New results for additive functionals of Markov chains

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PLAN OF TALK

Symmetric projective conditions. Open problems

New result for the variance of partial sums

A new central limit theorem for additive functionals of Markov chains

Stationary Markov chains

We consider a **stationary Markov chain** (ξ_n) defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $(S, \mathcal{A})$, with marginal distribution

$$\pi(A) = \mathbb{P}(\xi_0 \in A).$$

We assume that there is a regular conditional distribution $Q(x, A) = \mathbb{P}(\xi_1 \in A | \, \xi_0 = x)$. Let Q also denotes the Markov **transition operator** acting via

$$(Qf)(x) = \int_S f(s) Q(x, \mathrm{d}s)$$

on $\mathbb{L}_0^2(\pi)$, the set of measurable functions on S such that $\int f^2 d\pi < \infty$ and $\int f d\pi = 0$. For a function $f \in \mathbb{L}_0^2(\pi)$ let

$$X_i = f(\xi_i), \;\; S_n = \sum_{i=1}^{n} X_i.$$

Motivation: CLT for normal Markov chains

Denote $V_n = I + Q + ... + Q^n$. Note $E(X_k|\xi_0) = (Q^k f)(\xi_0)$. Denote by Q^* the adjoint of Q.

Gordin and Lifshitz (1981), Borodin and Ibragimov (1995), Derriennic and Lin (1996), Cuny (2011).

Theorem

Assume that the Markov chain is normal ($QQ^* = Q^*Q$), stationary and ergodic and satisfies

$$\sum_{n\geq 1} \frac{||E(S_n|\xi_0)||^2}{n^2} = \sum_{n\geq 1} \frac{||V_n(f)||^2}{n^2} < \infty.$$

Then

$$\lim_{n\to\infty} \frac{E(S_n^2)}{n} = \sigma^2$$

and

$$\frac{S_n}{\sqrt{n}} \Rightarrow N(0, \sigma^2).$$

Motivation: CLT for reversible Markov chains

Reversible Markov chains. Gordin and Lifshitz, Kipnis and Varadhan:

Theorem

Assume that the Markov chain is reversible ($Q = Q^*$), stationary and ergodic and satisfies

$$\sup_n \frac{E(S_n^2)}{n} < \infty.$$

Then the functional CLT holds ($W(t)$, standard Brownian motion)

$$\frac{S_{[nt]}}{\sqrt{n}} \Rightarrow \sigma W(t).$$

Identification of σ^2 : $\lim_{n\to\infty} \frac{E(S_n^2)}{n} = \sigma^2$.

This convergence is equivalent to $\sum_{n\geq 1} \frac{||V_n(f)||^2}{n^2} < \infty$.

General case question.

We may wonder if such a result holds without assuming normality?
More precisely, if a stationary and ergodic Markov chain satisfies

$$\sum_{n\geq 1} \frac{||V_n(f)||^2}{n^2} < \infty$$

does the CLT hold? The answer is **No.**

If the Markov chain is not normal there are numerous counter-examples (Peligrad Utev (2005), Volný (2010), Dedecker (2015) and Cuny and Lin (2016) and many others).

Example renewal process

Some counterexamples are constructed by **adjusting the parameters in the renewal process**. Whenever the chain hits 0, it regenerates according to probability $p_i = P(\tau = i) \geq 0$ with $\sum_{i\geq 0} p_i = 1$.

	0	1	2	3	4	5	...
0	p_0	p_1	p_2	p_3	p_4	p_5	...
1	1						...
2		1					...
3			1				...
4				1			...
5					1		...
...	...	...	...	...	...	...	...

The stationary distribution exists iff $E(\tau) < \infty$ $\left(\sum_{i\geq 0} ip_i < \infty\right)$.

What additional conditions we should impose?

A natural question is what **additional condition** should be added to

$$\sum_{n\geq 1} \frac{||V_n(f)||^2}{n^2} < \infty$$

for the CLT to hold?

