Breaking the Brownian chain: time and place

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joint work with:
Frank Aurzada and Volker Betz (both TU Darmstadt)

High Dimensional Probability IX
Będlewo $\rightsquigarrow$ Zoom
June 2020

A class of physical problems

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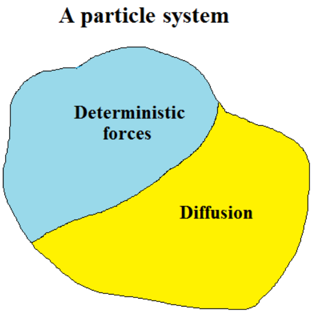

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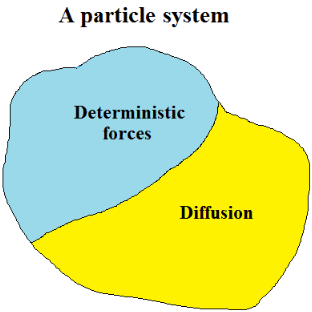

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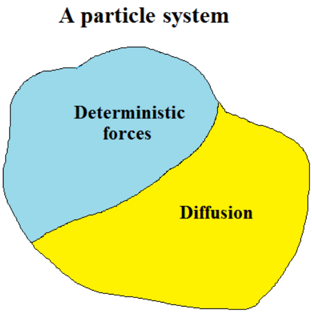

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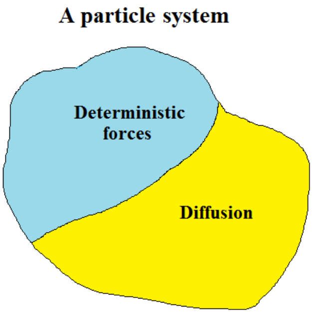

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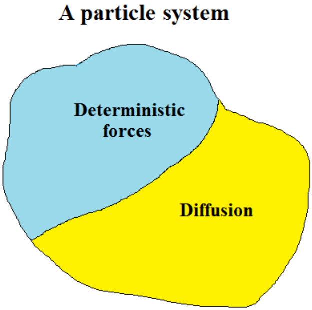

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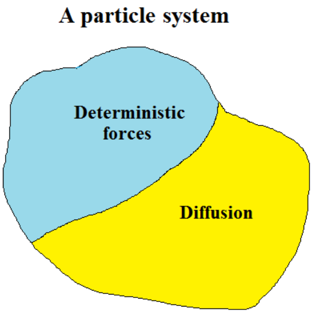

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We will explore in depth one such problem.

Problem setting: a Brownian chain

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- $d+1$ particles (named $0, 1, \ldots, d$) moving on the real line;
- initial location at time $t = 0$ is: $0, 1, \ldots, d$;
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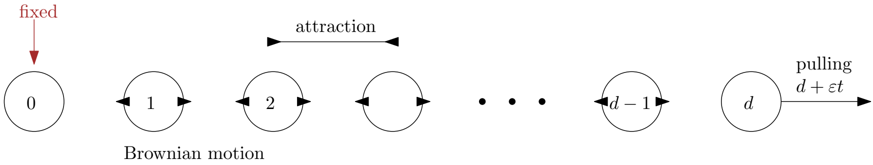

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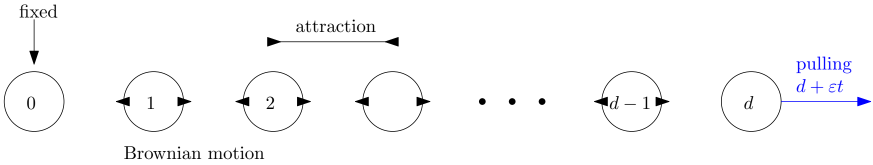

