

Maximal inequalities for centered norms of sums of independent random vectors

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Abstract

Let X_1, X_2, \dots, X_n be independent random variables and $S_k = \sum_{i=1}^k X_i$. We show that for any constants a_k ,

$$\mathbb{P}\left(\max_{1 \leq k \leq n} \|S_k\| - a_k > 11t\right) \leq 30 \max_{1 \leq k \leq n} \mathbb{P}(\|S_k\| - a_k > t).$$

We also discuss similar inequalities for sums of Hilbert and Banach space valued random vectors.

1 Introduction and Main Results

Let X_1, X_2, \dots be independent random vectors in a separable Banach space F . The Lévy-Octaviani maximal inequality (see e.g. Proposition 1.1.1 in [2]) states that for any $t > 0$,

$$\mathbb{P}\left(\max_{1 \leq k \leq n} \|S_k\| > 3t\right) \leq 3 \max_{1 \leq k \leq n} \mathbb{P}(\|S_k\| > t), \quad (1.1)$$

where here and in the rest of this note,

$$S_k = \sum_{i=1}^k X_i \quad \text{for } k = 1, 2, \dots$$

If, additionally, variables X_i are symmetric then the classical Lévy inequality gives the sharper bound

$$\mathbb{P}\left(\max_{1 \leq k \leq n} \|S_k\| > t\right) \leq 2\mathbb{P}(\|S_n\| > t).$$

Montgomery-Smith [4] showed that if we replace symmetry assumptions by the identical distribution then

$$\mathbb{P}\left(\max_{1 \leq k \leq n} \|S_k\| > C_1 t\right) \leq C_2 \mathbb{P}(\|S_n\| > t), \quad (1.2)$$

where one may take $C_1 = 30$ and $C_2 = 9$.

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Maximal inequalities are fundamental tools in the study of convergence of random series and limit theorems for sums of independent random vectors (see e.g. [2] and [3]).

In some applications one needs to investigate asymptotic behaviour of centered norms of sums, i.e. random variables of the form $(\|S_n\| - a_n)/b_n$ (cf. [1]). For such purpose it is natural to ask whether in (1.1) or (1.2) one may replace variables $\|S_k\|$ by $\|S_k\| - a_k$. The answer turns out to be positive in the real case.

Theorem 1.1. *Let X_1, X_2, \dots, X_n be independent real r.v.'s. Then for any numbers a_1, a_2, \dots, a_n and $t > 0$,*

$$\mathbb{P}\left(\max_{1 \leq k \leq n} \|S_k\| - a_k > 11t\right) \leq 30 \max_{1 \leq k \leq n} \mathbb{P}(\|S_k\| - a_k > t). \quad (1.3)$$

Example. Let Y_1, Y_2, \dots be i.i.d. r.v.'s such that $\mathbb{E}Y_i^2 = 1$ and $\text{Var}(Y_i^2) < \infty$. Let $S_k = \sum_{i=1}^k X_i$, where $X_i = e_i Y_i$ and (e_i) is an orthonormal system in a Hilbert space \mathcal{H} ; also let $|x|$ denotes the norm of a vector $x \in \mathcal{H}$. Then for $t > 0$,

$$\mathbb{P}(\|S_k\| - \sqrt{k} \geq t) \leq \mathbb{P}(\|S_k\|^2 - k \geq t\sqrt{k}) \leq \frac{\text{Var}(\|S_k\|^2)}{t^2 k} = \frac{\text{Var}(Y_1^2)}{t^2}.$$

