

Aspects of Coulomb gases

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Outline

Electrostatics

Gases

Dynamics for planar case

Conditioning

Jellium

Coulomb kernel in mathematical physics

- Coulomb kernel in \mathbb{R}^d , $d \geq 1$,

$$x \in \mathbb{R}^d \mapsto g(x) = \begin{cases} \log \frac{1}{|x|} & \text{if } d = 2 \\ \frac{1}{(d-2)|x|^{d-2}} & \text{if not} \end{cases} .$$

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- Fundamental solution of Poisson's equation

$$\Delta g = -c_d \delta_0 \quad \text{where} \quad c_d = |\mathbb{S}^{d-1}| = \frac{2\pi^{d/2}}{\Gamma(d/2)} .$$

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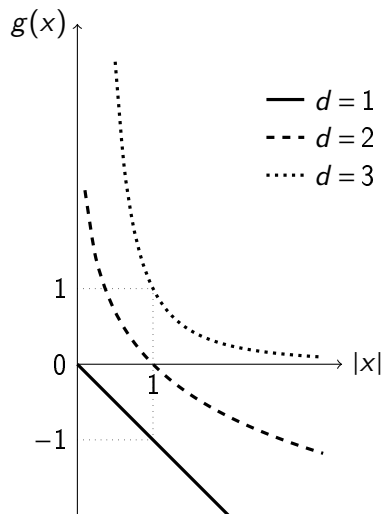
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- Riesz kernel $|x|^{-s}$, if $s = d - \alpha$ then fractional Laplacian Δ_α

From now on $d \geq 2$



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- Integration by parts and “carré du champ”, $\eta = \mu - \nu$,

$$\mathcal{E}(\eta) = \frac{1}{2} \int U_\eta d\eta = -\frac{1}{2c_d} \int U_\eta \Delta U_\eta dx = \frac{1}{2c_d} \int |\nabla U_\eta|^2 dx.$$

Confinement and equilibrium measure

- External confining potential $V : \mathbb{R}^d \rightarrow (-\infty, +\infty]$

$$\lim_{|x| \rightarrow \infty} (V(x) - \log|x| \mathbf{1}_{d=2}) > -\infty.$$

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- $\text{supp}(\mu_V)$ is compact if $\lim_{|x| \rightarrow \infty} (V(x) - \log|x| \mathbf{1}_{d=2}) = +\infty$

Convexity and Bochner positivity

- Convexity/Positivity for probability measures μ and ν

$$\begin{aligned} & \frac{t\mathcal{E}_V(\mu) + (1-t)\mathcal{E}_V(\nu) - \mathcal{E}_V(t\mu + (1-t)\nu)}{t(1-t)} \\ & = \mathcal{E}(\mu - \nu) = \frac{1}{2c_d} \int_{\mathbb{R}^d} |\nabla U_{\mu-\nu}|^2 dx. \end{aligned}$$

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- Euler-Lagrange: if $c_V = \mathcal{E}(\mu_V) - \int V d\mu_V$ then q.e.

$$U_{\mu_V} + V \begin{cases} = c_V & \text{on } \text{supp}(\mu_V) \\ \geq c_V & \text{outside} \end{cases}$$

Examples of equilibrium measures

Dimension d	Potential V	Equilibrium measure μ_V
≥ 1	$\infty \mathbf{1}_{ \cdot >r}$	Uniform on sphere of radius r
≥ 1	$< \infty$ and \mathcal{C}^2	$c_d^{-1} \Delta V$ on interior of support
≥ 1	$\frac{1}{2} \cdot ^2$	Uniform on unit ball
(Ginibre) 2	$\frac{1}{2} \cdot ^2$	Uniform on unit disc
(Spherical) 2	$\frac{1}{2} \log(1 + \cdot ^2)$	Heavy-tailed $\frac{1}{\pi(1+ \cdot ^2)^2}$