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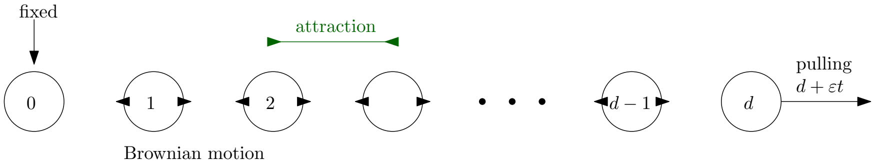

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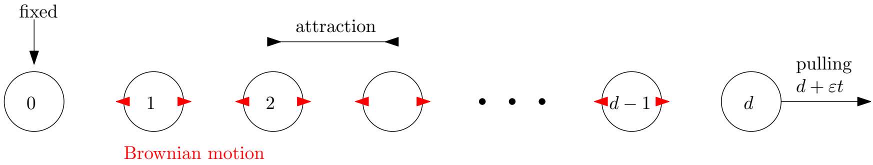

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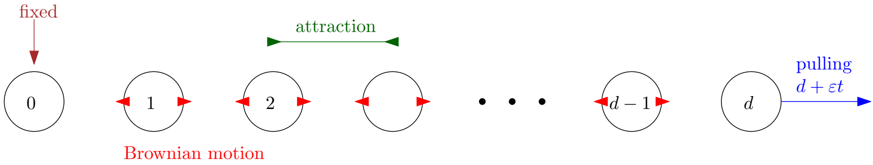

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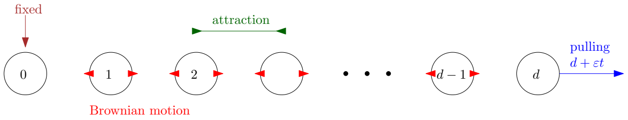

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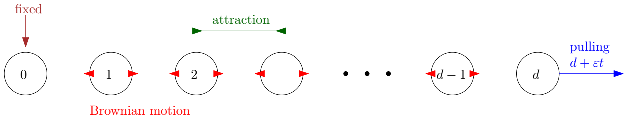
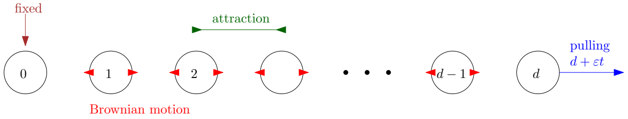


Let (B_t^i), $i = 1, \dots, d-1$, be independent Brownian motions.
We consider the system

$$\begin{cases} X_0^i = i & i = 0, 1, \dots, d, \\ X_t^0 = 0 & t \geq 0, \\ X_t^d = d + \varepsilon t & t \geq 0, \\ \mathrm{d}X_t^i = (X_t^{i+1} - 2X_t^i + X_t^{i-1})\mathrm{d}t + \sigma \mathrm{d}B_t^i, & i = 1, \dots, d-1, t \geq 0. \end{cases}$$

Problem setting: the break



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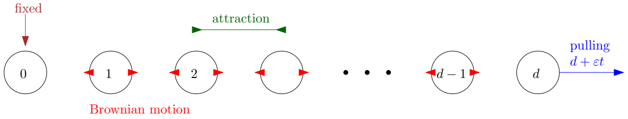

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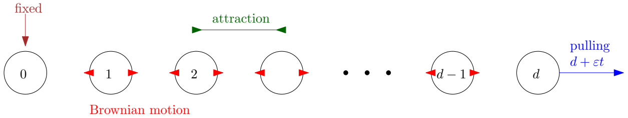

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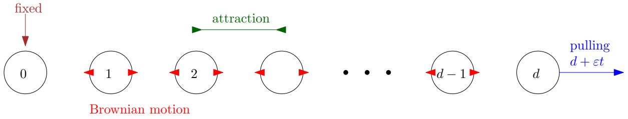

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$$\tau := \min_{1 \leq i \leq d} \tau^i.$$