On the other hand if we choose j_0 such that $2^{j_0/2} \geq t$, then for $n \geq 2^{j_0}$,

$$\begin{aligned} p_n &:= \mathbb{P}\left(\max_{1 \leq k \leq n} \|S_k\| - \sqrt{k} \geq t\right) \geq \mathbb{P}\left(\max_{2^{j_0} \leq k \leq n} (\|S_k\|^2 - k) \geq 3t\sqrt{k}\right) \\ &\geq \mathbb{P}\left(\bigcup_{j_0 \leq j \leq \log_2 n} \left\{\|S_{2^j}\|^2 - 2^j \geq 3 \cdot 2^{j/2} t\right\}\right) \\ &\geq \mathbb{P}\left(\bigcup_{j_0+1 \leq j \leq \log_2 n} \left\{2^{-j/2} \sum_{i=2^{j-1}+1}^{2^j} (Y_i^2 - 1) \geq 6t\right\}\right) \end{aligned}$$

and $\lim_{n \rightarrow \infty} p_n = 1$ for any $t > 0$ by the CLT. It is not hard to modify this example in such a way that X_i be an i.i.d. sequence.

Hence Theorem 1.1 does not hold in infinite dimensional Hilbert spaces even if we assume that X_i are symmetric and identically distributed. However a modification of (1.3) is satisfied in Hilbert spaces.

Proposition 1.2. *Let X_1, \dots, X_n be independent symmetric r.v.'s with values in a separable Hilbert space $(\mathcal{H}, |\cdot|)$. Then for any sequence of real numbers a_1, \dots, a_n and $t \geq 0$,*

$$\mathbb{P}\left(\max_{1 \leq k \leq n} \|S_k\|^2 - a_k \geq 3t\right) \leq 6 \max_{1 \leq k \leq n} \mathbb{P}(\|S_k\|^2 - a_k \geq t).$$

A first consequence of Proposition 1.2 is the following Hilbert-space version of (1.3) under a regularity assumption on coefficients (a_k) .

Corollary 1.3. *Let X_1, \dots, X_n be as in Proposition 1.2, $1 \leq i \leq n$ and non-negative real numbers a_i, \dots, a_n , α, β and t satisfy the condition*

$$a_k \leq \alpha a_l + \beta t \quad \text{for all } i \leq k, l \leq n. \quad (1.4)$$

Then

$$\mathbb{P}\left(\max_{i \leq k \leq n} \|S_k\| - a_k \geq (6\alpha + 2\beta + 1)t\right) \leq 6 \max_{i \leq k \leq n} \mathbb{P}(\|S_k\| - a_k \geq t).$$

In proofs of limit theorems one typically applies maximal inequalities to uniformly estimate $\|S_k\|$ for $cn \leq k \leq n$, where c is some constant. Next two corollaries show that if we restrict k to such a group of indices then, under i.i.d. and symmetry assumptions, (1.3) holds in Hilbert spaces.

Corollary 1.4. *Let X_1, X_2, \dots, X_n be symmetric i.i.d. r.v.'s with values in a separable Hilbert space $(\mathcal{H}, |\cdot|)$. Then for any integer i such that $\frac{n}{2} \leq i \leq n$ and any sequence of positive numbers a_i, \dots, a_n and $t \geq 0$ we have*

$$\mathbb{P}\left(\max_{i \leq k \leq n} \|S_k\| - a_k \geq 19t\right) \leq 6 \max_{i \leq k \leq n} \mathbb{P}(\|S_k\| - a_k \geq t).$$

Proof. We may obviously assume that

$$\max_{i \leq k \leq n} \mathbb{P}(\|S_k\| - a_k \geq t) \leq \frac{1}{6}.$$

Observe that for any $k < l$, the random variable $S_{k,l} := \sum_{i=k}^l X_i$ has the same distribution as S_{l-k+1} .

