polynomial $h$ of degree at most $C_1(p)n_lk$ such that $g\le h^p\le 2g$.*

*Proof.* There exist disjoint $I_1,I_2\subset\{1,\ldots,l\}$ such that

$$g:=2^{-|I_1|}\prod_{j\in I_1}\varphi_k^p(n_jt)\prod_{j\in I_2}\left(1-\frac{1}{2}\varphi_k^p(n_jt)\right).$$

Let $\varepsilon_j:=\frac{\ln 2}{p}2^{j-l-1}$ for $j\in I_2$ and

$$h:=2^{-\frac{|I_1|}{p}}\prod_{j\in I_1}\varphi_k(n_jt)\prod_{j\in I_2}w_{\varepsilon_j,p}(\varphi_k(n_jt)),$$

where $w_{\varepsilon_j,p}$ are polynomials given by Lemma 17. Then $h$ is a trigonometric polynomial of degree at most

$$\deg(h)\le\sum_{j\in I_1}n_jk+\sum_{j\in I_2}\lceil 4\varepsilon_j^{-2}\rceil n_jk \le \frac{8p^2}{\ln^2 2}\sum_{j=1}^{l}4^{l+1-j}n_jk \le \frac{64p^2}{\ln^2 2}n_lk.$$

Moreover,

$$g\le h^p\le g\prod_{j\in I_2}(1+\varepsilon_j)^p \le e^{p\sum_{j\in I_2}\varepsilon_j}g \le e^{\ln 2\sum_{j=1}^{l}2^{j-l-1}}g\le 2g.$$

□

**Lemma 19.** *Let $f_1$ and $f_2$ be vector-valued trigonometric polynomials of degree at most $d$. Then for $n > C_2(p)d$, we have*

$$\int_{\mathbb{T}} \|f_1(t) + f_2(t)\cos(nt)\|^p \mathrm{d}m \geq c_2(p) \int_{\mathbb{T}} \|f_2(t)(1+\cos(nt))\|^p \mathrm{d}m.$$

*Proof.* By (5) for $C_2(p) \geq 4\pi p$, we have $\|f_2(t)(1+\cos(nt))\|_p \sim_p \|f_2\|_p \sim_p \|f_2(t)\cos(nt)\|_p$, hence it is enough to show that

$$\|f_1(t) + f_2(t)\cos(nt)\|_p \geq \tilde{c}_2(p)\|f_2(t)\cos(nt)\|_p.$$

By the triangle inequality we have

$$\|f_1(t) + f_2(t)\cos(nt)\|_p \geq \|f_2(t)\cos(nt)\|_p - \|f_1\|_p,$$

so we can further assume that $\|f_1\|_p \geq \frac{1}{2}\|f_2(t)\cos(nt)\|_p \sim_p \|f_2\|_p$. Changing variables and evoking the triangle inequality yields,

$$\begin{aligned} 2\|f_1(t) + f_2(t)\cos(nt)\|_p &= \|f_1(t) + f_2(t)\cos(nt)\|_p + \|f_1(t+\pi/n) - f_2(t+\pi/n)\cos(nt)\|_p \\ &\geq \|f_1(t) + f_2(t)\cos(nt)\|_p + \|f_1(t) - f_2(t)\cos(nt)\|_p \\ &\qquad - \|f_1(t) - f_1(t+\pi/n)\|_p - \|(f_2(t) - f_2(t+\pi/n))\cos(nt)\|_p \\ &\geq 2\|f_1\|_p - \|f_1(t) - f_1(t+\pi/n)\|_p - \|f_2(t) - f_2(t+\pi/n)\|_p. \end{aligned}$$

Note that by (4) for $i = 1, 2$, we have

$$\|f_i(t) - f_i(t+\pi/n)\|_p = \left\| \int_0^{\pi/n} f_i'(t+s)\mathrm{d}s \right\|_p \leq \int_0^{\pi/n} \|f_i'(t+s)\|_p \mathrm{d}s = \frac{\pi}{n}\|f_i'\|_p \leq \frac{\pi d}{n}\|f_i\|_p.$$

This essentially finishes the proof. □

The next lemma is the announced analogue of Lemma 10.

**Lemma 20.** *Suppose that $k \geq 1$, $l \geq 0$ and $n_{j+1}/n_j \geq C_3(p)k$ for $j \geq 1$. Then for any $\mu \in \mathcal{F}^p_{k,l}$ and any vectors $v_0, \ldots, v_{l+1}$ in a normed space $(E, \|\cdot\|)$, we have*

$$\int_{\mathbb{T}} \Big\| \sum_{j=0}^{l+1} v_j R_j \Big\|^p \mathrm{d}\mu \geq c_3(p)\|v_{l+1}\|^p \int_{\mathbb{T}} R_{l+1}^p \mathrm{d}\mu.$$

*Proof.* We may assume that $C_3(p) \geq 8$. Let $g = \frac{\mathrm{d}\mu}{\mathrm{d}m}$ and $h$ be a trigonometric polynomial given by Lemma 18. We have

$$\int_{\mathbb{T}} \Big\| \sum_{j=0}^{l+1} v_j R_j \Big\|^p \mathrm{d}\mu \geq \frac{1}{2} \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l+1} v_j R_j \Big\|^p h^p \mathrm{d}m.$$

Observe that

$$\sum_{j=0}^{l+1} v_j R_j h = fh + v_{l+1}\cos(n_{l+1}t)R_l h,$$

where $f$ is a vector-valued trigonometric polynomial. Moreover,

$$\max\{\deg(R_l h), \deg(fh)\} \le \deg(h) + \sum_{j=1}^{l} n_j \le (C_1(p)+2)n_l k$$

and the assertion easily follows by Lemma 19. □

**Lemma 21.** *For any $p > 1$, there exists a real polynomial $w_p$ such that $x^{p-1} \le w_p^p(x)$ for $x \in [0,2]$ and*

$$\lambda_1(p) := \frac{\int_{\mathbb{T}} w_p^p(X_1)\mathrm{d}m}{\left(\int_{\mathbb{T}} X_1^p \mathrm{d}m\right)^{(p-1)/p}} < 1.$$

*Proof.* By the Weierstrass approximation theorem, for any $\varepsilon > 0$, there exists a polynomial $w_p$ such that $x^{(p-1)/p} \le w_p(x) \le x^{(p-1)/p} + \varepsilon$ for $x \in [0,2]$. It is enough to observe that

$$\left(\int_{\mathbb{T}} w_p^p(X_1)\mathrm{d}m\right)^{1/p} \le \left(\int_{\mathbb{T}} X_1^{p-1}\mathrm{d}m\right)^{1/p} + \varepsilon,$$

$$\left(\int_{\mathbb{T}} X_1^p\mathrm{d}m\right)^{1/p} = \left(\int_{\mathbb{T}} X_1^{p-1}\mathrm{d}m\right)^{1/(p-1)} + c_p,$$

where $c_p > 0$. The assertion follows by taking sufficiently small $\varepsilon = \varepsilon_p$.