The "additional" condition may be suggested by the **normal case** and by the following **philosophical question**. Note that a partial sum does not depend on the direction of time, i.e.

$$S_n = X_1 + X_2 + ... + X_n = X_n + X_{n-1} + ... + X_1.$$

Then, it is natural to look for **projection conditions which do not depend on the direction of time**.

There are examples in the literature of symmetric projective conditions: some mixing conditions, harnesses.

Open Problem 1

Problem

For a stationary and ergodic Markov chain is it true (or not) that conditions

$$\sum_{n\geq 1}\frac{||V_n(f)||^2}{n^2}<\infty \text{ and } \sum_{n\geq 1}\frac{||V_n^*(f)||^2}{n^2}<\infty.$$

imply that the CLT holds?

Above $V_n = I + Q + ... + Q^n$ and $V_n^* = I + Q^* + ... + (Q^*)^n$. So we can rewrite the conditions as

$$\sum\nolimits_{n\geq 1}\frac{||E(S_n|\xi_0)||^2}{n^2}<\infty \text{ and } \sum\nolimits_{n\geq 1}\frac{||E(S_n|\xi_n)||^2}{n^2}<\infty.$$

Square root condition.

Following Derriennic and Lin (2001), the operator $\sqrt{I-Q}$ is defined by

$$\sqrt{I-Q} := I - \sum_{n\geq 1} \delta_n Q^n,$$

where $\sqrt{1-x} = 1 - \sum_{n\geq 1} \delta_n x^n$, with $\delta_n > 0$, $n \geq 1$ and $\sum_{n\geq 1} \delta_n = 1$.

By a result in Cohen, Cuny and Lin (2017) we know that

$$f \in \sqrt{1-Q}\mathbb{L}_2(\pi) \text{ implies } \sum_{n\geq 1} \frac{||V_n(f)||^2}{n^2} < \infty$$

and also

$$f \in \sqrt{1-Q^*}\mathbb{L}_2(\pi) \text{ implies } \sum_{n\geq 1} \frac{||V_n^*(f)||^2}{n^2} < \infty.$$

Open Problem 2

These considerations suggest that the following problem deserves to be studied, of course, in case the answer to Problem 1 is negative.

Problem

If the Markov chain is stationary and ergodic is it true (or not) that $f \in \sqrt{1-Q}\mathbb{L}_2(\pi) \cap \sqrt{1-Q^*}\mathbb{L}_2(\pi)$ *implies that the CLT holds?*

Open Problem 3

Finally, if the answer to Problem 2 is negative, one could ask the following question:

Problem

If the Markov chain is stationary and ergodic is it true (or not) that the stronger conditions

$$\sum_{k\geq 1} ||Q^k f||^2 < \infty \text{ and } \sum_{k\geq 1} ||(Q^*)^k f||^2 < \infty$$

imply that the CLT holds?

These two last conditions can be reformulated as:

$$\sum_{k\geq 1} ||E(X_k|\xi_0)||^2 < \infty \text{ and } \sum_{k\geq 1} ||E(X_{-k}|\xi_0)||^2 < \infty.$$

The variance of partial sums result

Result on the variance of partial sums supporting the problems.
P. (2020). Recall $V_n = I + Q + ... + Q^n$ and $V_n^* = I + Q^* + ... + (Q^*)^n$

Theorem

Assume that

$$\sum_{n\geq 1} \frac{||V_n(f)||^2}{n^2} < \infty \text{ and } \sum_{n\geq 1} \frac{||V_n^*(f)||^2}{n^2} < \infty.$$

Then:

$$\lim_{n\to\infty} \frac{E(S_n^2)}{n} = \sigma^2.$$

The proof involves dyadic recurrence and subadditivity of $||V_n(f)||$ and $||V_n^*(f)||$.

A CLT with random centering. Non-ergodic case.

