Tangent martingales in Banach spaces

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Lévy-Khinchin formula

Let $L : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ Lévy martingale.

Theorem (Lévy-Khinchin formula)

There exist unique $\sigma \geq 0$ and measure ν on $\mathbb{R}$ s.t.

$$\mathbb{E}e^{i\theta L_t} = \exp\left\{ t\Big(-\frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta x \,\mathrm{d}\nu(x)\Big)\right\}, \quad t \geq 0, \quad \theta \in \mathbb{R}.$$

Features of σ and ν

- they are unique and deterministic
- they determine distribution of L

Is there analogue for general martingales?

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Is there analogue for general martingales?

Let $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ martingale. Then

$$[M]_t := \mathbb{P} - \lim_{\text{mesh}\to 0} \sum_{i=1}^{N} |M_{t_i} - M_{t_{i-1}}|^2,$$

$0 = t_0 < \ldots < t_N = t$, called quadratic variation of M

Example

- Let W standard Brownian motion. Then $[W]_t = t$ a.s.
- Let N standard Poisson process, $\widetilde{N}_t := N_t - t$ compensated Poisson process. Then $\widetilde{N}$ martingale and $[\widetilde{N}]_t = N_t$

Remark

Both M and $[M]$ have càdlàg versions (i.e. right-continuous with left limits)

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Meyer-Yoeurp decomposition

Let $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ martingale. Then

- M called continuous if $[M]$ continuous ($\Leftrightarrow$ path of M continuous)
- M called purely discontinuous if $[M]$ pure jump, i.e. $[M]_t = \sum_{0 \le s \le t} \Delta[M]_s$

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Theorem (Meyer 1976, Yoeurp 1976)

Let $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ martingale. Then $\exists!$ local martingales M^c and M^d s.t. M^c continuous, M^d p.d., $M^c_0 = 0$, and $M = M^c + M^d$

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Local characteristics

Let $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ martingale. Define μ^M on $\mathbb{R}_+ \times \mathbb{R}$

$$\mu^M([0,t] \times B) = \sum_{0 \le s \le t} \mathbf{1}_{B \setminus \{0\}}(\Delta M_t), \quad t \ge 0, \quad B \subset \mathbb{R} \text{ Borel}$$

Let ν^M the compensator of μ^M, i.e. unique predictable r.m. such that for any predictable $F : \mathbb{R}_+ \times \mathbb{R} \times \Omega \to \mathbb{R}_+$ with $\mathbb{E} \int_{\mathbb{R}_+ \times \mathbb{R}} F \, \mathrm{d}\mu^M < \infty$

$$t \mapsto \int_{[0,t] \times \mathbb{R}} F \, \mathrm{d}(\mu^M - \nu^M) \text{ local martingale}$$

Definition

Let $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ martingale, $M = M^c + M^d$ Meyer-Yoeurp decomposition. Then $([M^c], \nu^M)$ called local characteristics of M

$[M^c]$ cares about M^c
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$([L^c], \nu^L) = (\sigma^2 t, \lambda_{\mathbb{R}_+} \otimes \nu)$ local characteristics of L

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Properties of local characteristics

- they are unique and predictable, but not deterministic
- they do not determine distribution of martingale

Definition

Let $M, N : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ martingales. Then M and N tangent if $([M^c], \nu^M) = ([N^c], \nu^N)$

Example

Let L^1, L^2 equidistributed Lévy martingales, $\Phi : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ elementary predictable. Then

$$t \mapsto \int_0^t \Phi(s)\, \mathrm{d}L_s^1 \quad \text{and} \quad t \mapsto \int_0^t \Phi(s)\, \mathrm{d}L_s^2$$

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Let

- $(\mathcal{F}_n)_{n=1}^{\infty}$ filtration,
- $d = (d_n)_{n\geq 1}$ and $e = (e_n)_{n\geq 1}$ $\mathbb{R}$-valued martingale difference sequences

Definition

d and e tangent iff

$$\mathbb{P}(d_n|\mathcal{F}_{n-1}) = \mathbb{P}(e_n|\mathcal{F}_{n-1}), \quad n \geq 1$$

Recall that

$$\mathbb{P}(\xi|\mathcal{F})(A) := \mathbb{E}(\mathbf{1}_A(\xi)|\mathcal{F}), \quad A \subset \mathbb{R} \text{ Borel}$$

Example

Let $(\xi_n)_{n\geq 1}$ independent mean-zero, $(v_n)_{n\geq 1}$ predictable sequence, and let $(\xi'_n)_{n\geq 1}$ independent copy of $(\xi_n)_{n\geq 1}$. Then $(v_n\xi_n)$ and $(v_n\xi'_n)$ tangent

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Theorem (Zinn 1985, Hitczenko 1988)

