Sampling discretization of L^p norms in finite dimensional subspaces

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The sampling discretization problem

L — N-dimensional subspace of $L^p = L^p(\mu)$
Find the smallest possible m: $\exists\ x_1, \dots, x_m$ and numbers $c, C > 0$ such that

$$c\|f\|_p^p \leq \frac{1}{m}\sum_{j=1}^{m}|f(x_j)|^p \leq C\|f\|_p^p$$

for every $f \in L$, where

$$\|f\|_p := \left(\int |f|^p\, d\mu\right)^{1/p} = (\mathbb{E}|f(X)|^p)^{1/p}, \ X \sim \mu.$$

Clarly $m \geq N \Rightarrow$ we are interested in the assumptions on L such that m is close to N.

Some known results: $p = 2$

M. Rudelson (1999) $\Rightarrow$

Theorem. Assume that $L \subset L^2$ is such that $\exists M > 0$: $\|f\|_\infty \leq M N^{1/2} \|f\|_2 \; \forall f \in L$.

Then one can take $m = C(M)\, N \log N$.

Theorem (V.N. Temlyakov, 2017). Let $p = 2$ and let L be a subspace of trigonometric polynomials with frequencies from the set $Q \subset \mathbb{Z}^d$.

Then one can take $m = C\,|Q|$ (here $N = |Q|$).

Some known results: $p < 2$

Theorem (F. Dai, A. Prymak, A. Shadrin, V. Temlyakov, S. Tikhonov, 2020).
Let $p \in [1, 2)$,
L — N-dimensional subspace of L^p.

Assume that $\forall f \in L$:

$$\|f\|_\infty \leq MN^{1/2}\|f\|_2.$$

Then one can take

$m = C(M)\,N[\log N]^3.$

The main result

Theorem 1. $p \in (1, \infty)$,
L — N-dimensional subspace of L^p.

Assume that $\forall f \in L$:

$$\|f\|_\infty \leq MN^{1/\max\{p,2\}} \|f\|_{\max\{p,2\}}.$$

Then there are

$$m = \begin{cases} C(M)\, N[\log N]^p, \; p > 2 \\ C(M)\, N[\log N]^2, \; p \in (1,2) \end{cases}$$

points $x_1, \ldots, x_m$ such that

$$\frac{1}{2}\|f\|_p^p \leq \frac{1}{m}\sum_{j=1}^{m} |f(x_j)|^p \leq \frac{3}{2}\|f\|_p^p \quad \forall\, f \in L.$$

The symmetrization argument

B — set of functions. Let $R_p(f) = \sum_{j=1}^{m} |f(X_j)|^p$ and

$$V_p(B) := \sup_{f\in B}\left|\frac{1}{m}\sum_{j=1}^{m}|f(X_j)|^p - \|f\|_p^p\right|$$

Lemma. Assume $\forall\ X := (X_1, \ldots, X_m)$ — fixed:

$$\mathbb{E}_\varepsilon \sup_{f\in B}\left|\sum_{j=1}^{m}\varepsilon_j|f(X_j)|^p\right| \le \Theta \sup_{f\in B}\big(R_p(f)\big)^{1-r},$$

$r \in (0,1)$,

$\varepsilon_1, \ldots, \varepsilon_m$ — i.i.d ± 1 symmetric Bernoulli r. v.

Then

$$\mathbb{E}\, V_p(B) \le C(r)\left[\frac{\Theta^{1/r}}{m} + \left(\frac{\Theta^{1/r}}{m}\right)^r \left(\sup_{f\in B}\mathbb{E}|f(X_1)|^p\right)^{1-r}\right].$$

Let $B = B_p = \{f \in L \colon \|f\|_p \leq 1\}$.

Assume that for some m one has $\mathbb{E} V_p(B) < \varepsilon$.

