Maximal inequalities for centered norms of sums of independent random vectors

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Abstract

Let $X_1, X_2, \ldots, X_n$ be independent random variables and $S_k = \sum_{i=1}^k X_i$. We show that for any constants a_k,

$$\mathbb{P}(\max_{1\le k\le n} ||S_k| - a_k| > 11t) \le 30 \max_{1\le k\le n} \mathbb{P}(||S_k| - a_k| > t).$$

We also discuss similar inequalities for sums of Hilbert and Banach space valued random vectors.

1 Introduction and Main Results

Let $X_1, X_2, \ldots$ be independent random vectors in a separable Banach space F. The Lévy-Octaviani maximal inequality (see e.g. Proposition 1.1.1 in [2]) states that for any $t > 0$,

$$\mathbb{P}\Big(\max_{1\le k\le n} \|S_k\| > 3t\Big) \le 3 \max_{1\le k\le n} \mathbb{P}(\|S_k\| > t), \tag{1.1}$$

where here and in the rest of this note,

$$S_k = \sum_{i=1}^k X_i \quad \text{ for } k = 1, 2, \ldots.$$

If, additionally, variables X_i are symmetric then the classical Lévy inequality gives the sharper bound

$$\mathbb{P}\Big(\max_{1\le k\le n} \|S_k\| > t\Big) \le 2\mathbb{P}(\|S_n\| > t).$$

Montgomery-Smith [4] showed that if we replace symmetry assumptions by the identical distribution then

$$\mathbb{P}\Big(\max_{1\le k\le n} \|S_k\| > C_1 t\Big) \le C_2\mathbb{P}(\|S_n\| > t), \tag{1.2}$$

where one may take $C_1 = 30$ and $C_2 = 9$.

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Maximal inequalities are fundamental tools in the study of convergence of random series and limit theorems for sums of independent random vectors (see e.g. [2] and [3]).

In some applications one needs to investigates asymptotic behaviour of centered norms of sums, i.e. random variables of the form $(\|S_n\| - a_n)/b_n$ (cf. [1]). For such purpose it is natural to ask whether in (1.1) or (1.2) one may replace variables $\|S_k\|$ by $|\|S_k\| - a_k|$. The answer turns out to be positive in the real case.

Theorem 1.1. *Let $X_1, X_2, \ldots, X_n$ be independent real r.v.'s. Then for any numbers $a_1, a_2, \ldots, a_n$ and $t > 0$,*

$$\mathbb{P}\Big(\max_{1\le k\le n} ||S_k| - a_k| > 11t\Big) \le 30 \max_{1\le k\le n} \mathbb{P}(||S_k| - a_k| > t). \tag{1.3}$$

Example. Let $Y_1, Y_2, \ldots$ be i.i.d. r.v.'s such that $\mathbb{E}Y_i^2 = 1$ and $\mathrm{Var}(Y_i^2) < \infty$. Let $S_k = \sum_{i=1}^k X_i$, where $X_i = e_i Y_i$ and (e_i) is an orthonormal system in a Hilbert space $\mathcal{H}$; also let $|x|$ denotes the norm of a vector $x \in \mathcal{H}$. Then for $t > 0$,

$$\mathbb{P}(||S_k| - \sqrt{k}| \ge t) \le \mathbb{P}(||S_k|^2 - k| \ge t\sqrt{k}) \le \frac{\mathrm{Var}(|S_k|^2)}{t^2 k} = \frac{\mathrm{Var}(Y_1^2)}{t^2}.$$

On the other hand if we choose j_0 such that $2^{j_0/2} \ge t$, then for $n \ge 2^{j_0}$,

$$\begin{aligned}
p_n &:= \mathbb{P}\Big(\max_{1\le k\le n} ||S_k| - \sqrt{k}| \ge t\Big) \ge \mathbb{P}\Big(\max_{2^{j_0}\le k\le n} (|S_k|^2 - k) \ge 3t\sqrt{k}\Big) \\
&\ge \mathbb{P}\Big(\bigcup_{j_0\le j\le \log_2 n} \Big\{|S_{2^j}|^2 - 2^j \ge 3\cdot 2^{j/2} t\Big\}\Big) \\
&\ge \mathbb{P}\Big(\bigcup_{j_0+1\le j\le \log_2 n} \Big\{2^{-j/2} \sum_{i=2^{j-1}+1}^{2^j} (Y_i^2 - 1) \ge 6t\Big\}\Big)
\end{aligned}$$

and $\lim_{n\to\infty} p_n = 1$ for any $t > 0$ by the CLT. It is not hard to modify this example in such a way that X_i be an i.i.d. sequence.