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(CUE) 2	$\infty \mathbf{1}_{([a,b] \times \{0\})^c}$	Arcsine $s \mapsto \frac{\mathbf{1}_{s \in [a,b]}}{\pi \sqrt{(s-a)(b-s)}}$
(GUE) 2	$\frac{ \cdot ^2}{2} \mathbf{1}_{\mathbb{R} \times \{0\}} + \infty \mathbf{1}_{(\mathbb{R} \times \{0\})^c}$	Semicircle $s \mapsto \frac{\sqrt{4-s^2}}{2\pi} \mathbf{1}_{s \in [-2,2]}$

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Coulomb gas or one-component plasma

- Particles subject to confinement and singular pair repulsion

$$\begin{aligned}\beta E_n(x_1, \dots, x_n) &= \beta n^2 \left(\frac{1}{n} \sum_{i=1}^n V(x_i) + \frac{1}{n^2} \sum_{i < j} g(x_i - x_j) \right) \\ &= \beta n^2 \mathcal{G}_V^\neq(\mu_{x_1, \dots, x_n})\end{aligned}$$

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- Empirical measure $\mu_{x_1, \dots, x_n} = \frac{1}{n} \sum_{i=1}^n \delta_{x_i}$ and

$$\mathcal{E}_V^\neq(\mu) = \int V d\mu + \frac{1}{2} \iint_{\neq} g(u-v) d\mu(u) d\mu(v)$$

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- Boltzmann–Gibbs measure when $e^{-n\beta(V - \log(1+|\cdot|))\mathbf{1}_{d=2}} \in L^1(dx)$

$$dP_n(x_1, \dots, x_n) = \frac{e^{-\beta E_n(x_1, \dots, x_n)}}{Z_n} dx_1 \cdots dx_n$$

Examples: exactly solvable RMT and Coulomb gases

- $d = 2$ gives

$$e^{-\sum_i V(x_i) - \sum_{i < j} g(x_i - x_j)} = e^{-\sum_i V(x_i)} \prod_{i < j} |x_i - x_j|$$

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- $M \in \mathcal{M}_{n,n}(\mathbb{C})$, $M \propto e^{-\text{Trace}(MM^*)}$

Ensemble	Random Matrix	Potential V ($d = \beta = 2$)
Ginibre	M	$\frac{1}{2} \cdot ^2$
(GUE) Hermite	$\frac{1}{\sqrt{2}}(M + M^*)$	$\frac{1}{2} \cdot ^2 \mathbf{1}_{\mathbb{R} \times \{0\}} + \infty \mathbf{1}_{(\mathbb{R} \times \{0\})^c}$
(LUE) Laguerre	MM^*	$\frac{1}{2} \cdot \mathbf{1}_{\mathbb{R}_+ \times \{0\}} + \infty \mathbf{1}_{(\mathbb{R}_+ \times \{0\})^c}$
Spherical	$M_1 M_2^{-1}$	$\frac{1}{2} \frac{n+1}{n} \log(1 + \cdot ^2)$

Laplace method point of view

- Laplace point of view

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- Large Deviation Principle (also works if $\beta = \beta_n$ with $n\beta_n \rightarrow \infty$)

$$P_n(\mu_{x_1, \dots, x_n} \in B) \underset{n \rightarrow \infty}{\approx} e^{-\beta n^2 \inf_B(\mathcal{E}_V - \mathcal{E}_V(\mu_V))}$$

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- Law of Large Numbers : if $X \sim P_n$ then almost surely

$$\mu_{X_1, \dots, X_n} \xrightarrow[n \rightarrow \infty]{} \mu_V = \arg \min \mathcal{E}_V$$

..., Voiculescu, Ben Arous–Guionnet, Hiai–Petz, ...

..., Serfaty et al, C.–Gozlan–Zitt, Berman, García-Zelada, ...