P. (2020)

Theorem

If $(\xi_n)_{n\in\mathbb{Z}}$ *is a stationary Markov chain such that*

$$\sup_{n\geq 1}\frac{E(S_n^2)}{n}<\infty,$$

then there is a random variable A^2 *such that*

$$\frac{S_n-E(S_n|\xi_0,\xi_n)}{\sqrt{n}}\Rightarrow A^2N(0,1),$$

where A^2 *is independent of* $N(0,1)$*.*

A CLT with random centering. Ergodic case.

P. (2020)

Theorem

If $(\xi_n)_{n\in\mathbb{Z}}$ is a stationary and ergodic Markov chain such that

$$\sup_{n\geq 1} \frac{E(S_n^2)}{n} < \infty,$$

then, the following limit exists

$$\lim_{n\to\infty} \frac{1}{n}||S_n - E(S_n|\xi_0,\xi_n)||^2 = \eta^2$$

and

$$\frac{S_n - E(S_n|\xi_0,\xi_n)}{\sqrt{n}} \Rightarrow N(0,\eta^2).$$

(Ergodic: for every $k \geq 1$ we have $Q^k f = f$ implies f is constant a.s.)

CLT for absolutely regular Markov chains

Corollary

Assume that the sequence is stationary, Harris recurrent and aperiodic (absolutely regular), satisfying

$$\sup_{n\geq 1} \frac{E(S_n^2)}{n} < \infty.$$

Then

$$\frac{S_n}{\sqrt{n}} \Rightarrow N(0, \eta^2).$$

with

$$\eta^2 = \lim_{n\to\infty} \frac{\pi \left(E|S_n|\right)^2}{2n}.$$

(see also Xia Chen (1999), Nummelin's splitting technique).

Corollary related to Problem 1

Corollary

Assume that

$$\sum_{n\geq 1} \frac{||V_n(f)||^2}{n^2} < \infty \text{ and } \sum_{n\geq 1} \frac{||V_n^*(f)||^2}{n^2} < \infty.$$

Then

$$\lim_{n\to\infty} \frac{E(S_n^2)}{n} = \sigma^2, \quad \lim_{n\to\infty} \frac{1}{n}||E(S_n|\xi_0,\xi_n)||^2 = c^2$$

and

$$\frac{S_n - E(S_n|\xi_0,\xi_n)}{\sqrt{n}} \Rightarrow N(0,\sigma^2 - c^2).$$

Corollary

Corollary

Assume that

$$\sum_{n\geq 1} \frac{||E(S_n|\xi_0,\xi_n)||^2}{n^2} < \infty \ .$$

Then

$$\lim_{n\to\infty} \frac{E(S_n^2)}{n} = \sigma^2$$

and

$$\frac{S_n}{\sqrt{n}} \Rightarrow N(0,\sigma^2).$$

This sufficient condition is implied by

$$\sum_{n\geq 1} ||E(X_0|\xi_{-n},\xi_n)||^2 < \infty.$$

Steps in the proof of the CLT (ergodic case)

1. A blocking argument
2. Martingale construction with "a special compensator"
3. Martingale approximation
4. CLT
5. The characterization of the limiting variance.

Blocking argument

Blocking

Fix m $(m < n)$ a positive integer and make consecutive blocks of size m. Let $u = u_n(m) = [n/m]$.

Denote by Y_k the sum of variables in the k'th block.

So, for $k = 0, 1, ..., u - 1$, we have

$$Y_k = Y_k(m) = (X_{km+1} + ... + X_{(k+1)m}).$$

We also have a last block

$$Y_u = Y_u(m) = (X_{um+1} + ... + X_n).$$

Martingale construction

For $k = 0, 1, ..., u - 1$ let us consider the random variables

$$D_k = D_k(m) = \frac{1}{\sqrt{m}}(Y_k - E(Y_k|\xi_{km}, \xi_{(k+1)m})).$$

Let $\mathcal{F}_n = \sigma(\xi_j; j \leq n)$ (past) and $\mathcal{F}^n = \sigma(\xi_j; j \geq n)$ (future). Conditioning by $\sigma(\xi_{km}, \xi_{(k+1)m})$ is equivalent to conditioning by $\mathcal{F}_{km} \vee \mathcal{F}^{(k+1)m}$.