Hence Theorem 1.1 does not hold in infinite dimensional Hilbert spaces even if we assume that X_i are symmetric and identically distributed. However a modification of (1.3) is satisfied in Hilbert spaces.

Proposition 1.2. *Let $X_1, \ldots, X_n$ be independent symmetric r.v.'s with values in a separable Hilbert space $(\mathcal{H}, |\ |)$. Then for any sequence of real numbers $a_1, \ldots, a_n$ and $t \ge 0$,*

$$\mathbb{P}\Big(\max_{1\le k\le n} \big||S_k|^2 - a_k\big| \ge 3t\Big) \le 6 \max_{1\le k\le n} \mathbb{P}\big(\big||S_k|^2 - a_k\big| \ge t\big).$$

A first consequence of Proposition 1.2 is the following Hilbert-space version of (1.3) under a regularity assumption on coefficients (a_k).

Corollary 1.3. *Let $X_1, \dots, X_n$ be as in Proposition 1.2, $1 \le i \le n$ and non-negative real numbers $a_i, \dots, a_n$, α, β and t satisfy the condition*

$$a_k \le \alpha a_l + \beta t \quad \text{for all } i \le k, l \le n. \tag{1.4}$$

Then

$$\mathbb{P}\Big(\max_{i \le k \le n} ||S_k| - a_k| \ge (6\alpha + 2\beta + 1)t \Big) \le 6 \max_{i \le k \le n} \mathbb{P}\big(||S_k| - a_k| \ge t \big).$$

In proofs of limit theorems one typically applies maximal inequalities to uniformly estimate $\|S_k\|$ for $cn \le k \le n$, where c is some constant. Next two corollaries show that if we restrict k to such a group of indices then, under i.i.d. and symmetry assumptions, (1.3) holds in Hilbert spaces.

Corollary 1.4. *Let $X_1, X_2, \dots, X_n$ be symmetric i.i.d. r.v.'s with values in a separable Hilbert space $(\mathcal{H}, |\ |)$. Then for any integer i such that $\frac{n}{2} \le i \le n$ and any sequence of positive numbers $a_i, \dots, a_n$ and $t \ge 0$ we have*

$$\mathbb{P}\Big(\max_{i \le k \le n} ||S_k| - a_k| \ge 19t \Big) \le 6 \max_{i \le k \le n} \mathbb{P}(||S_k| - a_k| \ge t).$$

Proof. We may obviously assume that

$$\max_{i \le k \le n} \mathbb{P}(||S_k| - a_k| \ge t) \le \frac{1}{6}.$$

Observe that for any $k < l$, the random variable $S_{k,l} := \sum_{i=k}^{l} X_i$ has the same distribution as S_{l-k+1}.

Take $k, l \in \{i, \dots, n\}$, then

$$\begin{aligned} \mathbb{P}(|S_{2k}| \ge 2a_k + 2t) &\le \mathbb{P}(|S_k| \ge a_k + t) + \mathbb{P}(|S_{k+1,2k}| \ge a_k + t) \\ &= 2\mathbb{P}(|S_k| \ge a_k + t) \le \frac{1}{3}. \end{aligned}$$