Take $k, l \in \{i, \dots, n\}$, then

$$\begin{aligned} \mathbb{P}(\|S_{2k}\| \geq 2a_k + 2t) &\leq \mathbb{P}(\|S_k\| \geq a_k + t) + \mathbb{P}(\|S_{k+1,2k}\| \geq a_k + t) \\ &= 2\mathbb{P}(\|S_k\| \geq a_k + t) \leq \frac{1}{3}. \end{aligned}$$

Therefore

$$\begin{aligned} &\mathbb{P}(a_l - t \leq |S_l| \leq 2a_k + 2t) \\ &\geq \mathbb{P}(a_l - t \leq |S_l|, |S_l + S_{l+1,2k}| \leq 2a_k + 2t, |S_l - S_{l+1,2k}| \leq 2a_k + 2t) \\ &\geq 1 - \mathbb{P}(|S_l| < a_l - t) - 2\mathbb{P}(\|S_{2k}\| > 2a_k + 2t) \geq 1 - \frac{1}{6} - \frac{2}{3} > 0, \end{aligned}$$

where in the second inequality we used the symmetry of X_i . Hence we get $a_l \leq 2a_k + 3t$ and we may apply Corollary 1.3 with $\alpha = 2$ and $\beta = 3$. \square

Corollary 1.5. *Let X_1, X_2, \dots, X_n be as before. Then for any $\frac{n}{2j} \leq i \leq n$ and any sequence of positive numbers a_i, \dots, a_n and $t \geq 0$ we have*

$$\mathbb{P}\left(\max_{i \leq k \leq n} \|S_k\| - a_k \geq 19t\right) \leq 6j \max_{i \leq k \leq n} \mathbb{P}(\|S_k\| - a_k \geq t).$$

Corollary 1.4 naturally leads to the formulation of the following open question.

Question. Characterize all separable Banach spaces $(E, \|\cdot\|)$ with the following property. There exist constants $C_1, C_2 < \infty$ such that for any symmetric i.i.d. r.v.'s X_1, X_2, \dots, X_n with values in E , any $\frac{n}{2} \leq i \leq n$, any positive constants a_i, \dots, a_n and $t > 0$,

$$\mathbb{P}\left(\max_{i \leq k \leq n} \|\|S_k\| - a_k\| \geq C_1 t\right) \leq C_2 \max_{i \leq k \leq n} \mathbb{P}(\|\|S_k\| - a_k\| \geq t). \quad (1.5)$$

In particular does the above inequality hold in L^p with $1 < p < \infty$?

In the last section of the paper we present an example showing that in a general separable Banach space estimate (1.5) does not hold.

2 Proofs

Below we will use the following notation. By $\tilde{X}_1, \tilde{X}_2, \dots$ we will denote the independent copy of the random sequence X_1, X_2, \dots . We put

$$\tilde{S}_k := \sum_{i=1}^k \tilde{X}_i, \quad S_{k,n} := S_n - S_{k-1} = \sum_{i=k}^n X_i.$$

We start with the following simple lemma.

Lemma 2.1. *Suppose that real numbers x, y, a, b and u satisfy the conditions $\|x - a\| \leq u, \|y - a\| \leq u, \|x + s\| - b\| \leq u, \|y + s\| - b\| \leq u$ and $|x - y| > 2u$. Then $|a - b| \leq 2u$ and $|s| \leq 4u$.*

Proof. If $a < 0$ then $|x|, |y| < u$ and $|x - y| < 2u$. So $a \geq 0$ and in the same way we show that $b \geq 0$. Without loss of generality we may assume $x < y$, hence $x \in (-a - u, -a + u), y \in (a - u, a + u), x + s \in (-b - u, -b + u), y + s \in (b - u, b + u)$. Thus $2a - 2u \leq y - x \leq 2a + 2u$ and $2b - 2u \leq (y + s) - (x + s) \leq 2b + 2u$ and therefore $|a - b| \leq 2u$. Moreover, $-b + a - 2u \leq s \leq -b + a + 2u$ and we get $|s| \leq |a - b| + 2u \leq 4u$. \square

Proof of Theorem 1.1. We may and will assume that

$$p := \max_{1 \leq k \leq n} \mathbb{P}(\|\|S_k\| - a_k\| > t) \in (0, 1/30).$$

Let

$$I_1 := \{k: a_k \leq 2t\}, \quad I_2 := \{k: \mathbb{P}(\|\|S_k - \tilde{S}_k\| > 2t) > 5p\}$$

and

$$I_3 := \{1, \dots, n\} \setminus (I_1 \cup I_2).$$

First we show that

$$\mathbb{P}\left(\max_{k \in I_1} \|\|S_k\| - a_k\| > 11t\right) \leq 3p. \quad (2.1)$$