□

**Lemma 22.** *For any $p \ge 1$ there exists a constant $C_4(p)$ such that for any $j, k \ge 1$,*

$$\int_{\mathbb{T}} X_j^p \varphi_k^p(n_j t)\mathrm{d}m \le \frac{C_4(p)}{k}\int_{\mathbb{T}} X_j^p \mathrm{d}m \int_{\mathbb{T}} \varphi_k^p \mathrm{d}m.$$

*Proof.* Let $l = \lfloor kp \rfloor$. Then $\varphi_{l+1} \le \varphi_k^p \le \varphi_l$. Moreover, $X_j^p \le 2^{p-1}X_j$, so by Lemma 15

$$\int_{\mathbb{T}} X_j^p \varphi_k^p(n_j t)\mathrm{d}m \le 2^{p-1}\int_{\mathbb{T}} X_j \varphi_l(n_j t)\mathrm{d}m = 2^{p-1}4^{-l}\binom{2l}{l}\frac{1}{l+1}.$$

On the other hand using again Lemma 15 we get

$$\int_{\mathbb{T}} \varphi_k^p \mathrm{d}m \ge \int_{\mathbb{T}} \varphi_{l+1}\mathrm{d}m = 4^{-l-1}\binom{2(l+1)}{l+1} \ge \frac{1}{2}4^{-l}\binom{2l}{l}.$$

Since $\int_{\mathbb{T}} X_j^p \mathrm{d}m = \int_{\mathbb{T}} X_1^p \mathrm{d}m$, it is enough to take $C_4(p) = 2^p/(p\int_{\mathbb{T}} X_1^p \mathrm{d}m)$. □

**Lemma 23.** *For $p > 1$, there exist constants $C_5(p), C_6(p), C_7(p)$ and $\lambda_2(p) < 1$ with the following property. If $n_{j+1}/n_j \ge C_5(p)k$ for $j \ge 1$, $k \ge 1, l \ge 0$, then for any $\mu \in \mathcal{F}_{k,l}^p$, any $N \ge l+1$ and any vector valued polynomial $f$ of order at most $2n_l$, we have*

$$\int_{\mathbb{T}} \|f\|^p \varphi_k^p(n_{l+1}t)\mathrm{d}\mu \ge \frac{1}{4}\int_{\mathbb{T}} \|f\|^p \mathrm{d}\mu \int_{\mathbb{T}} \varphi_k^p \mathrm{d}m \tag{11}$$

*and*

$$\begin{aligned}&\int_{\mathbb{T}} \|f\| R_N^{p-1} \varphi_k^p(n_{l+1}t)\mathrm{d}\mu \\ &\leq \frac{C_6(p)}{k^{(p-1)/p}} \lambda_2(p)^{N-l-1} \left(\int_{\mathbb{T}} \|f\|^p \mathrm{d}\mu\right)^{1/p} \left(\int_{\mathbb{T}} \varphi_k^p \mathrm{d}m\right) \left(\int_{\mathbb{T}} R_N^p \mathrm{d}\mu\right)^{(p-1)/p}. \end{aligned} \tag{12}$$

*Moreover for any* $v_{l+1}, \ldots, v_N$ *we have*

$$\begin{aligned}&\int_{\mathbb{T}} \|f\| \left\| \sum_{j=l+1}^N v_j R_j \right\|^{p-1} \varphi_k^p(n_{l+1}t)\mathrm{d}\mu \\ &\leq \frac{C_7(p)}{k^{(p-1)/p}} \left(\int_{\mathbb{T}} \|f\|^p \mathrm{d}\mu\right)^{1/p} \left(\int_{\mathbb{T}} \varphi_k^p \mathrm{d}m\right) \left(\sum_{j=l+1}^N \lambda_2(p)^{j-l-1} \|v_j\|^p \int_{\mathbb{T}} R_j^p \mathrm{d}\mu\right)^{(p-1)/p}. \end{aligned} \tag{13}$$

*Proof.* Let $g = \frac{\mathrm{d}\mu}{\mathrm{d}m}$ and $h$ be a trigonometric polynomial given by Lemma 18. Notice that $hf$ is a vector-valued trigonometric polynomial with degree at most $(C_1(p)+2)n_l k$. Thus by (5) we have for sufficiently large $C_5(p)$,

$$\begin{aligned}\int_{\mathbb{T}} \|f\|^p \varphi_k^p(n_{l+1}t)\mathrm{d}\mu &\geq \frac{1}{2}\int_{\mathbb{T}} \|fh\|^p \varphi_k^p(n_{l+1}t)\mathrm{d}m \geq \frac{1}{4}\int_{\mathbb{T}} \|fh\|^p \mathrm{d}m \int_{\mathbb{T}} \varphi_k^p \mathrm{d}m \\ &\geq \frac{1}{4}\int_{\mathbb{T}} \|f\|^p \mathrm{d}\mu \int_{\mathbb{T}} \varphi_k^p \mathrm{d}m.\end{aligned}$$

To establish (12), let us define $\mathrm{d}\tilde{\mu} = h^p(t)\varphi_k^p(n_{l+1}t)\mathrm{d}m$. By Hölder's inequality, we have

$$\begin{aligned}\int_{\mathbb{T}} \|f\| R_N^{p-1} \varphi_k^p(n_{l+1}t)\mathrm{d}\mu &\leq \int_{\mathbb{T}} \|f\| R_N^{p-1} \mathrm{d}\tilde{\mu} \\ &\leq \left(\int_{\mathbb{T}} \|f\|^p R_{l+2,N}^{p-1} \mathrm{d}\tilde{\mu}\right)^{1/p} \left(\int_{\mathbb{T}} R_{l+1}^p R_{l+2,N}^{p-1} \mathrm{d}\tilde{\mu}\right)^{(p-1)/p}.\end{aligned}$$

