Itô-Nisio's theorem revisited and related topics

Jan Rosiński
University of Tennessee

High Dimensional Probability IX
Będlewo, Paris, Warsaw, Knoxville, June 15-19, 2020

This talk is based on a joint work with Andreas Basse-O'Connor and Jørgen Hoffmann-Jørgensen.

Outline:

1. Introduction
2. A non-separable extension of the Itô-Nisio theorem
3. Wiener classes
4. Path variation of Lévy processes

1. Introduction

The Itô-Nisio theorem describes a situation where the pointwise a.s. convergence of processes implies the uniform convergence.

Theorem (Itô-Nisio)

Let $\{S_n(t)\}_{t\in T}$ be partial sums of independent symmetric stochastic processes with continuous paths over a compact metrizable set T. Suppose there exists a process $\{S(t)\}_{t\in T}$ with continuous paths such that

$$\lim_{n\to\infty} S_n(t) = S(t) \quad a.s. \quad \textit{for each } t \in T.$$

Then

$$\lim_{n\to\infty} \|S_n - S\|_\infty = 0 \quad a.s.$$

This can be viewed as a stochastic version of Dini's Theorem.

Since $C(T)$ is an universal space for all separable Banach spaces we have

Corollary ((∗))

Let X_n be independent symmetric r.v.'s in a separable Banach space F and $S_n = \sum_{i=1}^n X_i$. Suppose there exists a r.v. S in F such that

$$\xi(S_n) \to \xi(S) \quad a.s. \quad \text{for each } \xi \in F^*.$$

Then $\|S_n - S\|_F \to 0$ a.s.

(∗) Itô, K. and M. Nisio. On the convergence of sums of independent Banach space valued random variables. Osaka J. Math., 5:35-48, 1968.

Theorem (The martingale case (**))

Let $\{M_n\}$ *be an* L^1*-bounded martingale in a Banach space* F*. Suppose there exist a strongly measurable r.v.* M *in* F *and a total subspace* $H \subset F^*$ *such that*

$$\xi(M_n) \to \xi(M) \quad a.s. \quad \text{for each } \xi \in H.$$

Then $\|M_n - M\|_F \to 0$ *a.s.*

Note: The strong measurability of M implies that M and M_n a.s. belong to a separable subspace of F.

(**) Davis, W. J., N. Ghoussoub, W. B. Johnson, S. Kwapień, and B. Maurey. Weak convergence of vector valued martingales. In Probability in Banach spaces VI, 41-50, Springer 1990.

Many processes live in non-separable Banach spaces and it is desirable to consider their approximation and convergence in the corresponding strong norms. In such setting, an extension of the Itô-Nisio theorem has its place.

In particular, series expansions of Gaussian processes have independent symmetric increments, while series expansions of Lévy processes, and more generally, of infinitely divisible random fields, have conditionally independent increments. This makes such extensions of the Itô-Nisio theorem directly applicable.

The non-separable case: $(D[0,1], \|\cdot\|_\infty)$.

Theorem (Basse-O'Connor, JR)

Let $X_n(t)$ *be independent symmetric càdlàg processes on* $[0,1]$ *and* $S_n(t) = \sum_{i=1}^n X_i(t)$. *Suppose there is a càdlàg process* $S(t)$ *and a dense subset* T *of* $[0,1]$ *with* $1 \in T$ *such that*

$$S_n(t) \to S(t) \quad a.s. \quad \text{for each } t \in T.$$

Then

$$\lim_{n\to\infty} \|S_n(t) - S(t)\|_\infty = 0 \quad a.s.$$

Basse-O'Connor, A. and Rosiński, J. On the uniform convergence of random series in Skorohod space and representations of cadlag infinitely divisible processes, *Ann. Probab.* 41, no. 6, 4317-4341, 2013.

Other known cases:

Remarks (∗)

(a) Itô-Nisio's theorem holds in $BV_1[0,1]$, the space of càdlàg functions of bounded p-variation;

(b) Itô-Nisio's theorem fails in the following spaces

1. ℓ^∞;
2. $BV_p[0,1]$ for all $p > 1$;
3. Hölder space $C^{0,\alpha}[0,1]$, for all $\alpha \in (0,1]$.

