Nodal domains of $G(n, p)$ graphs]

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joint work with Han Huang

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What are nodal domains?

Let M be a compact manifold. Let Δ be the Laplacian on M, and consider an eigenfunction f_k of $-\Delta$ corresponding to the eigenvalue λ_k.

Definition

A connected component of the set where f_k is positive or negative is called a nodal domain of f_k.

Basic questions

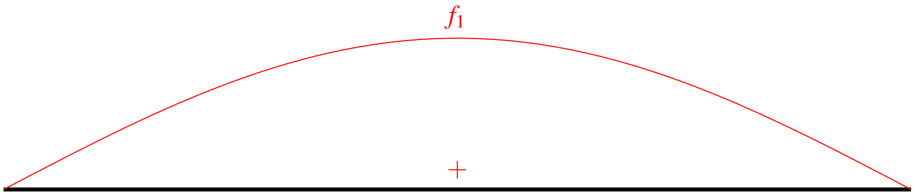
- Number of nodal domains.
- Geometry.

Typical behavior

Example

Let $M = [0, 1]$ and $\Delta f = f''$. Consider the Dirichlet boundary conditions $f(0) = f(1) = 0$.

$$\lambda_k = (\pi k)^2 \quad f_k(x) = \sin(\pi k x)$$



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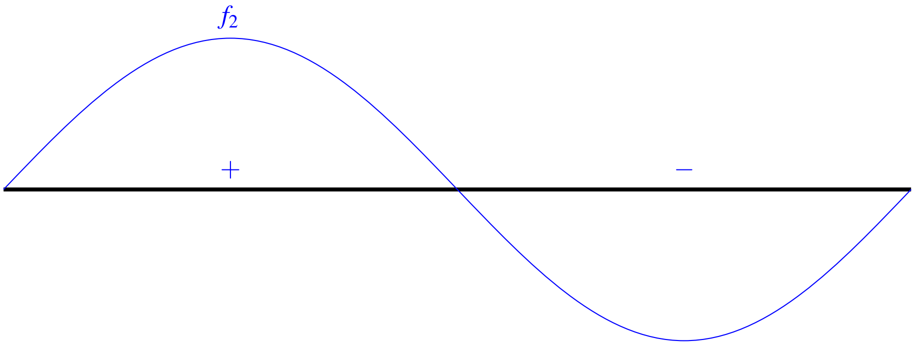

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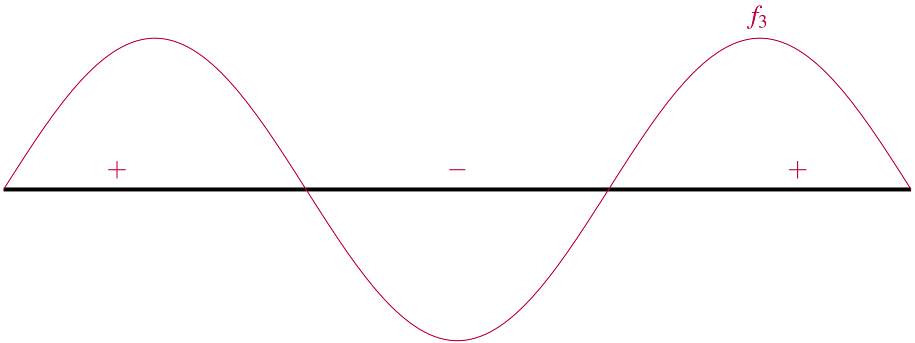

Manifolds vs graphs

Theorem (Courant)

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Let $G = (V, E)$ be a graph with the adjacency matrix A:

$$A_{u,v} = 1 \quad \text{if and only if } (u, v) \in E.$$

The graph Laplacian is defined as

$$\Delta f(u) = \sum_{v \sim u} f(v) - \deg(u) \cdot f(u) = (A - D)f,$$

where $D = \deg(v_1, \ldots, v_n)$.

Nodal domains of $G(n, p)$ graphs

Theorem (Dekel, Lee, Linial, $\sim$ 2007)

Let $p \in (0, 1)$. Then with high probability, any eigenvector of the adjacency matrix of a $G(n, p)$ graph has $C(p)$ nodal domains. Here $C(p)$ is independent of n and uniform over the eigenvectors.

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Linial's questions

Geography of nodal domains. View the positive nodal domain as earth, and the negative one as water.

- What is the distribution of heights?
- What is the length of the shoreline?

Delocalization of eigenvectors

Let $x \in S^{n-1}$ be a unit vector of an $n \times n$ Gaussian matrix (i.i.d. or i.i.d. symmetric). Then x is uniformly distributed over S^{n-1}.