Indeed, notice that for all k , $a_k > -t$ (otherwise $p = 1$). Therefore by the Lévy-Octaviani inequality (1.1),

$$\begin{aligned} \mathbb{P}\left(\max_{k \in I_1} \|S_k - a_k\| > 11t\right) &\leq \mathbb{P}\left(\max_{k \in I_1} |S_k| > 9t\right) \leq 3 \max_{k \in I_1} \mathbb{P}(|S_k| > 3t) \\ &\leq 3 \max_{k \in I_1} \mathbb{P}(\|S_k - a_k\| > t) \leq 3p. \end{aligned}$$

Next we prove that

$$\mathbb{P}\left(\max_{k \in I_2} \|S_k - a_k\| > 11t\right) \leq 5p. \quad (2.2)$$

Let us take $k \in I_2$ and define the following events

$$A_1 := \{|S_k - \tilde{S}_k| > 2t\}, \quad A_2 := A_1 \cap \{|S_{k+1,n}| > 4t\}$$

and

$$B := \{|S_k - a_k| \leq t, \|\tilde{S}_k - a_k\| \leq t, \|S_n - a_n\| \leq t, \|\tilde{S}_k + S_{k+1,n} - a_n\| \leq t\}.$$

We have $\mathbb{P}(A_1) + \mathbb{P}(B) > 5p + 1 - 4p > 1$, hence $A_1 \cap B \neq \emptyset$ and by Lemma 2.1, $|a_k - a_n| \leq 2t$. Also by Lemma 2.1, $A_2 \cap B = \emptyset$, hence $\mathbb{P}(A_2) + \mathbb{P}(B) \leq 1$. Therefore $5p\mathbb{P}(|S_{k+1,n}| > 4t) \leq \mathbb{P}(A_2) \leq 4p$. Thus for all $k \in I_2$, $|a_k - a_n| \leq 2t$ and $\mathbb{P}(|S_{k+1,n}| \leq 4t) \geq 1/5$. Let

$$\tau := \inf\{k \in I_2 : \|S_k - a_k\| > 11t\}.$$

Then

$$\begin{aligned} \frac{1}{5}\mathbb{P}(\tau = k) &\leq \mathbb{P}(\tau = k, |S_{k+1,n}| \leq 4t) \\ &\leq \mathbb{P}(\tau = k, \|S_n - a_n\| > 11t - 4t - |a_k - a_n|) \\ &\leq \mathbb{P}(\tau = k, \|S_n - a_n\| > t) \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}\left(\max_{k \in I_2} \|S_k - a_k\| > 11t\right) &= \sum_{k \in I_2} \mathbb{P}(\tau = k) \leq 5 \sum_{k \in I_2} \mathbb{P}(\tau = k, \|S_n - a_n\| > t) \\ &\leq 5\mathbb{P}(\|S_n - a_n\| > t) \leq 5p. \end{aligned}$$

Finally we show

$$\mathbb{P}\left(\max_{k \in I_3} \|S_k - a_k\| > 11t\right) \leq 21p. \quad (2.3)$$

To this end take any $k \in I_3$ and notice that

$$\begin{aligned} 2 \max\{\mathbb{P}(|S_k - a_k| \leq t), \mathbb{P}(|S_k + a_k| \leq t)\} \\ &\geq \mathbb{P}(|S_k - a_k| \leq t) + \mathbb{P}(|S_k + a_k| \leq t) \\ &\geq \mathbb{P}(\|S_k - a_k\| \leq t) \geq 1 - p \geq \frac{29}{30}. \end{aligned}$$