(∗) See Remark 2.4 in Basse-O'Connor and JR (2013),
Jain, N. C. and Monrad, D. Gaussian quasimartingales. Z. Wahrsch. Verw. Gebiete 59 139-159, 1982,

Jain, N. C. and Monrad, D. Gaussian measures in Bp. Ann. Probab. 11 46-57, 1983.

One has several examples but no theory.

2. An extension of the Itô-Nisio theorem

Known issues with non-separable Banach spaces in probability:

1. Too many Borel sets.
   - Basic stochastic processes may be not measurable.
     Take $(D[0,1], \|\cdot\|_\infty)$, for example. Empirical distribution functions and Poisson processes on $[0,1]$ belong to $D[0,1]$ but are not Borel-measurable.

2. Too many functionals.
   - Functionals of stochastic processes may not be measurable.
     Take $\ell^\infty(\mathbb{N})$. Let ν be a bounded finitely additive measure on $(\mathbb{N}, 2^{\mathbb{N}})$ taking only values 0 or 1 with $\nu(\mathbb{N}) = 1$.
     Then $\nu \in ba$, the dual of $\ell^\infty(\mathbb{N})$. Consider a Rademacher sequence $\boldsymbol{\epsilon} = (\epsilon_n : n \in \mathbb{N}) \in \ell^\infty(\mathbb{N})$. By a result of Sierpiński

$$\omega \mapsto \nu(\boldsymbol{\epsilon}(\omega)) = \int_{\mathbb{N}} \epsilon_n(\omega)\, \nu(dn) \in \{-1, 1\}$$

     is not measurable.

Definition

A Banach space $(F, \|\cdot\|)$ is said to be of *countable type* if there exists countable set Ξ of functionals in the unit ball of the dual F^* such that

$$\|f\| = \sup\{\xi(f) : \xi \in \Xi\}, \quad f \in F.$$

Ξ is called a *norming set* for F.

Under the topology of pointwise convergence on Ξ, F becomes (usually not complete) separable metric space, with a metric

$$d(f,g) = \sum_{i=1}^{\infty} 2^{-i}(|\xi_i(f-g)| \wedge 1) \quad f, g \in F, \quad \{\xi_i\} = \Xi.$$

Such F will be denoted by (F, Ξ), in a contrast with $(F, \|\cdot\|)$.

Examples of non-separable Banach spaces of countable type include:

- $D([0,1]^d)$ under supremum norm,
- space $C^{0,\alpha}[0,1]$ of α-Hölder continuous functions,
- space $BV_p([0,1])$ of càdlàg functions with bounded p-variation,
- space $\ell^\infty(\mathbb{N})$ of bounded sequences,
- space $\ell^1(U)$ of discrete signed measures on a Borel space U under the total variation norm,
- $F = E^*$, where E is a separable Banach space,
- $\mathcal{M}(S,V)$, the space of V-valued measures on $(S,\mathcal{A})$, where V is a Banach space of countable type and $\mathcal{A}$ is an infinite countably generated σ-algebra on S.

The Borel σ-algebra of (F, Ξ) is $\mathcal{B}^{\Xi}(F) = \sigma(\xi : \xi \in \Xi)$.

Given Ξ, by a r.v. in F we mean a $\mathcal{B}^{\Xi}(F)$-measurable map from Ω into F.

The distribution of X is a probability measure $\mathbb{P} \circ X^{-1}$ on (F, Ξ).

We will always assume that that (F, Ξ) is universally measurable (*u.m.*), so that every probability measure on (F, Ξ) is tight. $(*)$ This assumption holds in all our examples.

Notice that our spaces are slightly more general than representable Banach spaces of G. Godefroy and M. Talagrand $(**)$, which are of countable type and analytic.

$(*)$ Richard M. Dudley, R.M. *Real Analysis and Probability*, Cambridge University Press, 2004.

$(**)$ Godefroy, G. and Talagrand, M. Espaces de Banach représentables, *Israel J. Math.* 41 (1982), 321-330.

