Pisier's inequality and the discrete cube

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Thm. (Pisier 1986) g, g' indep $N(0, I_n)$.

$f : \mathbb{R}^n \longrightarrow X$ (X = Banach sp.).

$$\left(\mathbb{E}\,\|f(g) - \mathbb{E}f(g)\|_X^p\right)^{1/p} \le \frac{\pi}{2}\left(\mathbb{E}\,\Big\|\sum_{i=1}^n g_i' \frac{\partial f}{\partial x_i}(g)\Big\|_X^p\right)^{1/p}.$$

Ex. Scalar $f : \mathbb{R}^n \to \mathbb{R}$.

$$\|f(g) - \mathbb{E}f(g)\|_{L^p} \lesssim \sqrt{p}\;\|\nabla f(g)\|_p.$$

Ex. $$\mathbb{E}\,|f(g) - \mathbb{E}f(g)| \le \sqrt{\frac{\pi}{2}}\;\mathbb{E}\,\|\nabla f(g)\|$$

Cheeger's ineq.
in Gauss sp.

sharp!

Ex. $g_1, \dots, g_n$ iid $N(0,1)$.
$A_1, \dots, A_n$ det. $d \times d$ mtcs. $(\in H_d)$.

Noncomm. Khintchine (Lust-Piquard):

$$\left[\mathbb{E}\,\mathrm{Tr}\left(\sum_{i=1}^{n} g_i A_i\right)^{2p}\right]^{1/2p} \lesssim \sqrt{p}\left[\mathrm{Tr}\left(\sum_{i=1}^{n} A_i^2\right)^{p}\right]^{1/2p}$$

Pisier: $f : \mathbb{R}^n \to H_d$ X

$$\left(\mathbb{E}\,\mathrm{Tr}\left[\left(f(g) - \mathbb{E}f(g)\right)^{2p}\right]\right)^{1/2p} \le \frac{\pi}{2}\left(\mathbb{E}\,\mathrm{Tr}\left[\left(\sum_i g_i' \frac{\partial f}{\partial x_i}(g)\right)^{2p}\right]\right.$$

$$\lesssim \sqrt{p}\left(\mathbb{E}\,\mathrm{Tr}\left[\left(\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2(g)\right)^{p}\right]\right)^{1/2p}.$$

linear $\Rightarrow$ nonlinear

Enflo's question (1969–1978)

X Banach sp is of type p if $\forall n \geq 1$, $\forall x_1, \ldots, x_n$

$$\left(\mathbb{E} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\|_X^p \right)^{1/p} \leq T_p(X) \left(\sum_{i=1}^{n} \|x_i\|_X^p \right)^{1/p}$$

ε_i iid random signs

Enflo proposes a nonlinear analogue:

$$\left(\mathbb{E} \| f(\varepsilon) - f(-\varepsilon) \|_X^p \right)^{1/p} \leq T_p^E(X) \left(\mathbb{E} \sum_{i=1}^{n} \| f(\varepsilon) - f(\varepsilon_1, \ldots, -\varepsilon_i, \ldots, \varepsilon_n) \|_X^p \right)^{1/p}$$

Clear: Enflo type $\Rightarrow$ type $\left(f(\varepsilon) = \sum_{i=1}^{n} \varepsilon_i x_i \right)$

Q. type $p \Rightarrow$ Enflo type p ?

(Partial prog: Enflo, Pisier, B-M-W, Naor-Schechtman, Eskenazis, ...)

Pisier ineq. on cube: (Pisier 1986)

<u>Thm.</u> (Pisier 1986) ε_i, δ_i iid random signs, $f: \{\pm 1\}^n \to X$.

$$\left(\mathbb{E} \| f(\varepsilon) - \mathbb{E} f(\varepsilon) \|_X^p \right)^{1/p} \lesssim \log(n) \left(\mathbb{E} \left\| \sum_{i=1}^n \underline{\delta_i} D_i f(\varepsilon) \right\|_X^p \right)^{1/p}$$

where $$D_i f(\varepsilon) := \frac{f(\varepsilon) - f(\varepsilon_1, \dots, -\varepsilon_i, \dots, \varepsilon_n)}{2}$$

and $\log(n)$ is sharp for $X = \ell_\infty$ (Talagrand).

Thm. (I-vH-V '20)

$\mathbb{P}\{\xi_i(t) = \pm 1\} = \frac{1 \pm e^{-t}}{2}$. $\delta_i(t) = \frac{\xi_i(t) - \mathbb{E}\xi_i(t)}{(\mathrm{Var}\,\xi_i(t))^{1/2}}$,

$$\left(\mathbb{E}\|f(\varepsilon) - \mathbb{E}f(\varepsilon)\|_X^p\right)^{1/p} \le \frac{\pi}{2}\int_0^\infty \left(\mathbb{E}\left\|\sum_{i=1}^n \delta_i(t) D_i f(\varepsilon)\right\|_X^p\right)^{1/p} \mu$$

Prob. meas

Rem: this ineq implies Gauss. ineq. by CLT.

Applications

1) Enflo's conjecture.

2) Pisier's ineq. on cube $\left(\delta_i \leftarrow \delta_i(t) \right)$

3) $\mathbb{E}\left| f(g) - \mathbb{E} f(g) \right| \le \sqrt{\frac{\pi}{2}}\ \mathbb{E}\|\nabla f(g)\|$ (scalar)

sharp

Conj: $\mathbb{E}\left| f(\varepsilon) - \mathbb{E} f(\varepsilon) \right| \le \sqrt{\frac{\pi}{2}}\ \mathbb{E}\|Df(\varepsilon)\|$.

????

Best known: $\frac{\pi}{2}$ (Lust-Piquard

We can show: $\frac{\pi}{2} - \varepsilon$

4) Various Riesz-transf type ineq.