If $|x - a_k| \leq t$ and $|y + a_k| \leq t$ then $|x - y| \geq 2a_k - 2t > 2t$. Therefore

$$\begin{aligned} 5p &\geq \mathbb{P}(|S_k - \tilde{S}_k| > 2t) \\ &\geq \mathbb{P}(|S_k - a_k| \leq t, |\tilde{S}_k + a_k| \leq t) + \mathbb{P}(|S_k + a_k| \leq t, |\tilde{S}_k - a_k| \leq t) \\ &= 2\mathbb{P}(|S_k - a_k| \leq t)\mathbb{P}(|S_k + a_k| \leq t). \end{aligned}$$

So for any $k \in I_3$ we may choose $b_k = \pm a_k$ such that

$$\mathbb{P}(|S_k - b_k| \leq t) \leq \frac{30}{29}5p \leq 6p.$$

Therefore

$$\mathbb{P}(|S_k + b_k| > t) \leq \mathbb{P}(|S_k| - a_k > t) + \mathbb{P}(|S_k - b_k| \leq t) \leq 7p$$

and by the Lévy-Octaviani inequality (1.1),

$$\begin{aligned} \mathbb{P}(\max_{k \in I_3} |S_k| - a_k > 11t) &\leq \mathbb{P}(\max_{k \in I_3} |S_k + b_k| > 11t) \\ &\leq 3 \max_{k \in I_3} \mathbb{P}(|S_k + b_k| > \frac{11}{3}t) \leq 21p. \end{aligned}$$

This shows (2.3).

Inequalities (2.1), (2.2) and (2.3) imply (1.3). \square

Proof of Proposition 1.2. It is enough to consider the case when

$$p := \max_{1 \leq k \leq n} \mathbb{P}(|S_k|^2 - a_k| \geq t) < \frac{1}{6}.$$

Notice that

$$\mathbb{P}(|S_n|^2 - |S_k|^2 - (a_n - a_k)| \geq 2t) \leq \mathbb{P}(|S_n|^2 - a_n| \geq t) + \mathbb{P}(|S_k|^2 - a_k| \geq t)$$

Therefore

$$\mathbb{P}(|S_{k+1,n}|^2 + 2\langle S_k, S_{k+1,n} \rangle - (a_n - a_k)| \geq 2t) \leq 2p$$

and by the symmetry

$$\mathbb{P}(|S_{k+1,n}|^2 - 2\langle S_k, S_{k+1,n} \rangle - (a_n - a_k)| \geq 2t) \leq 2p.$$

Thus by the triangle inequality

$$\mathbb{P}(|S_{k+1,n}|^2 - (a_n - a_k)| \geq 2t) \leq 4p.$$

Now let $x \in \mathcal{H}$ be such that $|x|^2 - a_k| \geq 3t$ then by the triangle inequality and symmetry

$$\begin{aligned} 1 - 4p &\leq \mathbb{P}(|x|^2 + |S_{k+1,n}|^2 - a_n| \geq t) \\ &\leq \mathbb{P}(|x|^2 + |S_{k+1,n}|^2 + 2\langle x, S_{k+1,n} \rangle - a_n| \geq t) \\ &\quad + \mathbb{P}(|x|^2 + |S_{k+1,n}|^2 - 2\langle x, S_{k+1,n} \rangle - a_n| \geq t) \\ &= 2\mathbb{P}(|x|^2 + |S_{k+1,n}|^2 + 2\langle x, S_{k+1,n} \rangle - a_n| \geq t) \\ &= 2\mathbb{P}(|x + S_{k+1,n}|^2 - a_n| \geq t). \end{aligned}$$

So for any $x \in \mathcal{H}$ and $k = 1, 2, \dots, n$,

$$\|x\|^2 - a_k \geq 3t \Rightarrow \mathbb{P}(\|x + S_{k+1,n}\|^2 - a_n \geq t) \geq \frac{1}{2}(1 - 4p) \geq \frac{1}{6}. \quad (2.4)$$