Let $w_p$ be given by Lemma 21 and $\varepsilon = \varepsilon_p$ be a small positive number to be chosen later. By (5), if $C_5(p)$ is sufficiently large, we have

$$\begin{aligned}\int_{\mathbb{T}} \|f\|^p R_{l+2,N}^{p-1} \mathrm{d}\tilde{\mu} &\le \int_{\mathbb{T}} \|f\|^p \prod_{j=l+2}^{N} w_p^p(X_j) \mathrm{d}\tilde{\mu} \\ &\le (1+\varepsilon) \int_{\mathbb{T}} \|f\|^p \prod_{j=l+2}^{N-1} w_p^p(X_j) \mathrm{d}\tilde{\mu} \int_{\mathbb{T}} w_p^p(X_N) \mathrm{d}m \le \dots \\ &\le (1+\varepsilon)^{N-l-1} \int_{\mathbb{T}} \|f\|^p \mathrm{d}\tilde{\mu} \prod_{j=l+2}^{N} \int_{\mathbb{T}} w_p^p(X_j) \mathrm{d}m \\ &\le (1+\varepsilon)^{N-l} \int \|fh\|^p \mathrm{d}m \int_{\mathbb{T}} \varphi_k^p \mathrm{d}m \prod_{j=l+2}^{N} \int_{\mathbb{T}} w_p^p(X_j) \mathrm{d}m \\ &\le 2(1+\varepsilon)^{N-l} \lambda_1(p)^{N-l-1} \int_{\mathbb{T}} \|f\|^p \mathrm{d}\mu \int_{\mathbb{T}} \varphi_k^p \mathrm{d}m \prod_{j=l+2}^{N} \left( \int_{\mathbb{T}} X_j^p \mathrm{d}m \right)^{(p-1)/p}.\end{aligned}$$

In the same way we show that

$$\begin{aligned}&\int_{\mathbb{T}} R_{l+1}^p R_{l+2,N}^{p-1} \mathrm{d}\tilde{\mu} \\ &\quad \le 2(1+\varepsilon)^{N-l} \lambda_1(p)^{N-l-1} \int_{\mathbb{T}} R_l^p \mathrm{d}\mu \int_{\mathbb{T}} X_{l+1}^p \varphi_k^p(n_{l+1}t) \mathrm{d}m \prod_{j=l+2}^{N} \left( \int_{\mathbb{T}} X_j^p \mathrm{d}m \right)^{(p-1)/p}.\end{aligned}$$

The above estimates together with Lemma 22 yield

$$\begin{aligned}\int_{\mathbb{T}} \|f\| R_N^{p-1} \varphi_k^p(n_{l+1}t) \mathrm{d}\mu \le 2 \left( \frac{C_4(p)}{k} \right)^{(p-1)/p} &(1+\varepsilon)^{N-l} \lambda_1(p)^{N-l-1} \left( \int_{\mathbb{T}} \|f\|^p \mathrm{d}\mu \right)^{1/p} \\ &\times \left( \int_{\mathbb{T}} \varphi_k^p \mathrm{d}m \right) \left( \int_{\mathbb{T}} R_l^p \mathrm{d}\mu \prod_{j=l+1}^{N} \int_{\mathbb{T}} X_j^p \mathrm{d}m \right)^{(p-1)/p}.\end{aligned}$$

Estimate (5) implies however that for sufficiently large $C_5(p)$,

$$\int_{\mathbb{T}} R_N^p d\mu \ge \frac{1}{2}(1-\varepsilon) \int_{\mathbb{T}} R_{N-1}^p h^p \mathrm{d}m \int_{\mathbb{T}} X_N^p \mathrm{d}m \ge \dots \ge \frac{1}{2}(1-\varepsilon)^{N-l} \int_{\mathbb{T}} R_l^p g \mathrm{d}m \prod_{j=l+1}^{N} \int_{\mathbb{T}} X_j^p \mathrm{d}m.$$

To derive (12) we choose $\varepsilon = \varepsilon_p$ in such a way that

$$\lambda_2(p) := (1+\varepsilon)(1-\varepsilon)^{(1-p)/p} \lambda_1(p) < 1.$$

To show (13) we consider two cases. First assume that $1 < p \leq 2$. By (12), we have

$$\begin{aligned}
&\int_{\mathbb{T}} \|f\| \left\| \sum_{j=l+1}^{N} v_j R_j \right\|^{p-1} \varphi_k^p(n_{l+1}t)\mathrm{d}\mu \\
&\leq \int_{\mathbb{T}} \|f\| \sum_{j=l+1}^{N} \|v_j R_j\|^{p-1} \varphi_k^p(n_{l+1}t)\mathrm{d}\mu \\
&\leq \frac{C_6(p)}{k^{(p-1)/p}} \left(\int_{\mathbb{T}} \|f\|^p \mathrm{d}\mu\right)^{1/p} \left(\int \varphi_k^p \mathrm{d}m\right) \sum_{j=l+1}^{N} \lambda_2(p)^{j-l-1} \|v_j\|^{p-1} \left(\int_{\mathbb{T}} R_j^p \mathrm{d}\mu\right)^{(p-1)/p}.
\end{aligned}$$

However

$$\begin{aligned}
&\sum_{j=l+1}^{N} \lambda_2(p)^{j-l-1} \|v_j\|^{p-1} \left(\int_{\mathbb{T}} R_j^p \mathrm{d}\mu\right)^{(p-1)/p} \\
&\qquad \leq \left(\sum_{j=l+1}^{N} \lambda_2(p)^{j-l-1}\right)^{1/p} \left(\sum_{j=l+1}^{N} \lambda_2(p)^{j-l-1} \|v_j\|^p \int_{\mathbb{T}} R_j^p \mathrm{d}\mu\right)^{(p-1)/p} \\
&\qquad \leq (1-\lambda_2(p))^{-1/p} \left(\sum_{j=l+1}^{N} \lambda_2(p)^{j-l-1} \|v_j\|^p \int_{\mathbb{T}} R_j^p \mathrm{d}\mu\right)^{(p-1)/p}.
\end{aligned}$$

Finally, if $p > 2$, we have by the triangle inequality in $L_{p-1}$ and (12)

$$\begin{aligned}
&\int_{\mathbb{T}} \|f\| \left\| \sum_{j=l+1}^{N} v_j R_j \right\|^{p-1} \varphi_k^p(n_{l+1}t)\mathrm{d}\mu \\
&\leq \left(\sum_{j=l+1}^{N} \|v_j\| \left(\int_{\mathbb{T}} \|f\| R_j^{p-1} \varphi_k^p(n_{l+1}t)\mathrm{d}\mu\right)^{1/(p-1)}\right)^{p-1} \\
&\leq \frac{C_6(p)}{k^{(p-1)/p}} \left(\int_{\mathbb{T}} \|f\|^p \mathrm{d}\mu\right)^{1/p} \left(\int_{\mathbb{T}} \varphi_k^p \mathrm{d}m\right) \left(\sum_{j=l+1}^{N} \|v_j\| \lambda_2(p)^{(j-l-1)/(p-1)} \left(\int_{\mathbb{T}} R_j^p \mathrm{d}\mu\right)^{1/p}\right)^{p-1}.
\end{aligned}$$