Therefore

$$\begin{aligned} &\mathbb{P}(a_l - t \le |S_l| \le 2a_k + 2t) \\ &\quad \ge \mathbb{P}(a_l - t \le |S_l|,\ |S_l + S_{l+1,2k}| \le 2a_k + 2t,\ |S_l - S_{l+1,2k}| \le 2a_k + 2t) \\ &\quad \ge 1 - \mathbb{P}(|S_l| < a_l - t) - 2\mathbb{P}(|S_{2k}| > 2a_k + 2t) \ge 1 - \frac{1}{6} - \frac{2}{3} > 0, \end{aligned}$$

where in the second inequality we used the symmetry of X_i. Hence we get $a_l \le 2a_k + 3t$ and we may apply Corollary 1.3 with $\alpha = 2$ and $\beta = 3$. $\square$

Corollary 1.5. *Let $X_1, X_2, \dots, X_n$ be as before. Then for any $\frac{n}{2^j} \le i \le n$ and any sequence of positive numbers $a_i, \dots, a_n$ and $t \ge 0$ we have*

$$\mathbb{P}\Big(\max_{i \le k \le n} ||S_k| - a_k| \ge 19t \Big) \le 6j \max_{i \le k \le n} \mathbb{P}(||S_k| - a_k| \ge t).$$

Corollary 1.4 naturally leads to the formulation of the following open question.

Question. Characterize all separable Banach spaces $(E, \| \ \|)$ with the following property. There exist constants $C_1, C_2 < \infty$ such that for any symmetric i.i.d. r.v.'s $X_1, X_2, \ldots, X_n$ with values in E, any $\frac{n}{2} \leq i \leq n$, any positive constants $a_i, \ldots, a_n$ and $t > 0$,

$$\mathbb{P}\Big(\max_{i \leq k \leq n} |\|S_k\| - a_k| \geq C_1 t) \leq C_2 \max_{i \leq k \leq n} \mathbb{P}(|\|S_k\| - a_k| \geq t). \tag{1.5}$$

In particular does the above inequality hold in L^p with $1 < p < \infty$?

In the last section of the paper we present an example showing that in a general separable Banach space estimate (1.5) does not hold.

2 Proofs

Below we will use the following notation. By $\tilde{X}_1, \tilde{X}_2, \ldots$ we will denote the independent copy of the random sequence $X_1, X_2, \ldots$. We put

$$\tilde{S}_k := \sum_{i=1}^{k} \tilde{X}_i, \quad S_{k,n} := S_n - S_{k-1} = \sum_{i=k}^{n} X_i.$$

We start with the following simple lemma.

Lemma 2.1. *Suppose that real numbers x, y, a, b and u satisfy the conditions $||x| - a| \leq u$, $||y| - a| \leq u$, $||x + s| - b| \leq u$, $||y + s| - b| \leq u$ and $|x - y| > 2u$. Then $|a - b| \leq 2u$ and $|s| \leq 4u$.*

Proof. If $a < 0$ then $|x|, |y| < u$ and $|x - y| < 2u$. So $a \geq 0$ and in the same way we show that $b \geq 0$. Without loss of generality we may assume $x < y$, hence $x \in (-a-u, -a+u)$, $y \in (a-u, a+u)$, $x+s \in (-b-u, -b+u)$, $y+s \in (b-u, b+u)$. Thus $2a - 2u \leq y - x \leq 2a + 2u$ and $2b - 2u \leq (y + s) - (x + s) \leq 2b + 2u$ and therefore $|a - b| \leq 2u$. Moreover, $-b + a - 2u \leq s \leq -b + a + 2u$ and we get $|s| \leq |a - b| + 2u \leq 4u$. $\square$

Proof of Theorem 1.1. We may and will assume that

$$p := \max_{1 \leq k \leq n} \mathbb{P}(||S_k| - a_k| > t) \in (0, 1/30).$$

Let

$$I_1 := \{k\colon\ a_k \leq 2t\}, \quad I_2 := \{k\colon\ \mathbb{P}(|S_k - \tilde{S}_k| > 2t) > 5p\}$$

and

$$I_3 := \{1, \ldots, n\} \setminus (I_1 \cup I_2).$$

First we show that

$$\mathbb{P}\Big(\max_{k \in I_1} ||S_k| - a_k| > 11t\Big) \leq 3p. \tag{2.1}$$

Indeed, notice that for all k, $a_k > -t$ (otherwise $p = 1$). Therefore by the Lévy-Octaviani inequality (1.1),

$$\mathbb{P}\Big(\max_{k\in I_1}||S_k| - a_k| > 11t\Big) \le \mathbb{P}(\max_{k\in I_1}|S_k| > 9t) \le 3\max_{k\in I_1}\mathbb{P}(|S_k| > 3t) \\ \le 3\max_{k\in I_1}\mathbb{P}(||S_k| - a_k| > t) \le 3p.$$