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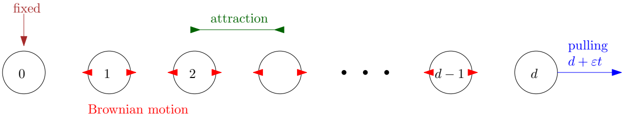

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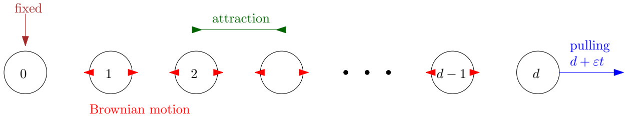

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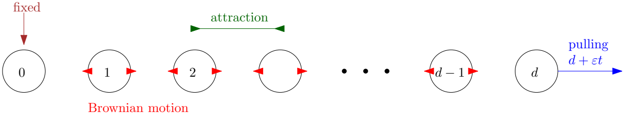
Next, find the distribution of the break position:

$$\mathbb{P}(\tau = \tau^i), \qquad i = 1, \ldots, d.$$

Problem setting: asymptotics



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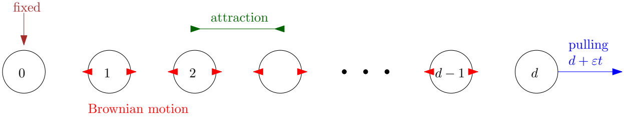

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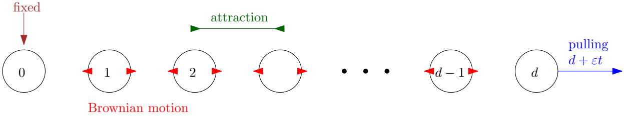

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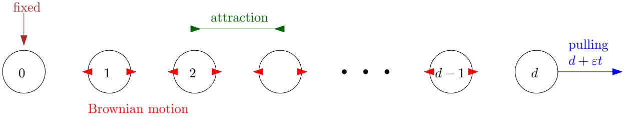

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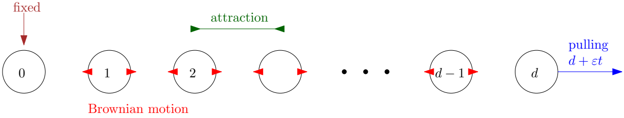

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Mathematically, this model was studied by Allman and Betz (2009), then by Allman, Betz, and Hairer (2011).

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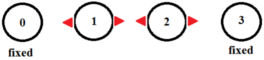

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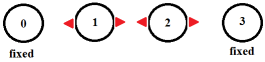
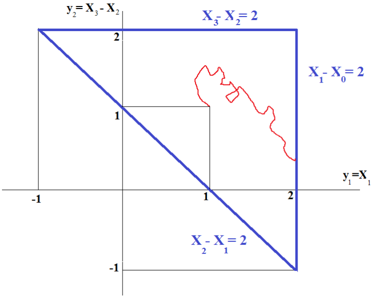


Then the break is an exit of a $2D$-diffusion from a triangle:



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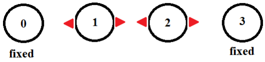

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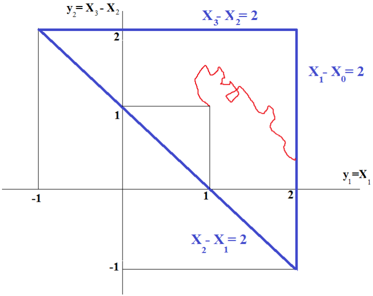

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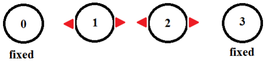

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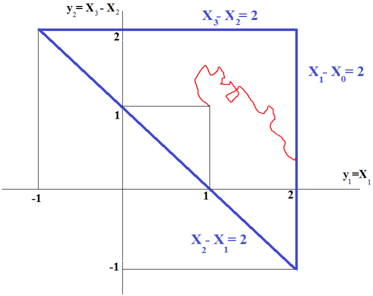

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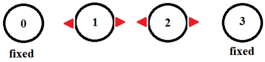