To finish the proof of (13) in this case it is enough to observe that by Hölder's inequality

$$\begin{aligned}
&\sum_{j=l+1}^{N} \|v_j\| \lambda_2(p)^{(j-l-1)/(p-1)} \left( \int_{\mathbb{T}} R_j^p \mathrm{d}\mu \right)^{1/p} \\
&\leq \left( \sum_{j=l+1}^{N} \lambda_2(p)^{(j-l-1)/(p-1)^2} \right)^{(p-1)/p} \left( \sum_{j=l+1}^{N} \lambda_2(p)^{j-l-1} \|v_j\|^p \int_{\mathbb{T}} R_j^p \mathrm{d}\mu \right)^{1/p} \\
&\leq \left( 1 - \lambda_2(p)^{1/(p-1)^2} \right)^{(1-p)/p} \left( \sum_{j=l+1}^{N} \lambda_2(p)^{j-l-1} \|v_j\|^p \int_{\mathbb{T}} R_j^p \mathrm{d}\mu \right)^{1/p}.
\end{aligned}$$

□

**Proposition 24.** *If $k \geq 2, N \geq l \geq 0$, $n_{j+1}/n_j \geq \max\{C_3(p), C_5(p), 8\}k$ for $j \geq 1$, then for any $\mu \in \mathcal{F}_{k,l}^p$ and any vectors $v_0, v_1, \ldots, v_N$ in a normed space $(E, \|\cdot\|)$ we have*

$$\int_{\mathbb{T}} \Big\| \sum_{j=0}^{N} v_j R_j \Big\|^p \mathrm{d}\mu \geq \alpha_p \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l} v_j R_j \Big\|^p \mathrm{d}\mu + \sum_{j=l+1}^{N} (\beta_p - c_{p,j-l}) \|v_j\|^p \int_{\mathbb{T}} R_j^p \mathrm{d}\mu,$$

*where*

$$\alpha_p = \frac{1}{16 \cdot 3^p} \int \varphi_k^p \mathrm{d}m, \quad \beta_p = \frac{c_3(p)}{2} \alpha_p, \quad \gamma_p = (16p3^p C_7(p))^{\frac{p}{p-1}} \frac{\alpha_p}{k} \quad \textit{and} \quad c_{p,j} = \gamma_p \sum_{i=0}^{j-1} \lambda_2(p)^i.$$

*Proof.* We proceed by induction on $N - l$. If $N - l = 0$ the assertion is obvious, since $\alpha_p \leq 1$. To show the induction step we may assume that $l$ is fixed and we increased $N$. We consider two cases.

**Case 1.** $\alpha_p \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l} v_j R_j \Big\|^p \mathrm{d}\mu \leq \gamma_p \sum_{j=l+1}^{N+1} \lambda_2(p)^{j-l-1} \|v_j\|^p \int_{\mathbb{T}} R_j^p \mathrm{d}\mu.$

By the induction assumption (applied to $N+1$ and $l+1$), we have

$$\begin{aligned}\int_{\mathbb{T}} \Big\| \sum_{j=0}^{N+1} v_j R_j \Big\|^p \mathrm{d}\mu &\geq \alpha_p \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l+1} v_j R_j \Big\|^p \mathrm{d}\mu + \sum_{j=l+2}^{N+1} (\beta_p - c_{p,j-l-1}) \|v_j\|^p \int_{\mathbb{T}} R_j^p \mathrm{d}\mu \\ &\geq \beta_p \|v_{l+1}\|^p \int_{\mathbb{T}} R_{l+1}^p \mathrm{d}\mu + \sum_{j=l+2}^{N+1} (\beta_p - c_{p,j-l-1}) \|v_j\|^p \int_{\mathbb{T}} R_j^p \mathrm{d}\mu \\ &\geq \alpha_p \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l} v_j R_j \Big\|^p \mathrm{d}\mu - \gamma_p \sum_{j=l+1}^{N+1} \lambda_2(p)^{j-l-1} \|v_j\|^p \int_{\mathbb{T}} R_j^p \mathrm{d}\mu \\ &\quad + \beta_p \|v_{l+1}\|^p \int_{\mathbb{T}} R_{l+1}^p \mathrm{d}\mu + \sum_{j=l+2}^{N+1} (\beta_p - c_{p,j-l-1}) \|v_j\|^p \int_{\mathbb{T}} R_j^p \mathrm{d}\mu \\ &= \alpha_p \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l} v_j R_j \Big\|^p \mathrm{d}\mu + \sum_{j=l+1}^{N+1} (\beta_p - c_{p,j-l}) \|v_j\|^p \int_{\mathbb{T}} R_j^p \mathrm{d}\mu,\end{aligned}$$

where the second inequality follows by Lemma 20.

**Case 2.** $\alpha_p \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l} v_j R_j \Big\|^p \mathrm{d}\mu > \gamma_p \sum_{j=l+1}^{N+1} \lambda_2(p)^{j-l-1} \|v_j\|^p \int_{\mathbb{T}} R_j^p \mathrm{d}\mu$.

Let

$$\mathrm{d}\mu_1 = \Big(1 - \frac{1}{2}\varphi_k^p(n_{l+1}t)\Big)\,\mathrm{d}\mu \quad \text{and} \quad \mathrm{d}\mu_2 = \frac{1}{2}\varphi_k^p(n_{l+1}t)\mathrm{d}\mu.$$

The induction assumption applied to $l+1$ and $N+1$ with the measure $\mu_1 \in \mathcal{F}_{k,l+1}^p$ yields

$$\int_{\mathbb{T}} \Big\| \sum_{j=0}^{N+1} v_j R_j \Big\|^p \mathrm{d}\mu_1 \geq \alpha_p \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l+1} v_j R_j \Big\|^p \mathrm{d}\mu_1 + \sum_{j=l+2}^{N+1} (\beta_p - c_{p,j-l-1}) \|v_j\|^p \int_{\mathbb{T}} R_j^p \mathrm{d}\mu_1.$$