Note that D_k is adapted to $\mathcal{F}_{(k+1)m} = \mathcal{G}_k$ and $E(D_k|\mathcal{G}_{k-1}) = 0$ a.s.

So $(D_k, \mathcal{G}_k)_{k\geq 0}$ is a **stationary and ergodic** sequence of square integrable **martingale differences.**

CLT for the martingale difference array

By the classical central limit theorem for ergodic martingales, for every m, a fixed positive integer, we have

$$\frac{1}{\sqrt{u}} M_u(m) := \frac{1}{\sqrt{u}} \sum_{k=0}^{u-1} D_k(m) \Rightarrow N_m \text{ as } n \to \infty,$$

where N_m is a normally distributed random variable with mean 0 and variance

$$m^{-1}||Y_0 - E(Y_0|\xi_0, \xi_m)||^2 = m^{-1}||S_m - E(S_m|\xi_0, \xi_m)||^2.$$

Denote **the compensator** by $Z_k = m^{-1/2} E(Y_k|\xi_{km}, \xi_{(k+1)m})$ and let $R_u(m) = \sum_{k=0}^{u-1} Z_k$. So

$$\frac{1}{\sqrt{n}} S_n \approx \frac{1}{\sqrt{u}} M_u(m) + \frac{1}{\sqrt{u}} R_u(m).$$

We can show that $M_n(m)$ and $R_n(m)$ are orthogonal but $R_u(m)/\sqrt{u}$ **is not negligible**.

Martingale approximation

With the **notation** $T_n = S_n - E(S_n|\xi_0, \xi_n)$, for m fixed

$$||\frac{1}{\sqrt{n}}T_n - \frac{1}{\sqrt{u}}M_u(m)||^2 \approx \left(\frac{1}{n}||T_n||^2 - \frac{1}{m}||T_m||^2\right) \text{ as } n \to \infty.$$

We do not know (yet) whether the limit of $||T_n||^2/n$ **exists**. But clearly

$$\liminf_m \limsup_{n\to\infty} ||\frac{1}{\sqrt{n}}T_n - \frac{1}{\sqrt{u}}M_u(m)||^2 = 0.$$

Since the martingale satisfies the CLT, we obtain

$$\frac{1}{\sqrt{n}}T_n \Rightarrow N(0, \eta^2).$$

where $\eta^2 = \limsup ||T_n||^2/n$. Finally, by Skorokhod's theorem and Fatou's lemma, we identify $\eta^2 = \lim ||T_n||^2/n$.

Idea of proof for absolutely regular Markov chains.

For a stationary Markov chain $(\xi_k)_{k\in Z}$, with values in a Borel space, the coefficient of absolute regularity is

$$\beta_n = \beta(\xi_0, \xi_n) = ||\mu_{(\xi_0,\xi_n)} - \mu_{\xi_0} \times \mu_{\xi_n}||_{\text{tot var}}$$

where $\mu_{(\xi_0,\xi_n)}$ is the joint distribution, μ_{ξ_0}, μ_{ξ_n} are the distributions of ξ_0 and ξ_n.

A Markov chain is Harris recurrent and aperiodic iff $\beta_n \to 0$.