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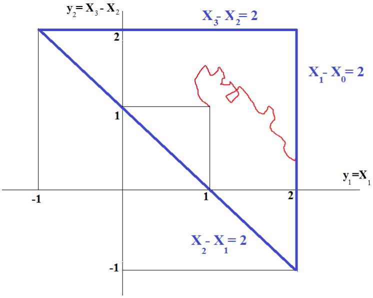

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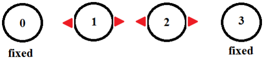

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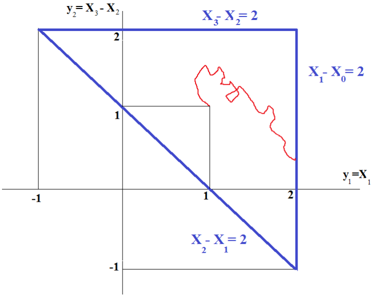

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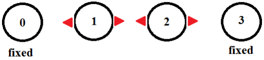

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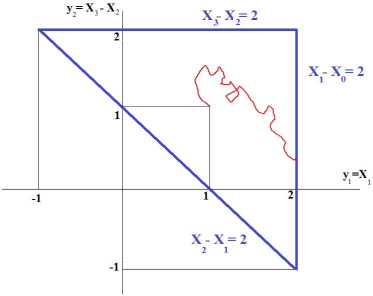

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If $\varepsilon > 0$, the triangle shrinks in time.

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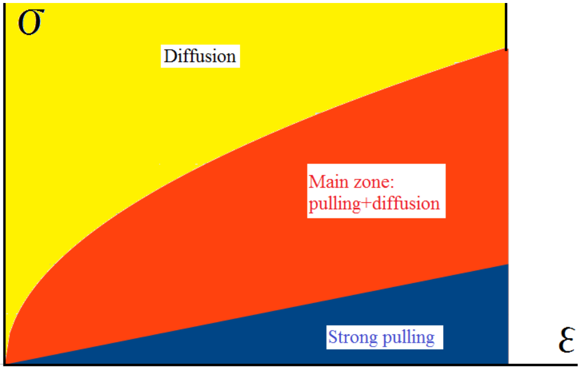

There are three zones, each with its own behavior of the model.

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Let $\varepsilon, \sigma \to 0$ *and* $\varepsilon \gg \sigma$. *Then*

$$\mathbb{P}(\tau = \tau^d) \to 1, \qquad \mathbb{P}(\tau = \tau^i) \to 0, \ i = 1, \dots, d-1.$$

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$$\tau = t_* - \frac{(d-1)(2d-1)}{6} + o_P(1).$$

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  half less probable on the first and the last link $0 \leftrightarrow 1$, $d-1 \leftrightarrow d$;
- the break time: when properly rescaled, has a Gumbel (double exponential) distribution defined by $\mathbb{P}(\mathcal{G} \le r) = \exp\{-g_1 \cdot \exp(-g_2 r)\}$.

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where $\mathcal{G}$ *has a Gumbel distribution:* $\mathbb{P}(\mathcal{G} \le r) = \exp\{-g_1 \cdot \exp(-g_2 r)\}$ *with* $g_1 = \frac{2vd^2}{\sqrt{2\pi}}, g_2 = \frac{\sqrt{2}}{vd}$ *and* $v^2 := \frac{d-1}{2d}$.

The intermediate (main) regime: $\varepsilon \ll \sigma$, $|\log \varepsilon|\sigma^2 \to 0$ (the red zone in )

Formally:

Theorem

Let $\varepsilon, \sigma \to 0$ *be such that* $\varepsilon \ll \sigma$ *but* $|\log \varepsilon|\sigma^2 \to 0$. *Then*

$$\mathbb{P}(\tau = \tau^i) \to \begin{cases} \frac{1}{2(d-1)} & i = 1, d, \\ \frac{1}{d-1} & i = 2, \dots, d-1, \end{cases}$$

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Remark: $|\log \varepsilon|\sigma^2 \to 0$ ensures that $t_* \gg \frac{\sigma}{\varepsilon}\sqrt{\log(\sigma/\varepsilon)}$.