Since $1 - \frac{1}{2}\varphi_k^p \geq \frac{1}{2}$, we get by Lemma 20

$$\int_{\mathbb{T}} \Big\| \sum_{j=0}^{l+1} v_j R_j \Big\|^p \mathrm{d}\mu_1 \geq \frac{1}{2} \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l+1} v_j R_j \Big\|^p \mathrm{d}\mu \geq \frac{1}{2} c_3(p) \|v_{l+1}\|^p \int_{\mathbb{T}} R_{l+1}^p \mathrm{d}\mu,$$

hence

$$\int_{\mathbb{T}} \Big\| \sum_{j=0}^{N+1} v_j R_j \Big\|^p \mathrm{d}\mu_1 \geq \beta_p \|v_{l+1}\|^p \int_{\mathbb{T}} R_{l+1}^p d\mu + \sum_{j=l+2}^{N+1} (\beta_p - c_{p,j-l-1}) \|v_j\|^p \int_{\mathbb{T}} R_j^p \mathrm{d}\mu_1. \tag{14}$$

Define

$$f = \sum_{j=0}^{l} v_j R_j \quad \text{and} \quad g = \sum_{j=l+1}^{N+1} v_j R_j.$$

Estimate (11) and the assumptions of Case 2 yield

$$\int_{\mathbb{T}} \|f\|^p \mathrm{d}\mu_2 \geq \frac{1}{8}\int_{\mathbb{T}} \|f\|^p \mathrm{d}\mu \int_{\mathbb{T}} \varphi_k^p \mathrm{d}m$$
$$\geq \frac{1}{8}\left(\int_{\mathbb{T}} \|f\|^p \mathrm{d}\mu\right)^{1/p}\left(\int_{\mathbb{T}} \varphi_k^p \mathrm{d}m\right)\left(\frac{\gamma_p}{\alpha_p}\sum_{j=l+1}^{N+1}\lambda_2(p)^{j-l-1}\|v_j\|^p\int_{\mathbb{T}} R_j^p \mathrm{d}\mu\right)^{(p-1)/p}.$$

On the other hand, by (13) we get

$$\int_{\mathbb{T}} \|f\|\|g\|^{p-1}\mathrm{d}\mu_2$$
$$\leq \frac{C_7(p)}{2k^{(p-1)/p}}\left(\int_{\mathbb{T}} \|f\|^p \mathrm{d}\mu\right)^{1/p}\left(\int_{\mathbb{T}} \varphi_k^p \mathrm{d}m\right)\left(\sum_{j=l+1}^{N+1}\lambda_2(p)^{j-l-1}\|v_j\|^p\int_{\mathbb{T}} R_j^p \mathrm{d}\mu\right)^{(p-1)/p}.$$

Thus

$$\int_{\mathbb{T}} \|f\|\|g\|^{p-1}\mathrm{d}\mu_2 \leq \frac{1}{4p3^p}\int_{\mathbb{T}} \|f\|^p \mathrm{d}\mu_2$$

and Lemma 4 gives

$$\int_{\mathbb{T}} \Big\|\sum_{j=0}^{N+1} v_j R_j\Big\|^p \mathrm{d}\mu_2 = \int_{\mathbb{T}} \|f+g\|^p \mathrm{d}\mu_2 \geq \frac{1}{2\cdot 3^p}\int_{\mathbb{T}} \|f\|^p \mathrm{d}\mu_2 + \int_{\mathbb{T}} \|g\|^p \mathrm{d}\mu_2.$$

Inequality (11) gives

$$\frac{1}{2\cdot 3^p}\int_{\mathbb{T}} \|f\|^p \mathrm{d}\mu_2 \geq \frac{1}{16\cdot 3^p}\int_{\mathbb{T}} \|f\|^p \mathrm{d}\mu \int \varphi_k^p \mathrm{d}m = \alpha_p \int_{\mathbb{T}} \left\|\sum_{j=0}^{l} v_j R_j\right\|^p \mathrm{d}\mu.$$

The induction assumption applied to $l+1, N+1$ and measure $\mu_2 \in \mathcal{F}_{k,l+1}^p$ yields

$$\int_{\mathbb{T}} \|g\|^p \mathrm{d}\mu_2 \geq \alpha_p \int_{\mathbb{T}} \|v_{l+1}R_{l+1}\|^p \mathrm{d}\mu_2 + \sum_{j=l+2}^{N+1}(\beta_p - c_{p,j-l-1})\|v_j\|^p \int_{\mathbb{T}} R_j^p \mathrm{d}\mu_2.$$

Thus

$$\int_{\mathbb{T}} \Big\|\sum_{j=0}^{N+1} v_j R_j\Big\|^p \mathrm{d}\mu_2 \geq \alpha_p \int_{\mathbb{T}} \left\|\sum_{j=0}^{l} v_j R_j\right\|^p \mathrm{d}\mu + \sum_{j=l+2}^{N+1}(\beta_p - c_{p,j-l-1})\|v_j\|^p \int_{\mathbb{T}} R_j^p \mathrm{d}\mu_2. \tag{15}$$

Adding (14) and (15), we obtain

$$\begin{aligned}\int_{\mathbb{T}} \Big\| \sum_{j=0}^{N+1} v_j R_j \Big\|^p \mathrm{d}\mu \geq& \alpha_p \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l} v_j R_j \Big\|^p \mathrm{d}\mu + \beta_p \|v_{l+1}\|^p \int_{\mathbb{T}} R_{l+1}^p \mathrm{d}\mu \\ &+ \sum_{j=l+2}^{N+1} (\beta_p - c_{p,j-l-1}) \|v_j\|^p \int_{\mathbb{T}} R_j^p \mathrm{d}\mu \\ \geq& \alpha_p \int_{\mathbb{T}} \Big\| \sum_{j=0}^{l} v_j R_j \Big\|^p \mathrm{d}\mu + \sum_{j=l+1}^{N+1} (\beta_p - c_{p,j-l}) \|v_j\|^p \int_{\mathbb{T}} R_j^p \mathrm{d}\mu.\end{aligned}$$

□

*Proof of Theorem 1 (lower bound).* The proof is similar to the proof of Theorem 2. Let $k = k(p)$ be the smallest integer such that $\frac{\gamma_p}{1-\lambda_2(p)} \leq \frac{\beta_p}{2}$. Then $c_{p,m} \leq \frac{\gamma_p}{1-\lambda_2(p)} \leq \frac{\beta_p}{2}$ and thus applying Proposition 24 with $l = 0$ and $\mu = m$ yields the result with the constant $c_p = \min(\alpha_p, \beta_p/2)$. □

## 6. Proof of the upper bound

All the integrals over the one dimensional torus $\mathbb{T}$ appearing in this section are with respect to its (normalised) Haar measure $m$. We shall need three preparatory facts. The first two are immediate corollaries to Lemma 7.