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Let $x \in S^{n-1}$ be a unit vector of an $n \times n$ random matrix (i.i.d. or i.i.d. symmetric). Then x is approximately uniformly distributed over S^{n-1}.

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ℓ_∞ delocalization – rules out large coordinates

- $\|x\|_\infty = O\left(\frac{\sqrt{\log n}}{\sqrt{n}}\right)$ with high probability.
- Erdős, Knowles, Yau, Yin for i.i.d. symmetric matrices,
  R'-Vershynin for i.i.d. matrices

$$\|x\|_\infty = O\left(\frac{\log^c n}{\sqrt{n}}\right) \quad \text{with probability at least } 1 - n^{-C}$$

Delocalization of eigenvectors

No-gaps delocalization – rules out small coordinates

- With high probability, any $J \subset [n]$ supports a non-negligible mass:

$$\|x|_J\|_2 \geq c \left(\frac{|J|}{n} \right)^{\alpha}$$

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- R'-Vershynin for i.i.d. i.i.d. symmetric, i.i.d. skew-symmetric etc. matrices
  If $|J| \geq n^{6/7}$ then

$$\|x|_J\|_2 \geq c \left(\frac{|J|}{n} \right)^{6} \quad \text{with probability at least } 1 - \exp(-|J|).$$

The two types for delocalization are complementary and are frequently used in combination.

Dekel, Lee, Linial, Arora, Bhaskara theorem

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Proof (Step 1).

Denote by P and N the largest positive and negative nodal domains. Set $W = [n] \setminus (P \cup N)$. Then

$$|W| = O\left(\frac{\log^2 n}{p^2}\right).$$

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- If $V_1, V_2 \subset [n]$ are not connected, then one of them has $O\left(\frac{\log n}{p}\right)$ elements.
- Take one vertex from each positive (negative) nodal domain. They form an independent set. The cardinality of an independent set is $O\left(\frac{\log n}{p}\right)$.

□

Dekel, Lee, Linial, Arora, Bhaskara theorem

Assume that $p = \text{const}$.
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Assume that $W \neq \varnothing$. Let $w \in W, x_w < 0$. Then $w \not\sim N$, so

$$\lambda x_w = (Ax)_w = \sum_{\substack{v \in P \\ v \sim w}} x_v + \sum_{\substack{v \in W \\ v \sim w}} x_v$$

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This contradicts no-gaps delocalization since $|\{v \in P,\ v \sim w\}| \geq cn$.

Geography of nodal domains

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A similar statement holds for any vertex of the negative domain.

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Conclusion: A $G(n,p)$ graph is a swamp.

Size of nodal domains

Sizes of nodal domains

Is it true that $|P| \approx |N|$?

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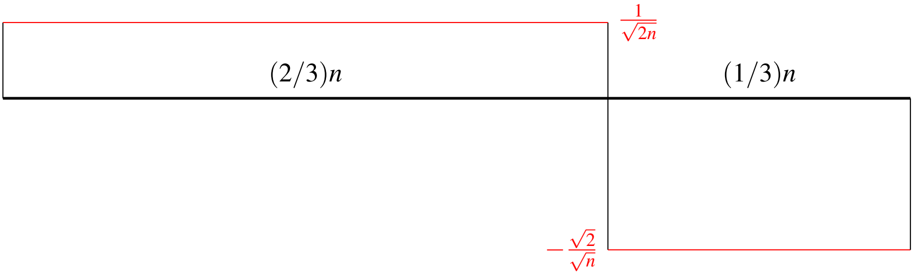
This would not follow from delocalization.
Indeed, imagine that the eigenvector x has the following coordinates:



Size of nodal domains, bulk case

Eigenvalues:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

Theorem (Huang, R')

Let $\varepsilon, \kappa \in (0, 1)$, *and let* $p \in \left[n^{-c}, \frac{1}{2}\right]$.
Let u_α *be an eigenvector of a* $G(n, p)$ *graph with* $\alpha \in [\kappa n, n - \kappa n]$.
Denote by P *and* N *the nodal domains of this eigenvector. Then*

$$\mathbb{P}\left(|P| \vee |N| \geq \left(\frac{1}{2} + \varepsilon\right) n\right) \leq n^{-\eta}$$

for some $\eta = \eta(\varepsilon, \kappa) > 0$.