Now let

$$\tau := \inf \{k \leq n: \|\|S_k\|^2 - a_k\| \geq 3t\},$$

then since $\{\tau = k\} \in \sigma(X_1, \dots, X_k)$ we get by (2.4),

$$\mathbb{P}(\tau = k, \|\|S_n\|^2 - a_n\| \geq t) \geq \frac{1}{6}\mathbb{P}(\tau = k).$$

Hence

$$\mathbb{P}(\|\|S_n\|^2 - a_n\| \geq t) \geq \frac{1}{6} \sum_{k=1}^n \mathbb{P}(\tau = k) = \frac{1}{6} \mathbb{P}\left(\max_{1 \leq k \leq n} \|\|S_k\|^2 - a_k\| \geq 3t\right)$$

and Proposition follows. \square

Proof of Corollary 1.3. We may consider variables S_i, X_{i+1}, \dots, X_n instead of X_1, \dots, X_n and assume that $i = 1$. Let $a := \min_{1 \leq k \leq n} a_k$. We will analyze two cases.

Case 1. $a \leq 3t$. Then by (1.4) we get $a_k \leq (3\alpha + \beta)t$ for all k . Thus by the Lévy inequality,

$$\begin{aligned} \mathbb{P}\left(\max_k \|\|S_k\| - a_k\| \geq (6\alpha + 2\beta + 1)t\right) &\leq \mathbb{P}\left(\max_k \|S_k\| \geq (3\alpha + \beta + 1)t\right) \\ &\leq 2\mathbb{P}(\|S_n\| \geq (3\alpha + \beta + 1)t) \\ &\leq 2\mathbb{P}(\|\|S_n\| - a_n\| \geq t) \\ &\leq 2 \max_k \mathbb{P}(\|\|S_k\| - a_k\| \geq t). \end{aligned}$$

Case 2. $a \geq 3t$. Notice first that for any $s > 0$ we have

$$\{\|\|S_k\|^2 - a_k^2\| \geq s(2a_k + s)\} \subset \{\|\|S_k\| - a_k\| \geq s\} \subset \{\|\|S_k\|^2 - a_k^2\| \geq sa_k\}. \quad (2.5)$$

Indeed, the last inclusion follows since $\|\|S_k\|^2 - a_k^2\| = (\|S_k\| + a_k)\|\|S_k\| - a_k\| \geq a_k\|\|S_k\| - a_k\|$. To see the first inclusion in (2.5) observe that

$$\begin{aligned} \{\|\|S_k\|^2 - a_k^2\| \geq s(2a_k + s)\} &\subset \{\|\|S_k\| - a_k\| \geq s\} \cup \{\|S_k\| + a_k \geq 2a_k + s\} \\ &\subset \{\|\|S_k\| - a_k\| \geq s\}. \end{aligned}$$

Now by (2.5) we get

$$\begin{aligned} &\mathbb{P}\left(\max_k \|\|S_k\| - a_k\| \geq (6\alpha + 2\beta + 1)t\right) \\ &\leq \mathbb{P}\left(\max_k \|\|S_k\|^2 - a_k^2\| \geq (6\alpha + 2\beta + 1)at\right). \end{aligned}$$

Hence by Proposition 1.2,

$$\mathbb{P}\left(\max_k \|S_k - a_k\| \geq (6\alpha + 2\beta + 1)t\right) \leq 6 \max_k \mathbb{P}(|S_k|^2 - a_k^2| \geq \frac{1}{3}(6\alpha + 2\beta + 1)at).$$

But $\frac{1}{3}(6\alpha + 2\beta + 1)a \geq 2(\alpha + \beta)t + t \geq 2a_k + t$ for all k by (1.4). Therefore by (2.5),

$$\begin{aligned} \mathbb{P}\left(\max_k \|S_k - a_k\| \geq (6\alpha + 2\beta + 1)t\right) &\leq 6 \max_k \mathbb{P}(|S_k|^2 - a_k^2| \geq t(2a_k + t)) \\ &\leq 6 \max_k \mathbb{P}(\|S_k - a_k\| \geq t). \end{aligned}$$