**Corollary 25.** *For $p \geq 1$ and a nonzero integer $n$, we have*

$$\left| \int_{\mathbb{T}} R_k(t)^p e^{int} \right| \leq \frac{2\pi p \deg R_k}{|n|} \int_{\mathbb{T}} R_k^p, \qquad k \geq 0. \tag{16}$$

**Corollary 26.** *Let $p \geq 1$, $d \geq 2\pi p + 1$ and $n_{j+1}/n_j \geq d$, $j \geq 1$. For positive integers $k < l$, we have*

$$1 - \frac{2\pi p}{d-1} \leq \frac{\int_{\mathbb{T}} R_{k,l}^p X_{l+1}^p}{\int_{\mathbb{T}} R_{k,l}^p \int_{\mathbb{T}} X_{l+1}^p} \leq 1 + \frac{2\pi p}{d-1}. \tag{17}$$

*In particular, for $k \geq 0$, $l \geq 1$,*

$$\left(1 - \frac{2\pi p}{d-1}\right)^{l-1} \leq \frac{\int_{\mathbb{T}} X_{k+1}^p \dots X_{k+l}^p}{\int_{\mathbb{T}} X_{k+1}^p \dots \int_{\mathbb{T}} X_{k+l}^p} \leq \left(1 + \frac{2\pi p}{d-1}\right)^{l-1}. \tag{18}$$

*Proof.* Note that

$$\frac{\deg(R_{k,l})}{n_{l+1}} = \frac{n_k + \ldots + n_l}{n_{l+1}} \leq \frac{1}{d^{l-k+1}} + \ldots + \frac{1}{d} < \frac{1}{d-1},$$

hence applying Lemma 7 for $f = R_{k,l}$, $h(t) = (1 + \cos t)^p$ and $n = n_{l+1}$ gives

$$\left| \int_{\mathbb{T}} R_{k,l}^p X_{l+1}^p - \int_{\mathbb{T}} R_{k,l}^p \int_{\mathbb{T}} X_{l+1}^p \right| \leq \frac{2\pi p}{d-1} \int_{\mathbb{T}} R_{k,l}^p \int_{\mathbb{T}} X_{l+1}^p.$$

Iterating (17) yields (18). □

**Lemma 27.** *Let $p \geq 1$, $d > 2p+1$ and $n_{j+1}/n_j \geq d$, $j \geq 1$. Then for every $k \geq 0$, $m \geq 1$ and nonnegative integers $l_1, \ldots, l_m \leq p$, we have*

$$\int_{\mathbb{T}} R_k^p X_{k+1}^{l_1} \ldots X_{k+m}^{l_m} \leq (1+\epsilon) \int_{\mathbb{T}} R_k^p \int_{\mathbb{T}} X_{k+1}^{l_1} \ldots X_{k+m}^{l_m}, \tag{19}$$

*where $\epsilon = \frac{4\pi d}{d-1} \frac{p(2p+1)}{d-2p-1}$.*

*Proof.* For any $t$,

$$\left(1 + \frac{e^{it} + e^{-it}}{2}\right)^l = \frac{1}{2^l}\left(e^{it/2} + e^{-it/2}\right)^{2l} = \sum_{j=-l}^{l} \frac{1}{2^l}\binom{2l}{j+l} e^{itj}.$$

Thus,

$$X_k^l(t) = \sum_{j=-l}^{l} \frac{1}{2^l}\binom{2l}{j+l} e^{itn_k j}.$$

Define

$$f = X_{k+1}^{l_1} \ldots X_{k+m}^{l_m} = \sum_j \left[\frac{1}{2^{l_1}}\binom{2l_1}{j_1+l_1} \cdots \frac{1}{2^{l_m}}\binom{2l_m}{j_m+l_m}\right] e^{itN_j}, \tag{20}$$

where the sum is over all vectors $j = (j_1, \ldots, j_m) \in \mathsf{X}_{s=1}^m \{-l_s, \ldots, 0, \ldots, l_s\}$ and $N_j = n_{k+1}j_1 + \ldots + n_{k+m}j_m$.

Take now two such vectors $j \neq j'$ and let $r$ be the last index where they differ, that is the largest $s \leq m$ for which $j_s \neq j'_s$. Then

$$\begin{aligned} |N_{j'} - N_j| &\geq n_{k+r}|j'_r - j_r| - n_{k+r-1}|j'_{r-1} - j_{r-1}| - \ldots - n_{k+1}|j'_1 - j_1| \\ &\geq n_{k+r} - 2p(n_{k+r-1} + \ldots + n_{k+1}) \geq n_{k+r} - 2pn_{k+r}\left(\frac{1}{d} + \ldots + \frac{1}{d^{r-1}}\right) \\ &\geq n_{k+r}\left(1 - \frac{2p}{d-1}\right). \end{aligned}$$

Therefore, if $d > 2p+1$, then each vector $j$ corresponds to a different value of $N_j$, that is in the expansion (20) of $f$ all the phases $e^{itN_j}$ are different. Let us write

$$f = b_0 + \sum_{j \in \textsc{Comb}} b_j e^{itN_j},$$

where $b_j = \frac{1}{2^{l_1}}\binom{2l_1}{j_1+l_1} \ldots \frac{1}{2^{l_m}}\binom{2l_m}{j_m+l_m}$ and $\textsc{Comb}$ denotes the set $\mathsf{X}_{s=1}^m\{-l_s, \ldots, l_s\} \backslash \{(0, \ldots, 0)\}$ of all nonzero vectors $j$. It is clear that

$$b_0 = \int_{\mathbb{T}} f \qquad \text{and} \qquad b_j \leq b_0 \text{ for } j \in \textsc{Comb}.$$

Applying Corollary 25 yields

$$\int_{\mathbb{T}} R_k^p f = \left| \int_{\mathbb{T}} R_k^p b_0 + \sum_{j \in \textsc{Comb}} b_j \int_{\mathbb{T}} R_k^p e^{itN_j} \right| \le \int_{\mathbb{T}} R_k^p b_0 + \sum_{j \in \textsc{Comb}} b_j \frac{2\pi p \deg R_k}{|N_j|} \int_{\mathbb{T}} R_k^p$$
$$\le \left( \int_{\mathbb{T}} R_k^p \int_{\mathbb{T}} f \right) \left( 1 + 2\pi p \deg R_k \sum_{j \in \textsc{Comb}} \frac{1}{|N_j|} \right).$$