Next we prove that

$$\mathbb{P}\Big(\max_{k\in I_2}||S_k| - a_k| > 11t\Big) \le 5p. \tag{2.2}$$

Let us take $k \in I_2$ and define the following events

$$A_1 := \{|S_k - \tilde{S}_k| > 2t\}, \quad A_2 := A_1 \cap \{|S_{k+1,n}| > 4t\}$$

and

$$B := \{||S_k| - a_k| \le t, ||\tilde{S}_k| - a_k| \le t, ||S_n| - a_n| \le t, ||\tilde{S}_k + S_{k+1,n}| - a_n| \le t\}.$$

We have $\mathbb{P}(A_1) + \mathbb{P}(B) > 5p + 1 - 4p > 1$, hence $A_1 \cap B \neq \emptyset$ and by Lemma 2.1, $|a_k - a_n| \le 2t$. Also by Lemma 2.1, $A_2 \cap B = \emptyset$, hence $\mathbb{P}(A_2) + \mathbb{P}(B) \le 1$. Therefore $5p\mathbb{P}(|S_{k+1,n}| > 4t) \le \mathbb{P}(A_2) \le 4p$. Thus for all $k \in I_2$, $|a_k - a_n| \le 2t$ and $\mathbb{P}(|S_{k+1,n}| \le 4t) \ge 1/5$. Let

$$\tau := \inf\{k \in I_2 : ||S_k| - a_k| > 11t\}.$$

Then

$$\frac{1}{5}\mathbb{P}(\tau = k) \le \mathbb{P}(\tau = k, |S_{k+1,n}| \le 4t) \\ \le \mathbb{P}(\tau = k, ||S_n| - a_n| > 11t - 4t - |a_k - a_n|) \\ \le \mathbb{P}(\tau = k, ||S_n| - a_n| > t)$$

and

$$\mathbb{P}\Big(\max_{k\in I_2}||S_k| - a_k| > 11t\Big) = \sum_{k\in I_2}\mathbb{P}(\tau = k) \le 5\sum_{k\in I_2}\mathbb{P}(\tau = k, ||S_n| - a_n| > t) \\ \le 5\mathbb{P}(||S_n| - a_n| > t) \le 5p.$$

Finally we show

$$\mathbb{P}\Big(\max_{k\in I_3}||S_k| - a_k| > 11t\Big) \le 21p. \tag{2.3}$$

To this end take any $k \in I_3$ and notice that

$$2\max\{\mathbb{P}(|S_k - a_k| \le t), \mathbb{P}(|S_k + a_k| \le t)\} \\ \ge \mathbb{P}(|S_k - a_k| \le t) + \mathbb{P}(|S_k + a_k| \le t) \\ \ge \mathbb{P}(||S_k| - a_k| \le t) \ge 1 - p \ge \frac{29}{30}.$$

If $|x-a_k|\le t$ and $|y+a_k|\le t$ then $|x-y|\ge 2a_k-2t>2t$. Therefore

$$\begin{aligned}5p &\ge \mathbb{P}(|S_k-\tilde{S}_k|>2t)\\ &\ge \mathbb{P}(|S_k-a_k|\le t, |\tilde{S}_k+a_k|\le t)+\mathbb{P}(|S_k+a_k|\le t, |\tilde{S}_k-a_k|\le t)\\ &= 2\mathbb{P}(|S_k-a_k|\le t)\mathbb{P}(|S_k+a_k|\le t).\end{aligned}$$