□

3 Example

Let us fix a positive integer n and put

$$I_n = \left\{j \in \mathbb{Z}: \frac{n}{2} \leq j \leq n\right\}.$$

Let $t_j = \frac{n^2+j}{j}$ for $j = 1, 2, \dots, n$, then

$$jt_j = n^2 + j \quad \text{and} \quad (j-1)t_j \leq n^2 \quad \text{for } j \in I_n. \quad (3.1)$$

Let N be a large integer (to be fixed later) and let F be the space of all double-indexed sequences $a = (a_{i,j})_{0 \leq i \leq N, j \in I_n}$ with the norm

$$\left\| (a_{i,j})_{0 \leq i \leq N, j \in I_n} \right\| = \max_{j \in I_n} \left(|a_{0,j}| + t_j \sum_{1 \leq i_1 < i_2 < \dots < i_j \leq N} \sum_{s=1}^j |a_{i_s, j}| \right).$$

Let $(e_{i,j})$ be a standard basis of F , so that $(a_{i,j}) = \sum_{i,j} a_{i,j} e_{i,j}$.

Define random vectors X_1, X_2, \dots, X_n by the formula

$$X_l = \sum_{j \in I_n} (Y_{l,j} e_{0,j} + R_{l,j} e_{N_l, j}),$$

where $(Y_{l,j}, R_{l,j})_{l \leq n, j \in I_n}$ and $(N_l)_{l \leq n}$ are independent r.v.'s, $\mathbb{P}(R_{l,j} = \pm 1) = 1/2$, $Y_{l,j}$ are symmetric $\mathbb{P}(|Y_{l,j}| = \frac{1}{2n}) = 1 - \mathbb{P}(Y_{l,j} = 0) = p_n$ (with p_n a small positive number to be specified later) and N_l are uniformly sampled from the set $\{1, \dots, N\}$.

Obviously X_1, X_2, \dots, X_n are i.i.d. and symmetric. As usual we set $S_k = X_1 + X_2 + \dots + X_k$. Let

$$A = \{N_1, N_2, \dots, N_n \text{ are pairwise distinct}\}.$$

Notice that $\mathbb{P}(A) \rightarrow 0$ when $N \rightarrow \infty$. On the set A we have for $k \leq n$,

$$\|S_k\| = \max_{j \in I_n} \left(\left| \sum_{l=1}^k Y_{l,j} \right| + t_j \min\{k, j\} \right).$$

For $j > k$ we have by (3.1),

$$\left| \sum_{l=1}^k Y_{l,j} \right| + t_j \min\{k, j\} < 1 + t_j(j-1) \leq n^2 + 1,$$

hence on the set A , for $k \in I_n$ we get

$$\|S_k\| = \max_{j \in I_n, j \leq k} \left(\left| \sum_{l=1}^k Y_{l,j} \right| + n^2 + j \right) = \left| \sum_{l=1}^k Y_{l,k} \right| + n^2 + k.$$

Take $0 < t < \frac{1}{2nC_1}$ then for $k \in I_n$,

$$\mathbb{P}(\|S_k\| - (n^2 + k) \geq t) \leq \mathbb{P}(A) + \mathbb{P}\left(\sum_{l=1}^k Y_{l,k} \neq 0\right) \leq \mathbb{P}(A) + kp_n$$

and

$$\mathbb{P}\left(\max_{k \in I_n} \|S_k\| - (n^2 + k) \geq tC_1\right) \geq \mathbb{P}\left(\max_{k \in I_n} \left| \sum_{l=1}^k Y_{l,k} \right| \neq 0\right) - \mathbb{P}(A).$$

The last number is of order $n^2 p_n$ if N is large and p_n is small. This shows that if (1.5) holds for $i = \lceil n/2 \rceil$ in F then C_2 must be of order n . So (1.5) cannot hold with absolute constants C_1 and C_2 in (infinite dimensional) separable Banach spaces.

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