Rademacher measures:

We will say that a random element X in F is a *Rademacher series* if there exist $(f_j) \subset F$ and i.i.d. r.v.'s ϵ_j with $\mathbb{P}(\epsilon_j = \pm 1) = 1/2$ such that

$$\xi(X) = \sum_{j=1}^{\infty} \epsilon_j \xi(f_j) \quad a.s., \quad \text{for every } \xi \in \Xi.$$

The distribution $\mu = \mathbb{P} \circ X^{-1}$ of X will be called a *Rademacher measure* on (F, Ξ).

Condition (R): For every sequence $(\xi_i) \subset \Xi$ and a Rademacher measure μ on (F, Ξ) there exists a subsequence (ξ_{i_k}) such that $\lim_{k\to\infty} \xi_{i_k}(f)$ exists for μ-almost all $f \in F$.

Theorem (Basse-O'Connor, Hoffmann-Jørgensen, JR)

Let $(F, \|\cdot\|)$ be a Banach space of countable type with a norming set Ξ. Suppose that (F, Ξ) is u.m. and condition (R) holds.

Let X_n be independent r.v.'s in F and $S_n = \sum_{i=1}^{n} X_i$. Suppose there exists a r.v. S in F such that

$$\xi(S_n) \to \xi(S) \quad \text{a.s. for each } \xi \in \Xi \tag{1}$$

and for every $\delta > 0$

$$\limsup_{n>m\to\infty} \sup_{\xi\in\Xi} \mathbb{P}\Big(\xi(S_n - S_m) > \delta\Big) < 1\,. \tag{2}$$

Then

$$\|S_n - S\| \to 0 \quad a.s. \text{ as } n \to \infty. \tag{3}$$

Condition (2) trivially holds when X_i are symmetric. It also holds when $\{\xi(S) : \xi \in \Xi\}$ is uniformly integrable and $\mathbb{E}\xi(X_n) = 0$ for each $\xi \in \Xi$ and $n \in \mathbb{N}$.

Remarks

(i) *Obviously,* (R) *holds when* Ξ *is sequentially weak** *pre-compact. That is, if every sequence* $(\xi_i) \subset \Xi$ *contains a subsequence* (ξ_{i_k}) *such that* $\lim_{k\to\infty} \xi_{i_k}(f)$ *exists* $\forall\ f \in F$. *Every separable Banach space has this property.*

(ii) (R) *holds when* F *does not contain an isomorphic copy of* c_0.

(iii) (R) *is also necessary for* (1) $\wedge$ (2) $\Rightarrow$ (3) *to hold in general.*

Example

- Consider $(D([0,1]^d, \|\cdot\|_\infty)$. This is non-separable Banach space. Take $T_0 \subset [0,1]$, a countable dense set with $1 \in T_0$, and define

$$\Xi = \{\delta_{\mathbf{t}} : \mathbf{t} = (t_1, \ldots, t_d) \in T_0^d\}.$$

Then Ξ is norming. Every sequence $(\mathbf{t}_i) \subset T_0^d$ contains a subsequence $(\mathbf{t}_{i_k})$ which is monotone on each coordinate, so that for every $f \in D([0,1]^d)$

$$\lim_{k\to\infty} \delta_{\mathbf{t}_{i_k}}(f) = \lim_{k\to\infty} f(\mathbf{t}_{i_k}) \quad \text{exists.}$$

Thus (R) holds. Hence, Itô-Nisio theorem holds for random fields with paths in $D([0,1]^d)$ under the supremum norm.

Examples

- $(BV_1[0,1], \|\cdot\|_{BV_1})$, space of càdlàg functions having finite variation is non-separable but it does not contain an isomorphic copy of c_0. Thus (R) holds, as does the Itô-Nisio theorem.
- Suppose $F = E^*$, where E is a separable Banach space. If F does not contain an isomorphic copy of c_0, then the Itô-Nisio theorem holds. If it does contain c_0, then it contains ℓ^∞. This implies that Itô-Nisio theorem fails with respect to some norming sequence in F^*, which may be different from the original one in E.

$\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ a continuous nondecreasing such that $\phi(0) = 0$ and $\phi(u) > 0$ for $u > 0$. Sometimes we impose more conditions, such as ϕ is convex or $\phi(2u) \leq C\phi(u)\ \forall u \geq 0$ (the Δ_2-condition).