So for any $k\in I_3$ we may choose $b_k=\pm a_k$ such that

$$\mathbb{P}(|S_k-b_k|\le t)\le \frac{30}{29}5p\le 6p.$$

Therefore

$$\mathbb{P}(|S_k+b_k|>t)\le \mathbb{P}(||S_k|-a_k|>t)+\mathbb{P}(|S_k-b_k|\le t)\le 7p$$

and by the Lévy-Octaviani inequality (1.1),

$$\begin{aligned}\mathbb{P}(\max_{k\in I_3}||S_k|-a_k|>11t) &\le \mathbb{P}(\max_{k\in I_3}|S_k+b_k|>11t)\\ &\le 3\max_{k\in I_3}\mathbb{P}(|S_k+b_k|>\frac{11}{3}t)\le 21p.\end{aligned}$$

This shows (2.3).

Inequalities (2.1), (2.2) and (2.3) imply (1.3). $\square$

Proof of Proposition 1.2. It is enough to consider the case when

$$p:=\max_{1\le k\le n}\mathbb{P}\big(\big||S_k|^2-a_k\big|\ge t\big)<\frac{1}{6}.$$

Notice that

$$\mathbb{P}\big(\big||S_n|^2-|S_k|^2-(a_n-a_k)\big|\ge 2t\big)\le \mathbb{P}\big(\big||S_n|^2-a_n\big|\ge t\big)+\mathbb{P}\big(\big||S_k|^2-a_k\big|\ge t\big)$$

Therefore

$$\mathbb{P}\big(\big||S_{k+1,n}|^2+2\langle S_k,S_{k+1,n}\rangle-(a_n-a_k)\big|\ge 2t)\le 2p$$

and by the symmetry

$$\mathbb{P}\big(\big||S_{k+1,n}|^2-2\langle S_k,S_{k+1,n}\rangle-(a_n-a_k)\big|\ge 2t\big)\le 2p.$$

Thus by the triangle inequality

$$\mathbb{P}\big(\big||S_{k+1,n}|^2-(a_n-a_k)\big|\ge 2t\big)\le 4p.$$

Now let $x\in\mathcal{H}$ be such that $||x|^2-a_k|\ge 3t$ then by the triangle inequality and symmetry

$$\begin{aligned}1-4p &\le \mathbb{P}\big(\big||x|^2+|S_{k+1,n}|^2-a_n\big|\ge t\big)\\ &\le \mathbb{P}\big(\big||x|^2+|S_{k+1,n}|^2+2\langle x,S_{k+1,n}\rangle-a_n\big|\ge t\big)\\ &\quad+\mathbb{P}\big(\big||x|^2+|S_{k+1,n}|^2-2\langle x,S_{k+1,n}\rangle-a_n\big|\ge t\big)\\ &= 2\mathbb{P}\big(\big||x|^2+|S_{k+1,n}|^2+2\langle x,S_{k+1,n}\rangle-a_n\big|\ge t\big)\\ &= 2\mathbb{P}\big(\big||x+S_{k+1,n}|^2-a_n\big|\ge t\big).\end{aligned}$$

So for any $x \in \mathcal{H}$ and $k = 1, 2, \ldots, n$,

$$\big||x|^2 - a_k\big| \geq 3t \;\Rightarrow\; \mathbb{P}\big(\big||x + S_{k+1,n}|^2 - a_n\big| \geq t\big) \geq \frac{1}{2}(1 - 4p) \geq \frac{1}{6}. \tag{2.4}$$

Now let

$$\tau := \inf\big\{k \leq n\colon\; \big|\|S_k\|^2 - a_k\big| \geq 3t\big\},$$

then since $\{\tau = k\} \in \sigma(X_1, \ldots, X_k)$ we get by (2.4),

$$\mathbb{P}\big(\tau = k, \big||S_n|^2 - a_n\big| \geq t\big) \geq \frac{1}{6}\mathbb{P}(\tau = k).$$

Hence

$$\mathbb{P}\big(\big||S_n|^2 - a_n\big| \geq t\big) \geq \frac{1}{6}\sum_{k=1}^{n}\mathbb{P}(\tau = k) = \frac{1}{6}\mathbb{P}\Big(\max_{1\leq k\leq n}\big||S_k|^2 - a_k\big| \geq 3t\Big)$$

and Proposition follows. □

Proof of Corollary 1.3. We may consider variables $S_i, X_{i+1}, \ldots, X_n$ instead of $X_1, \ldots, X_n$ and assume that $i = 1$. Let $a := \min_{1\leq k\leq n} a_k$. We will analyze two cases.