Lemma

If

$$\sup_{n\geq 1} \frac{E(S_n^2)}{n} < \infty \textit{ and } \beta_n \to 0$$

then the random centering is negligible:

$$\frac{E|E(S_n|\xi_0,\xi_n)|}{\sqrt{n}} \to 0.$$

Idea of proof of Lemma

Fact

Let X, Z be two random variables on a probability space $(\Omega, \mathcal{K}, P)$ with values in a separable Banach space. Let $\mathcal{B} \subset \mathcal{K}$ be a sub $\sigma-$algebra, $\mathcal{A} = \sigma(X)$ and $\mathcal{C} = \sigma(Z)$. Assume that X and Z are conditionally independent given $\mathcal{B}$. Then

$$\beta(\mathcal{B}, \mathcal{A} \vee \mathcal{C}) \leq \beta(\mathcal{A}, \mathcal{B}) + \beta(\mathcal{C}, \mathcal{B}) + \beta(\mathcal{A}, \mathcal{C}),$$

Comment on the CLT for absolutely regular Markov chains.

The traditional CLT for absolutely regular Markov chains involves the estimation of the β_n coefficients. For instance, if $X_i = f(\xi_i)$ is bounded and centered, the minimal condition for the CLT is

$$\sum_{n\geq 1} \beta_n < \infty.$$

(Doukhan, Massard, Rio (1994), Bradley (2007))
However, the results presented here show that if one knows that

$$\sup_{n\geq 1} \frac{E(S_n^2)}{n} < \infty,$$

a CLT can be obtained without estimating the $\beta-$mixing rate.

An application

Let $\mathbf{Y} = (Y_j)$ and $\mathbf{Z} = (Z_j)$ be two stationary sequences of bounded random variables, independent among them and satisfying

$$\sum_{n\geq 1} \beta_n(Y) < \infty \text{ and } \beta_n(Z) \to 0.$$

Define $\mathbf{X} = (X_n)$, where for each n we set

$$X_n = Y_n Z_n$$

Then, we have

$$\beta_n(X) \leq \beta_n(Y) + \beta_n(Z).$$

As a consequence, $\beta_n(X) \to 0$. One can show for $S_n = \sum_{k=1}^n X_k$ that

$$E(S_n^2) \leq Cn.$$

By the corollary mentioned before, we obtain that the CLT holds for the sequence $(S_n/\sqrt{n})$.

An example

As a particular example of this kind let us consider **two stationary renewal processes** (ξ_i) and (η_i), with countable state space $\{0, 1, 2, ...\}$ and **independent among them**.

	0	1	2	3	4	5	...
0	p_0	p_1	p_2	p_3	p_4	p_5	...
1	1						...
2		1					...
3			1				...
4				1			...
5					1		...
...	...	...	...	...	...	...	...

$$p_i = \left(2i^3(\log(i+1))^2\right)^{-1} \ , \ i \geq 1.$$

$$p_i' = \left(2i^2(\log^2(i+1))\right)^{-1} \ , \ i \geq 1.$$

An example

From Davydov (1973), we know that the $\beta-$**mixing coefficients** for these sequences are of orders

$$\beta_n((\xi_i)_i) \leq c\frac{1}{n(\log(n+1))^2} \text{ and } \beta_n((\eta_i)_i) \leq c\frac{1}{\log^2(n+1)},$$

where c is a positive constant.

Now, let f and g be two bounded function and define the sequences $\mathbf{Y}$ and $\mathbf{Z}$ by $Y_i = f(\xi_i) - E(f(\xi_i))$ and $Z_i = g(\eta_i) - E(g(\eta_i))$ and set $X_i = Z_i Y_i$.

Clearly, for this example $\sum_{n\geq 1} \beta_n(Y) < \infty$ and $\beta_n(Z) \to 0$ and we obtain that the CLT holds for the sequence $(S_n/\sqrt{n})$.

Other directions

Other directions suggested by discussions with some colleagues:

1. Extension of these results to Markov random fields
2. Identifying the limiting distribution in non-ergodic case
3. In the strong mixing case is the random centering needed?
4. If the double tail sigma field is trivial is the random centering needed?
5. Similar results without assuming Markov property.

Thank you !