Π denotes the set of all finite partitions π of $[0, 1]$, $\pi : 0 = t_0 < t_1 < \cdots < t_n = 1$. The norm of π is given by

$$|\pi| = \max_{1 \leq i \leq n} |t_i - t_{i-1}|.$$

The ϕ-variation of $f : [0, 1] \to \mathbb{R}$ on $[0, 1]$ is defined as

$$V_\phi(f) = \sup_{\pi \in \Pi} \sum_{i=1}^{n} \phi(|f(t_i) - f(t_{i-1})|).$$

$$BV_\phi := \{f \in D[0,1] : V_\phi(f/k) < \infty \text{ for some } k > 0\}.$$

When ϕ is convex, BV_ϕ is a non-separable Banach space with the norm

$$\|f\|_{BV_\phi} = |f(0)| + \inf\{c > 0 : V_\phi(f/c) \le 1\}.$$

The space BV_p of bounded p-variation corresponds to $\phi(u) = u^p$, $1 \le p < \infty$, in which case

$$\|f\|_{BV_p} = |f(0)| + \sup_{\pi \in \Pi} \left(\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p \right)^{1/p}.$$

Analogously, V_p and BV_p correspond $\phi(u) = u^p$.

Definition

Assume that ϕ is convex. The closure of the space of càdlàg step functions in the $\|\cdot\|_{BV_\phi}$ norm is called the Wiener class and denoted by BV_ϕ^0. That is,

$$BV_\phi^0 = \overline{\{\text{càdlàg step functions on } [0,1]\}}_{BV_\phi}.$$

BV_ϕ^0 is a non-separable closed subspace of BV_ϕ .

Proposition

$f \in BV_\phi^0$ if and only if $f \in D[0,1]$ and for every $k > 0$

$$\lim_{\pi\in\Pi,\, |\pi|\to 0} \phi(k|f(t_i) - f(t_{i-1})|) = \sum_{s\in(0,1]} \phi(|k\Delta f(s)|) < \infty.$$

If ϕ also satisfies (Δ_2), then it is enough to take $k = 1$.

For every $p > 1$,

$$\bigcup_{q\colon q<p} BV_q \subseteq BV_p^0 \subseteq BV_p.$$

Let ϕ be convex, $\phi(u)/u \to 0$ as $u \to 0$ and $\phi(u)/u \to \infty$ as $u \to \infty$. The norming set Ξ for BV_ϕ the set of functionals of norm at most 1 of the form

$$\xi(f) = \beta_0 f(0) + \sum_{i=1}^{n} \beta_i (f(t_i) - f(t_{i-1})), \quad f \in BV_\phi,$$

where $\beta_j \in \mathbb{Q}$ and $t_i \in T$, where T is countable dense in $[0, 1]$ with $1 \in T$.

Proposition

For every sequence $\{\xi_i\} \subset \Xi$ *there is a subsequence* $\{\xi_{i_k}\}$ *such that for every* $f \in BV_\phi^0$

$$\lim_{k\to\infty} \xi_{i_k}(f) \quad \textit{exists.}$$

Thus (BV_ϕ^0, Ξ) *satisfies (R).*

Corollary

The Itô-Niso theorem holds in BV_ϕ^0*. That is, if* $X_n \in BV_\phi^0$ *are independent symmetric and* $S_n(t) = \sum_{i=1}^n X_i(t) \to S(t)$ *a.s.* $\forall t \in T$*, where* $S \in BV_\phi^0$*, then*

$$\lim_{n\to\infty} \|S_n - S\|_{BV_\phi} = 0 \quad a.s.$$

Brownian Motion.

Let $\{B_t : t \in [0,1]\}$ be a standard Brownian motion.

Let

$$\phi_2(u) = u^2/\log(1+\log(1+u^{-1})) \qquad u > 0,$$

$\phi_2(0) = 0$. ϕ_2 is convex and $\phi_2(u) \sim u^2/\log(\log u^{-1})$, $u \to 0$.