Case 1. $a \leq 3t$. Then by (1.4) we get $a_k \leq (3\alpha + \beta)t$ for all k. Thus by the Lévy inequality,

$$\begin{aligned}
\mathbb{P}\Big(\max_k ||S_k| - a_k| \geq (6\alpha + 2\beta + 1)t\Big) &\leq \mathbb{P}\Big(\max_k |S_k| \geq (3\alpha + \beta + 1)t\Big)\\
&\leq 2\mathbb{P}(|S_n| \geq (3\alpha + \beta + 1)t)\\
&\leq 2\mathbb{P}(||S_n| - a_n| \geq t)\\
&\leq 2\max_k \mathbb{P}(||S_k| - a_k| \geq t).
\end{aligned}$$

Case 2. $a \geq 3t$. Notice first that for any $s > 0$ we have

$$\{||S_k|^2 - a_k^2| \geq s(2a_k + s)\} \subset \{||S_k| - a_k| \geq s\} \subset \{||S_k|^2 - a_k^2| \geq sa_k\}. \tag{2.5}$$

Indeed, the last inclusion follows since $||S_k|^2 - a_k^2| = (|S_k| + a_k)||S_k| - a_k| \geq a_k||S_k| - a_k|$. To see the first inclusion in (2.5) observe that

$$\begin{aligned}
\{||S_k|^2 - a_k^2| \geq s(2a_k + s)\} &\subset \{||S_k| - a_k| \geq s\} \cup \{|S_k| + a_k \geq 2a_k + s\}\\
&\subset \{||S_k| - a_k| \geq s\}.
\end{aligned}$$

Now by (2.5) we get

$$\begin{aligned}
&\mathbb{P}\Big(\max_k ||S_k| - a_k| \geq (6\alpha + 2\beta + 1)t\Big)\\
&\qquad \leq \mathbb{P}\Big(\max_k ||S_k|^2 - a_k^2| \geq (6\alpha + 2\beta + 1)at\Big).
\end{aligned}$$

Hence by Proposition 1.2,

$$\mathbb{P}\Big(\max_k ||S_k|-a_k| \geq (6\alpha+2\beta+1)t\Big) \leq 6\max_k \mathbb{P}\big(||S_k|^2-a_k^2| \geq \frac{1}{3}(6\alpha+2\beta+1)at\big).$$

But $\frac{1}{3}(6\alpha+2\beta+1)a \geq 2(\alpha a+\beta t)+t \geq 2a_k+t$ for all k by (1.4). Therefore by (2.5),

$$\begin{aligned}\mathbb{P}\Big(\max_k ||S_k|-a_k| \geq (6\alpha+2\beta+1)t\Big) &\leq 6\max_k \mathbb{P}(||S_k|^2-a_k^2| \geq t(2a_k+t)) \\ &\leq 6\max_k \mathbb{P}(||S_k|-a_k| \geq t).\end{aligned}$$

□

3 Example

Let us fix a positive integer n and put

$$I_n = \Big\{j \in \mathbb{Z}\colon\ \frac{n}{2} \leq j \leq n\Big\}.$$

Let $t_j = \frac{n^2+j}{j}$ for $j = 1, 2, \ldots, n$, then

$$jt_j = n^2+j \quad \text{and} \quad (j-1)t_j \leq n^2 \quad \text{for } j \in I_n. \tag{3.1}$$

Let N be a large integer (to be fixed later) and let F be the space of all double-indexed sequences $a = (a_{i,j})_{0\leq i\leq N, j\in I_n}$ with the norm

$$\Big\|(a_{i,j})_{0\leq i\leq N, j\in I_n}\Big\| = \max_{j\in I_n}\Big(|a_{0,j}| + t_j \sum_{1\leq i_1<i_2<\ldots<i_j\leq N}\sum_{s=1}^{j}|a_{i_s,j}|\Big).$$

Let $(e_{i,j})$ be a standard basis of F, so that $(a_{i,j}) = \sum_{i,j} a_{i,j}e_{i,j}$.

