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High-Dimensional Probability IX
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Title: Hardy's inequalities: a unification via probability theory

Abstract: Over the period 1915 to 1925, G. H. Hardy proved and refined two interesting inequalities as follows. First the continuous (or integral form) inequality: for a non-negative function ψ on $(0,\infty)$ and $p>1$

$$\int_0^\infty \left(\frac{1}{x}\int_0^x \psi(y)dy\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_{\mathbb{R}} \psi^p(x)dx.$$

On the other hand, the discrete (or series form) inequality is: for a sequence $\{a_n\}$ of non-negative real numbers and $p>1$

$$\sum_{n=1}^\infty \left(\frac{1}{n}\sum_{k=1}^n a_k\right)^p \le \left(\frac{p}{p-1}\right)^p \sum_{n=1}^\infty a_n^p.$$

In this talk I will discuss the following unification of these two inequalities in probability terms: for any distribution function F on $\mathbb{R}$, non-negative function ψ on $\mathbb{R}$, and $p>1$

$$\int_{\mathbb{R}} \left(\frac{1}{F(x)}\int_{(-\infty,x]} \psi(y)dF(y)\right)^p dF(x) \le \left(\frac{p}{p-1}\right)^p \int_{\mathbb{R}} \psi^p(x)dF(x).$$

Alternatively, for independent random variables X and Y both having distribution function F on $\mathbb{R}$,

$$E\left(\Big[\frac{E(\psi(Y)1_{[Y\le X]}|X)}{F(X)}\Big]^p\right) \le \left(\frac{p}{p-1}\right)^p E(\psi^p(Y)).$$

(Recall that an arbitrary distribution function F on $\mathbb{R}$ can be decomposed as $F=F_d+F_c=F_d+F_s+F_{ac}$ where F_d is the discrete part of F, F_c is the continuous part of F, and where F_s and F_{ac} are the singular and the absolutely continuous parts of F_c.) It seems that the long-standing lack of such a unified version has much to do with the history of the development of probability theory within mathematics.

I will also discuss some related inequalities and their application to the theory of isoperimetric constants and optimal Poincaré inequalities.

This talk is based on joint work with C. A. J. Klaassen.