Then $P\Big(\sup_{f \in B_p} \Big| \frac{1}{m} \sum_{j=1}^{m} |f(X_j)|^p - \|f\|_p^p \Big| \geq 2\varepsilon \Big) \leq 1/2$
and there are points $x_1, \dots x_m$:

$$\Big| \frac{1}{m} \sum_{j=1}^{m} |f(x_j)|^p - \|f\|_p^p \Big| \leq 2\varepsilon \|f\|_p^p \quad \forall f \in L.$$

Thus, we want to estimate the expextation

$$\mathbb{E}_\varepsilon \sup_{f \in B} \Big| \sum_{j=1}^{m} \varepsilon_j |f(X_j)|^p \Big|.$$

Conditional theorem

Theorem 2. Let $X = \{X_1, \dots, X_m\}$ be fixed and let $\|f\|_{\infty,X} := \max_{1 \le j \le m} |f(X_j)|$. Assume, that

$$e_k(B_p, \|\cdot\|_{\infty,X}) \le A \begin{cases} 2^{-k/p}, & k \le \log N, \\ N^{-1/p} 2^{-2^k/N}, & k \ge \log N. \end{cases}$$

Then $\mathbb{E}_\varepsilon \sup_{f \in B_p} \left| \sum_{j=1}^m \varepsilon_j |f(X_j)|^p \right| \le$

$$C(p) A [\log N]^{1-1/\max\{p,2\}} \sup_{f \in B_p} \left(\sum_{j=1}^m |f(X_j)|^p \right)^{1-1/\max\{p,2\}}.$$

Here

$$e_k(F, \|\cdot\|) := \inf\left\{\varepsilon : \exists f_1, \dots, f_{2^{2^k}} \in F : F \subset \bigcup_j B_\varepsilon(f_j)\right\},$$

where $B_\varepsilon(f) := \{g : \|f - g\| < \varepsilon\}$.

Generic Chaining

The proof of the above result is based on the generic chaining and its recent development due to R. van Handel (2018).

Let X_f be a random process, $f \in (F, \varrho)$.
Definition. An admissible sequence of F is an increasing sequence $(\mathcal{F}_k)$ of partitions of F:
$|\mathcal{F}_k| \leq 2^{2^k}$ $\forall k \geq 1$ and $|\mathcal{F}_0| = 1$.
$F_k(f)$ — unique element of $\mathcal{F}_k$ that contains f.
Definition. Let $\alpha > 0$. Let

$$\gamma_{\alpha,1}(F, \varrho) := \inf \sup_{f \in F} \sum_{k=0}^{\infty} 2^{k/\alpha} \mathrm{diam}\big(F_k(f)\big),$$

where the infimum is taken over all admissible sequences of F.

Talagrand's theorem

Assume that there are numbers $K > 0$ and $\alpha > 0$:

$$P(|X_f - X_g| \geq K t^{1/\alpha} \varrho(f,g)) \leq 2e^{-t} \quad \forall t > 0.$$

Then $\forall\ f_0 \in F$ one has

$$\mathbb{E} \sup_{f \in F} |X_f - X_{f_0}| \leq C(\alpha) R K \gamma_{\alpha,1}(F, \varrho).$$

In our case $X_f = \sum_{j=1}^{m} \varepsilon_j |f(X_j)|^p,\ f \in B_p$.

The estimation of the γ-functional exploits recent results of R. van Handel (2018).

Estimates for the entropy numbers

Theorem 3. Let $p > 2$ and assume that

$$\|f\|_\infty \le M\|f\|_p \; \forall f \in L.$$

Then $A \le C(p)M[\log m]^{1/p}$.

Theorem 4. Let $p \in (1,2)$, and assume that there is $M \ge 2$ such that

$$\|f\|_\infty \le M\|f\|_2 \; \forall f \in L.$$

Then $A \le C(p)M^{2/p}[\log M]^{1/p-1/2}[\log m]^{1/2}$.

References

1) F. Dai, A. Prymak, A. Shadrin, V. Temlyakov, S. Tikhonov, Entropy numbers and Marcinkiewicz-type discretization theorem, arXiv:2001.10636.

2) R. van Handel, Chaining, interpolation and convexity II: The contraction principle, Ann. of Probab. 46(3) (2018) 1764–1805.

3) M. Rudelson, Random vectors in the isotropic position, J. Funct. Anal. 164(1) (1999) 60–72.

4) V.N. Temlyakov, The Marcinkiewicz-type discretization theorems for the hyperbolic cross polynomials, Jaen J. Approx. 9(1) (2017) 37–63.

Thank You!