To deal with the sum over $j$, we break $\textsc{Comb}$ into the sets $\textsc{Comb}_r$, $r = 1, \ldots, m$, of the vectors $j$ for which the largest index of a nonzero coordinate is $r$. We thus get

$$\sum_{j \in \textsc{Comb}} \frac{1}{|N_j|} \le \sum_{r=1}^{m} \sum_{j \in \textsc{Comb}_r} \frac{1}{n_{k+r} - p(n_{k+1} + \ldots + n_{k+r-1})}$$
$$\le \sum_{r=1}^{m} |\textsc{Comb}_r| \frac{1}{n_{k+r}\left(1 - \frac{p}{d-1}\right)} \le \sum_{r=1}^{m} (2p+1)^r \frac{1}{n_k d^r \left(1 - \frac{p}{d-1}\right)}$$
$$\le \frac{1}{n_k \left(1 - \frac{p}{d-1}\right)} \frac{2p+1}{d-2p-1} < \frac{2}{n_k} \frac{2p+1}{d-2p-1}.$$

Plugging this back into the previous estimate and noticing that $(\deg R_k)/n_k \le (n_1 + \ldots + n_k)/n_k \le 1/d^{k-1} + \ldots + 1 < d/(d-1)$ yields (19). □

*Proof of the upper bound of Theorem 1.* Let $a_k = \|v_k\|$. By the triangle inequality, we have

$$\left\| \sum_{k=0}^{N} v_k R_k \right\| \le \sum_{k=0}^{N} a_k R_k.$$

Therefore, it suffices to show that

$$\int_{\mathbb{T}} \left( \sum_{k=0}^{N} a_k R_k \right)^p \le C_p \sum_{k=0}^{N} a_k^p \int_{\mathbb{T}} R_k^p. \tag{21}$$

For $N = 0$ this is obvious. When $0 < p \le 1$ this instantly follows from the inequality $(x+y)^p \le x^p + y^p$, $x, y \ge 0$ (with $C_p = 1$). Let $N \ge 1$. Suppose that for some integer $m \ge 1$, (21) holds when $m - 1 < p \le m$ and we want to show it when $m < p \le m+1$. Iterating the inequality $(x+y)^p \le x^p + 2^p(yx^{p-1} + y^p)$, $x, y \ge 0$ (see [2], p. 1705), we find

$$\int_{\mathbb{T}} \left( \sum_{k=0}^{N} a_k R_k \right)^p \le a_N^p \int_{\mathbb{T}} R_N^p + 2^p \left( \sum_{k=0}^{N-1} a_k \int_{\mathbb{T}} R_k \left( \sum_{i=k+1}^{N} a_i R_i \right)^{p-1} + \sum_{k=0}^{N-1} a_k^p \int_{\mathbb{T}} R_k^p \right).$$

The challenge is to deal with the mixed term

$$\sum_{k=0}^{N-1} a_k \int_{\mathbb{T}} R_k \left( \sum_{i=k+1}^{N} a_i R_i \right)^{p-1} = \sum_{k=0}^{N-1} a_k \int_{\mathbb{T}} R_k^p F_k^{p-1},$$

where

$$F_k = \sum_{i=k+1}^{N} a_i R_{k+1,i}, \qquad k \geq 0.$$

We shall make several observations. Firstly, take $\alpha, \beta > 1$ with $1/\alpha + 1/\beta = 1$ and use Hölder's inequality,

$$\int_{\mathbb{T}} R_k^p F_k^{p-1} = \int_{\mathbb{T}} R_k^{p/\alpha}\left(R_k^{p/\beta} F_k^{p-1}\right) \leq \left(\int_{\mathbb{T}} R_k^p\right)^{1/\alpha} \left(\int_{\mathbb{T}} R_k^p F_k^{(p-1)\beta}\right)^{1/\beta}$$

(which holds trivially when $\beta = 1$). Choosing $\beta$ so that $(p-1)\beta = \lceil p \rceil - 1 = m$ gives us the natural power at $F_k$. Then brutally expanding yields

$$\begin{aligned}\int_{\mathbb{T}} R_k^p F_k^{(p-1)\beta} &= \int_{\mathbb{T}} R_k^p \left(\sum_{i=k+1}^{N} a_i R_{k+1,i}\right)^m \\ &= \sum_{m_{k+1}+\ldots+m_N = m} \binom{m}{m_{k+1}, \ldots, m_N} \int_{\mathbb{T}} R_k^p \prod_{i=k+1}^{N} a_i^{m_i} R_{k+1,i}^{m_i}.\end{aligned}$$

The integral $\int_{\mathbb{T}} R_k^p \prod_{i=k+1}^N R_{k+1,i}^{m_i}$ is of the form $\int_{\mathbb{T}} R_k^p X_{k+1}^{l_1} \ldots X_N^{l_N}$ with the nonnegative integer powers $l_{k+1}, \ldots, l_N$ not exceeding $m < p$. Therefore we can apply Lemma 27 to factor $R_k^p$ out,

$$\int_{\mathbb{T}} R_k^p \prod_{i=k+1}^{N} R_{k+1,i}^{m_i} \leq (1+\epsilon) \int_{\mathbb{T}} R_k^p \int_{\mathbb{T}} \prod_{i=k+1}^{N} R_{k+1,i}^{m_i},$$

provided that $d > 2p+1$, and then use the multinomial formula again to get back to $F_k^m$,

$$\int_{\mathbb{T}} R_k^p F_k^m \leq (1+\epsilon) \int_{\mathbb{T}} R_k^p \int_{\mathbb{T}} F_k^m.$$

Recall that $\epsilon = \frac{4\pi d}{d-1} \frac{p(2p+1)}{d-2p-1}$. We choose $d_p$ large enough to assure that for $d \geq d_p$ we have $\varepsilon < 1$. By the inductive assumption,

$$\int_{\mathbb{T}} F_k^m \leq C_m \sum_{i=k+1}^{N} a_i^m \int_{\mathbb{T}} R_{k+1,i}^m$$

with $C_m \geq 1$, provided that $d \geq d_m$. We finally get

$$\begin{aligned}\sum_{k=0}^{N-1} a_k \int_{\mathbb{T}} R_k^p F_k^{p-1} &\leq \sum_{k=0}^{N-1} a_k \left(\int_{\mathbb{T}} R_k^p\right)^{1/\alpha} \left(2 \int_{\mathbb{T}} R_k^p \cdot C_m \sum_{i=k+1}^{N} a_i^m \int_{\mathbb{T}} R_{k+1,i}^m\right)^{1/\beta} \\ &\leq 2C_m \sum_{k=0}^{N} \sum_{i=k+1}^{N} a_k a_i^{p-1} \int_{\mathbb{T}} R_k^p \left(\int_{\mathbb{T}} R_{k+1,i}^m\right)^{1/\beta}.\end{aligned}$$