Theorem (S.J. Taylor[‡])

$$V_{\phi_2}(B) = \sup_{\pi\in\Pi} \sum_{i=1}^{n} \phi_2(|B_{t_i} - B_{t_{i-1}}|) < \infty \quad a.s.$$

Moreover, if $\phi(u)$ *satisfies* $\phi(u)/\phi_2(u) \to \infty$ *as* $u \to 0$, *then* $V_\phi(B) = \infty$ *a.s.*

Theorem (S.J. Taylor[‡])

Let $\{B_t : t \in [0,1]\}$ *be a standard Brownian motion. Then*

$$\lim_{\delta\to 0} \sup_{|\pi|<\delta} \sum_{i=1}^{n} \phi_2(|B_{t_i} - B_{t_{i-1}}|) = 1 \quad a.s.$$

NOTE:

$$\lim_{\delta\to 0} \inf_{|\pi|<\delta} \sum_{i=1}^{n} \phi_2(|B_{t_i} - B_{t_{i-1}}|) = 0 \quad a.s.$$

Corollary

Paths of a Brownian motion lie in $BV_{\phi_2} \setminus BV_{\phi_2}^0$ *a.s.*

[‡] S.J. Taylor (1972). Exact asymptotic estimates of Brownian path variation. *Duke Math. J. 39*, 219–241.

Lemma

Let $(h_k)_{k\geq 0}$ *be the Haar basis in* $L^2[0,1]$ *and* $f_k(t) = \int_0^t h_k(s)\,ds$*.* *Let* $(Z_k)_{k\geq 0}$ *be i.i.d.* $N(0,1)$*. Then the series*

$$B_t = \sum_{k=0}^{\infty} Z_k f_k(t) \quad t \in [0,1]$$

converges a.s. uniformly to a standard Brownian motion but with probability 1 fails to converge in the ϕ_2*-variation norm.*

Recall $h_0 = \mathbf{1}_{[0,1]}$, $h_{2^n+i} = 2^{n/2}\big(\mathbf{1}_{[\frac{i}{2^n},\frac{2i+1}{2^{n+1}}]} - \mathbf{1}_{(\frac{2i+1}{2^{n+1}},\frac{i+1}{2^n}]}\big)$, $0 \leq i < 2^n$, $n \geq 0$.

Lévy process with no Gaussian part.

Let $\{X_t : t \in [0,1]\}$ be such Lévy process.

Using the embedding of martingales into Brownian motion and Taylor's result, we obtain

Theorem (Basse-O'Connor, Hoffmann-Jørgensen, JR)

If X does not have a Gaussian part then $X \in BV^0_{\phi_2}$ a.s.
Hence,

$$\lim_{\delta\to 0} \sup_{|\pi|<\delta} \sum_{i=1}^{n} \phi_2(|X_{t_i} - X_{t_{i-1}}|) = \sum_{0\le t\le 1} \phi_2(|\Delta X_t|) \quad a.s.$$

General Lévy process.

Let $Y_t = \sigma B_t + X_t$ be such Lévy process, where B_t is a standard Brownian motion independent of a Lévy process X_t with no Gaussian part. Recall that $B \in BV_{\phi_2} \setminus BV^0_{\phi_2}$ and $X \in BV^0_{\phi_2}$. Then

Theorem (Basse-O'Connor, Hoffmann-Jørgensen, JR)

Let $Y = \{Y_t : t \in [0,1]\}$ be a Lévy process was above. Then

$$\lim_{\delta \to 0} \sup_{|\pi| < \delta} \sum_{i=1}^{n} \phi_2(|X_{t_i} - X_{t_{i-1}}|) = \sigma^2 + \sum_{0 \le t \le 1} \phi_2(|\Delta X_t|) \quad a.s.$$

Recall that $\phi_2(u) \sim u^2 / \log(\log u^{-1})$, $u \to 0$ is neither additive nor quadratic.

Theorem (Basse-O'Connor, Hoffmann-Jørgensen, JR)

Let X be a Lévy process without Gaussian part and with Lévy measure ν satisfying $\int_{\{\|x\|\le 1\}} \|x\|^p \, \nu(dx) < \infty$ for some $p \le 2$. Then

$$X \in \begin{cases} BV_p^0 & \text{if } p < 2 \\ BV_{\phi_2}^0 & \text{if } p = 2. \end{cases}$$

Thus conditionally independent series representations of X converge a.s. in the corresponding Wiener classes.

Thank you!