Size of nodal domains, edge case

Theorem (Huang, R')

Let $p \in (0, 1)$.
Let u_α *be a non-leading eigenvector of a* $G(n, p)$ *graph with*

$$\min\{\alpha, n - \alpha\} \leq \textit{small}$$

Denote by P and N the nodal domains of this eigenvector.
Then, for any $\varepsilon > 0$,

$$\mathbb{P}\left(|P| \vee |N| \geq \left(\frac{1}{2} + n^{-\frac{1}{6}+\varepsilon}\right) n\right) \leq n^{-\delta}.$$

for large n, with $\delta = \delta(\varepsilon) > 0$ *independent of n and p.*

Ideas of proof, bulk case

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The latter equation follows if for $i \neq j$,

$$\mathbb{E}\,\operatorname{sign}(x_i x_j) = o(1).$$

Ideas of proof, bulk case

Asymptotic normality:

Theorem (Bourgade, J. Huang, Yau)

For any polynomial $f : \mathbb{R} \to \mathbb{R}$ *for any* $n \geq n(f)$, $\alpha \in [\kappa n : n - \kappa n]$ *and any* $q \in S^{n-1}$, $q \perp (1, \dots, 1)$, *there exists an* $\nu > 0$ *such that*

$$|\mathbb{E}f(n\langle q, x\rangle^2) - \mathbb{E}f(g^2)| \leq n^{-\nu}.$$

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$$|\mathbb{E} f(n\langle q, x\rangle^2) - \mathbb{E} f(g^2)| \leq n^{-\nu}.$$

1. Pass to a polynomial of 4 variables removing the condition $q \perp (1, \dots, 1)$.
2. Pass to an even polynomial of 2 variables.
3. Pass to an odd polynomial of $x_i x_j$:

$$|\mathbb{E} f(n \cdot x_i x_j) - \mathbb{E} f(g_1 g_2)| \leq n^{-\nu} \qquad \text{and} \qquad \mathbb{E} f(g_1 g_2) = 0.$$

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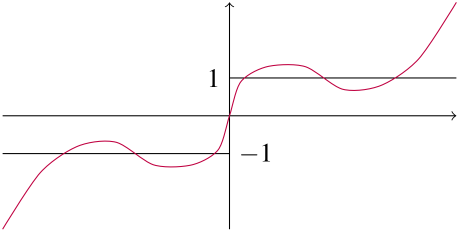
$$|\mathbb{E} f(n \cdot x_i x_j)| \le n^{-\nu} \quad \stackrel{?}{\Rightarrow} \quad |\mathbb{E}\, \mathrm{sign}\,(n \cdot x_i x_j)\,| \le n^{-\nu}$$

Approximate sign by an odd polynomial:



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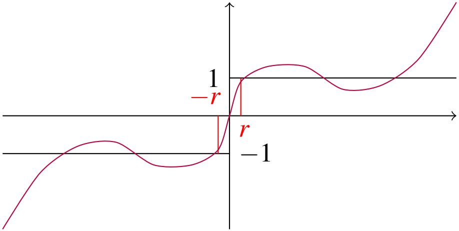

- Interval $(0, r]$ – few coordinates – no-gaps decollation.

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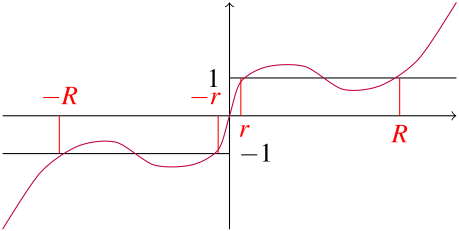

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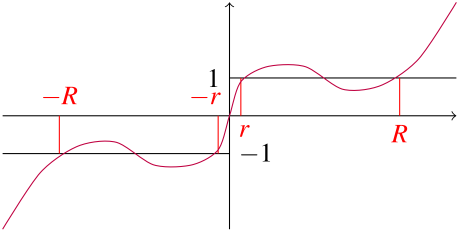

- Interval $(0, r]$ – few coordinates – no-gaps decollation.
- Interval $(r, R]$ – many coordinates – uniform approximation.
- Interval $(R, \infty]$ – few coordinates – L_2 bound with respect to the distribution of $g_1 g_2$.

The last two requirements are conflicting.

Open problems - sparse graphs

- Multiple nodal domains. Assume that $p \ll 1$.
  The proof of DLL+AB theorem was based on

$$|W| = O\left(\frac{\log^2 n}{p^2}\right).$$

  This is vacuous if $p \leq n^{-1/2}$.

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  Numerical experiments (Huang, Eldan): if p is small and λ is at the negative edge then there are multiple one vertex nodal domains.
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- Geography of nodal domains
  Numerical experiments (Huang): if p is small and λ is at the negative edge then the main nodal domains have non-trivial geography.

Open problems - dense graphs

Definition

Let $G = (V, E)$ be a graph. A partition $V = U \sqcup W$ is called

- inner if any vertex in U and W has more neighbors in the same set than in the other one.
- outer if both U and V have the opposite property.

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Conjecture (Linial)

- If $\lambda > 0$ then $V = P \sqcup N$ is an inner partition with high probability.
- If $\lambda < 0$ then $V = P \sqcup N$ is an outer partition with high probability.