Lastly, notice that we have $R_{k+1,i}$ to the power of $m$ but we want the $p$-th power. Since $m < p$, there is some room. Introduce the constant

$$\lambda_p = \left( \frac{(\int_{\mathbb{T}} X_1^m)^{1/m}}{(\int_{\mathbb{T}} X_1^p)^{1/p}} \right)^{p-1} < 1.$$

By (18) we obtain

$$\begin{aligned}
\left( \int_{\mathbb{T}} R_{k+1,i}^m \right)^{1/\beta} &\le \left( \left(1 + \frac{2\pi p}{d-1}\right)^{i-k} \int_{\mathbb{T}} X_{k+1}^m \cdots \int_{\mathbb{T}} X_i^m \right)^{1/\beta} \\
&\le \left( \left(1 + \frac{2\pi p}{d-1}\right)^{i-k} \left(\lambda_p^{m/(p-1)}\right)^{i-k} \left( \int_{\mathbb{T}} X_{k+1}^p \cdots \int_{\mathbb{T}} X_i^p \right)^{m/p} \right)^{1/\beta} \\
&= \left[ \left(1 + \frac{2\pi p}{d-1}\right)^{1/\beta} \lambda_p \right]^{i-k} \left( \int_{\mathbb{T}} X_{k+1}^p \cdots \int_{\mathbb{T}} X_i^p \right)^{(p-1)/p} \\
&\le \eta_p^{i-k} \left( \int_{\mathbb{T}} X_{k+1}^p \cdots \int_{\mathbb{T}} X_i^p \right)^{(p-1)/p},
\end{aligned}$$

where $\eta_p = \left(1 + \frac{2\pi p}{d-1}\right) \lambda_p$. Therefore,

$$\begin{aligned}
\sum_{k=0}^{N-1} a_k \int_{\mathbb{T}} R_k^p F_k^{p-1} &\le 2C_m \sum_{k=0}^{N} \sum_{i=k+1}^{N} \eta_p^{i-k} \left( \int_{\mathbb{T}} R_k^p \right) \cdot a_k a_i^{p-1} \left( \int_{\mathbb{T}} X_{k+1}^p \cdots \int_{\mathbb{T}} X_i^p \right)^{(p-1)/p} \\
&\le 2C_m \sum_{k=0}^{N} \sum_{i=k+1}^{N} \eta_p^{i-k} \left( \int_{\mathbb{T}} R_k^p \right) \cdot \left( \frac{1}{p} a_k^p + \frac{p-1}{p} a_i^p \int_{\mathbb{T}} X_{k+1}^p \cdots \int_{\mathbb{T}} X_i^p \right).
\end{aligned}$$

Provided that $\eta_p < 1$, the first bit can be easily estimated as desired,

$$\sum_{k=0}^{N} \sum_{i=k+1}^{N} \eta_p^{i-k} \left( \int_{\mathbb{T}} R_k^p \right) \cdot \frac{1}{p} a_k^p \le \frac{\eta_p}{p(1-\eta_p)} \sum_{k=0}^{N} a_k^p \int_{\mathbb{T}} R_k^p.$$

The second one requires some more work. With the aid of (17) with $k = 1$ and (18),

$$\int_{\mathbb{T}} R_k^p \int_{\mathbb{T}} X_{k+1}^p \cdots \int_{\mathbb{T}} X_i^p \le \left(1 - \frac{2\pi p}{d-1}\right)^{-(i-k)} \int_{\mathbb{T}} R_i^p,$$

so, provided that $\eta_p < 1 - \frac{2\pi p}{d-1}$, that is $\lambda_p\left(1+\frac{2\pi p}{d-1}\right) < \left(1-\frac{2\pi p}{d-1}\right)$, we obtain

$$\sum_{k=0}^{N}\sum_{i=k+1}^{N}\eta_p^{i-k}\left(\int_{\mathbb{T}} R_k^p\right)\cdot\frac{p-1}{p}a_i^p\int_{\mathbb{T}}X_{k+1}^p\cdots\int_{\mathbb{T}}X_i^p \le \frac{p-1}{p}\sum_{i=1}^{N}a_i^p\int_{\mathbb{T}}R_i^p\sum_{k=0}^{i-1}\left[\frac{\eta_p}{1-\frac{2\pi p}{d-1}}\right]^{i-k}$$

$$\le \left[\frac{p-1}{p}\left(1-\frac{n_p}{1-\frac{2\pi p}{d-1}}\right)^{-1}\right]\sum_{i=1}^{N}a_i^p\int_{\mathbb{T}}R_i^p.$$

Putting everything together,

$$\sum_{k=0}^{N-1}a_k\int_{\mathbb{T}}R_k\left(\sum_{i=k+1}^{N}a_iR_i\right)^{p-1} \le C\sum_{k=0}^{N}a_k^p\int_{\mathbb{T}}R_k^p,$$

where

$$C = 2C_m\left(\frac{\eta_p}{p(1-\eta_p)}+\frac{p-1}{p}\left(1-\frac{n_p}{1-\frac{2\pi p}{d-1}}\right)^{-1}\right).$$

Thus,

$$\int_{\mathbb{T}}\left(\sum_{k=0}^{N}a_kR_k\right)^p \le 2^p(1+C)\sum_{k=0}^{N}a_k^p\int_{\mathbb{T}}R_k^p,$$

which completes the proof. □

*Remark* 28. Even though we have not kept track of the values of the constants $d_p, c_p$ and $C_p$ in our arguments, with some extra work it can be shown that for the upper bound in Theorem 1 one can take

$$d_p^{(\text{upper})} = 80p^2 \qquad \text{and} \qquad C_p = (16p)^{p+1}, \qquad p > 1,$$

whereas for the lower bound it is enough to have

$$\begin{matrix} d_p^{(lower)} = \left(\frac{10^{12}}{p-1}\right)^{\frac{3}{p-1}} \\ c_p = \left(\frac{p-1}{10^{13}}\right)^{\frac{1}{p-1}} \end{matrix}, \quad p \in (1,2] \qquad \text{and} \qquad \begin{matrix} d_p^{(lower)} = 10^{10p^2} \\ c_p = 10^{-8p} \end{matrix}, \quad p > 2.$$

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