Let d and e tangent. Then

$$\mathbb{E} \sup_{1\leq n\leq N} \Big|\sum_{n=1}^{N} d_n\Big|^p \eqsim_p \mathbb{E} \sup_{1\leq n\leq N} \Big|\sum_{n=1}^{N} e_n\Big|^p, \quad 1 \leq p < \infty$$

Theorem (Kallenberg 2017, Kwapień-Woyczyński 1991)

Let $M, N : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be tangent martingales. Then

$$\mathbb{E} \sup_{0\leq t\leq\infty} |M_t|^p \eqsim_p \mathbb{E} \sup_{0\leq t\leq\infty} |N_t|^p, \quad 1 \leq p < \infty$$

What happens in Banach spaces?

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UMD Banach space

Let X Banach space. X called UMD if for any $p \in [1, \infty)$, X-valued m.d.s. $(d_n)_{n\geq 1}$, and for any $(\varepsilon_n)_{n\geq 1} \in \{-1, 1\}$

$$\mathbb{E} \sup_{N\geq 1} \Big\| \sum_{n=1}^{N} \varepsilon_n d_n \Big\|^p \eqsim_{p,X} \mathbb{E} \sup_{N\geq 1} \Big\| \sum_{n=1}^{N} d_n \Big\|^p, \quad 1 \leq p < \infty$$

Example

- Hilbert spaces,
- L^q, $q \in (1, \infty)$,
- reflexive Sobolev, Besov, Schatten class, Orlicz, variable $L^{p(\cdot)}$ spaces

Non reflexive are not UMD: L^1, L^∞, $C(K)$

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$$\mathbb{E} \sup_{N\geq 1} \Big\| \sum_{n=1}^{N} \varepsilon_n d_n \Big\|^p \eqsim_{p,X} \mathbb{E} \sup_{N\geq 1} \Big\| \sum_{n=1}^{N} d_n \Big\|^p, \quad 1 \leq p < \infty$$

Example

- Hilbert spaces,
- L^q, $q \in (1, \infty)$,
- reflexive Sobolev, Besov, Schatten class, Orlicz, variable $L^{p(\cdot)}$ spaces

Non reflexive are not UMD: L^1, L^∞, $C(K)$

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Martingales in Banach spaces are well-studied, see e.g. *Analysis in Banach spaces* by Hytönen, van Neerven, Veraar, Weis

Let $M : \mathbb{R}_+ \times \Omega \to X$ martingale

- $\langle M, x^* \rangle$ martingale for any $x^* \in X^*$
- M continuous if $\langle M, x^* \rangle$ continuous for any $x^* \in X^*$
- M purely discontinuous if $\langle M, x^* \rangle$ p.d. for any $x^* \in X^*$

Proposition (Y. 2017)

Let X Banach space. Then X UMD iff for any martingale $M : \mathbb{R}_+ \times \Omega \to X$ there $\exists!$ local martingales $M^c, M^d : \mathbb{R}_+ \times \Omega \to X$ such that M^c continuous, M^d p.d., $M^c_0 = 0$, and $M = M^c + M^d$.

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$$[\![M]\!](x^*) \quad \text{and} \quad [\langle M, x^* \rangle] \ \text{indistinguishable} \quad \forall x^* \in X^*$$

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Let $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ martingale. Define μ^M on $\mathbb{R}_+ \times \mathbb{R}$

$$\mu^M([0,t] \times B) = \sum_{0 \le s \le t} \mathbf{1}_{B \setminus \{0\}}(\Delta M_t), \quad t \ge 0, \ \ B \subset \mathbb{R} \ \text{ Borel}$$

Let ν^M *the compensator* of μ^M, i.e. unique predictable r.m. such that for any predictable $F : \mathbb{R}_+ \times \mathbb{R} \times \Omega \to \mathbb{R}_+$ with $\mathbb{E} \int_{\mathbb{R}_+ \times \mathbb{R}} F \, \mathrm{d}\mu^M < \infty$

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Definition

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Let X be a UMD Banach space, $M, N : \mathbb{R}_+ \times \Omega \to X$ martingales. Then M and N are tangent if they have the same local characteristics, i.e. if $([\![M^c]\!], \nu^M) = ([\![N^c]\!], \nu^N)$

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Theorem (McConnell 1989, Hitczenko unpubl)

Let X Banach space, $1 \leq p < \infty$. Then X UMD iff for any X-valued tangent m.d.s. $(d_n)_{n\geq 1}$ and $(e_n)_{n\geq 1}$

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Proof

- the canonical decomposition (Y. 2017)
- time-change
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Applications

- ϕ-inequalities
- decoupled tangent martingales, Jacod-Kwapień-Woyczyński approach
- stochastic integral inequalities
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Many thanks!