Define random vectors $X_1, X_2, \ldots, X_n$ by the formula

$$X_l = \sum_{j\in I_n}(Y_{l,j}e_{0,j} + R_{l,j}e_{N_l,j}),$$

where $(Y_{l,j}, R_{l,j})_{l\leq n, j\in I_n}$ and $(N_l)_{l\leq n}$ are independent r.v's, $\mathbb{P}(R_{l,j} = \pm 1) = 1/2$, $Y_{l,j}$ are symmetric $\mathbb{P}(|Y_{l,j}| = \frac{1}{2n}) = 1 - \mathbb{P}(Y_{k,j} = 0) = p_n$ (with p_n a small positive number to be specified later) and N_l are uniformly sampled from the set $\{1, \ldots, N\}$.

Obviously $X_1, X_2, \ldots, X_n$ are i.i.d. and symmetric. As usual we set $S_k = X_1 + X_2 + \ldots + X_k$. Let

$$A = \{N_1, N_2, \ldots, N_n \text{ are pairwise distinct}\}.$$

Notice that $\mathbb{P}(A) \to 0$ when $N \to \infty$. On the set A we have for $k \leq n$,

$$\|S_k\| = \max_{j\in I_n}\Big(\Big|\sum_{l=1}^{k} Y_{l,j}\Big| + t_j \min\{k, j\}\Big).$$

For $j > k$ we have by (3.1),

$$\Big|\sum_{l=1}^{k} Y_{l,j}\Big| + t_j \min\{k,j\} < 1 + t_j(j-1) \leq n^2 + 1,$$

hence on the set A, for $k \in I_n$ we get

$$\|S_k\| = \max_{j\in I_n, j\leq k}\Big(\Big|\sum_{l=1}^{k} Y_{l,j}\Big| + n^2 + j\Big) = \Big|\sum_{l=1}^{k} Y_{l,k}\Big| + n^2 + k.$$

Take $0 < t < \frac{1}{2nC_1}$ then for $k \in I_n$,

$$\mathbb{P}(|\|S_k\| - (n^2+k)| \geq t) \leq \mathbb{P}(A) + \mathbb{P}\Big(\sum_{l=1}^{k} Y_{l,k} \neq 0\Big) \leq \mathbb{P}(A) + kp_n$$

and

$$\mathbb{P}\Big(\max_{k\in I_n} |\|S_k\| - (n^2+k)| \geq tC_1\Big) \geq \mathbb{P}\Big(\max_{k\in I_n}\Big|\sum_{l=1}^{k} Y_{l,k}\Big| \neq 0\Big) - \mathbb{P}(A).$$

The last number is of order $n^2 p_n$ if N is large and p_n is small. This shows that if (1.5) holds for $i = \lceil n/2 \rceil$ in F then C_2 must be of order n. So (1.5) cannot hold with absolute constants C_1 and C_2 in (infinite dimensional) separable Banach spaces.

References

[1] E. Giné and D. Mason, *The law of the iterated logarithm for the integrated squared deviation of a kernel density estimator*, Bernoulli **10** (2004), 721–752.

[2] S. Kwapień and W.A. Woyczyński *Random series and stochastic integrals: single and multiple*, Birkhäuser, Boston, MA, 1992.

[3] M. Ledoux and M. Talagrand, *Probability in Banach spaces*, Springer-Verlag, Berlin, 1991.

[4] S. Montgomery-Smith, *Comparison of sums of independent identically distributed random vectors* Probab. Math. Statist. **14** (1993), 281–285.

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