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$$\textit{but}: \qquad \tau\,(\sigma v)^{-1} \exp(-(\sigma v)^{-2}/2) \cdot \frac{2d}{\sqrt{2\pi}} \Rightarrow \mathcal{E},$$

where $\mathcal{E}$ is standard exponential and $v^2 = \frac{d-1}{2d}$.

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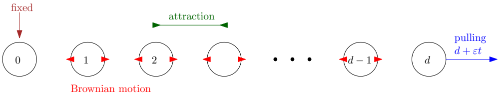
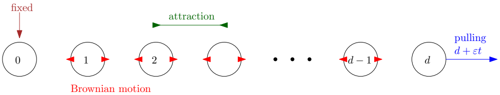
Further complication: If the winning probability of i-th player depends on the game number ℓ: $p_i \to \delta_\ell p_i$, then the winning probability remains the same but the series duration distribution may change (for example to Gumbel law).

Dictionary: from games to breaks





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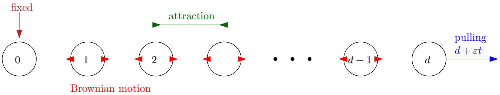

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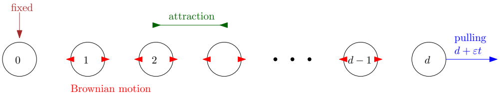

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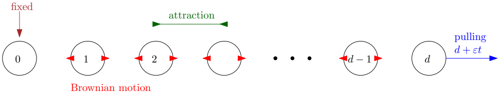

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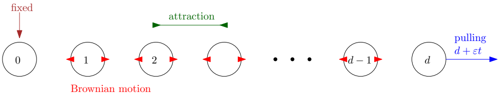

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- the time intervals should be separated by smaller but still long intervals in order to make the "games" almost independent;
- the exit probabilities are evaluated via Pickands–Piterbarg lemma (sharper than with usual LDP).

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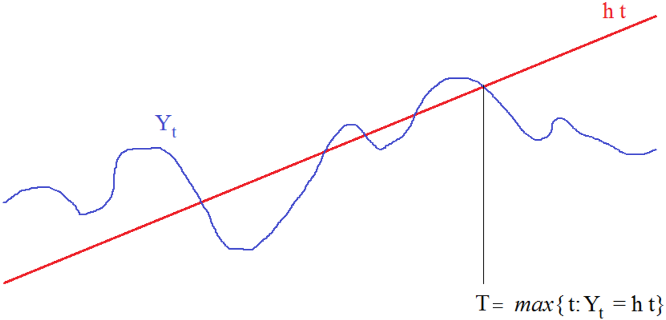

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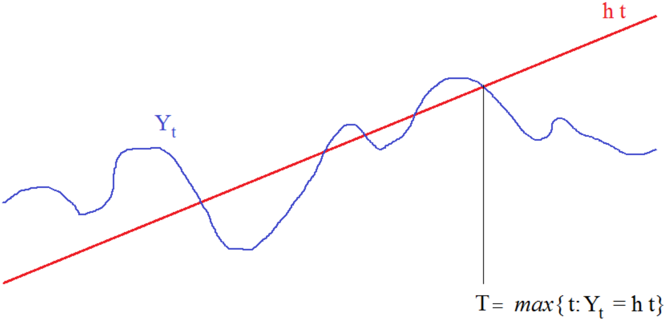

Under appropriate assumptions on Y (Pickands lemma applies, mixing), as $h \to 0$, T goes to infinity and has asymptotically Gumbel distribution.

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- Consider other systems (cf. Malyshev for second order equation).

Thank you for your attention!

<u>References:</u>

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