Asymptotic analysis of fluctuations

- Gaussian structure: $P_n \approx e^{-\frac{n^2}{2} \langle -\beta c_d \Delta^{-1} \mu, \mu \rangle - n^2 \beta \langle V, \mu \rangle}$

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- Asymptotics: Central Limit Theorem with Gaussian Free Field

$$\sum_{i=1}^n f(X_i) - \mathbb{E}(\dots) \xrightarrow[n \rightarrow \infty]{\text{law}} \mathcal{N}\left(0, \frac{1}{\beta c_d} \int_{\mathbb{R}^d} |\nabla f|^2 dx + \dots\right)$$

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- Quantitative: concentration of measure inequalities

$$\begin{aligned} \mathbb{P}(\text{dist}(\mu_{X_1, \dots, X_n}, \mu_V) \geq r) \\ \leq e^{-a\beta n^2 r^2 + \mathbf{1}_{d=2}(\frac{\beta}{4} n \log n) + b\beta n^{2-2/d} + c(\beta)n} \end{aligned}$$

..., Guionnet–Zeitouni, Rougerie–Serfaty, Hardy–C.–Maïda, Berman, ...

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Langevin dynamics

- Overdamped Langevin dynamics on $(\mathbb{R}^d)^n$: $X_t \xrightarrow[t \rightarrow \infty]{\text{law}} P_n$

$$dX_t = \sqrt{2 \frac{\alpha}{\beta}} dB_t - \alpha \nabla E_n(X_t) dt, \quad L = \alpha (\beta^{-1} \Delta - \nabla E_n \cdot \nabla)$$

Dyson, Bru, Lassalle, Rogers–Shi, Guionnet et al., ..., Erdős–Yau et al. ...
Bolley–C.–Fontbona, C.–Lehec, Akkeman–Byun, ...

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- Mean-field McKean–Vlasov limit: if $\sigma = \lim_{n \rightarrow \infty} \frac{\alpha_n}{\beta_n}$ then (?)

$$\lim_{n \rightarrow \infty} \mu_{X_t} = \nu_t \quad \text{where} \quad \partial_t \nu_t = \sigma \Delta \nu_t + \nabla \cdot ((\nabla V + \nabla g * \nu_t) \nu_t).$$

..., Carrillo–McCann–Villani, ..., Guionnet et al, ..., Jabin et al, ...

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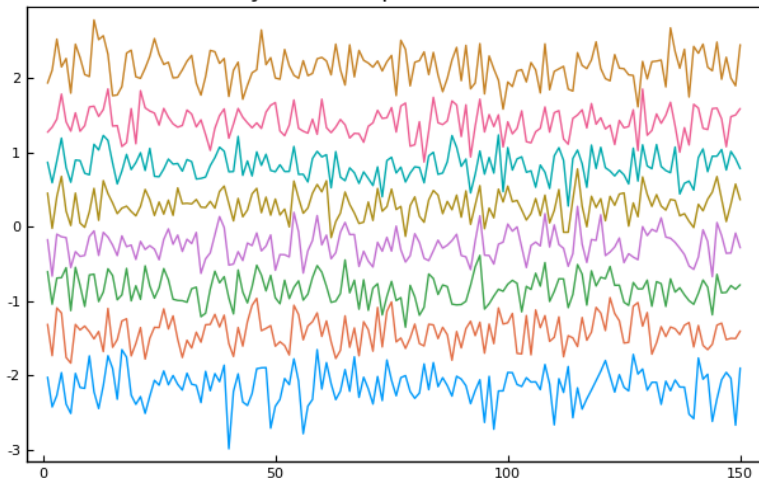
- Underdamped Langevin or kinetic **Dyson–Ornstein–Uhlenbeck**

$$dX_t = \alpha Y_t dt, \quad dY_t = -\alpha \nabla E_n(X_t) dt + \sqrt{2\frac{\gamma\alpha}{\beta}} dB_t - \gamma\alpha Y_t dt$$

Good for numerical simulation via Hamiltonian Monte Carlo

Ferré–C., Lu–Mattingly, Dolbeault et al

Example of kinetic Dyson HMC in 1D

Dyson HMC positions $N=8$ 

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- As a consequence if $X_n \sim P_n$ then

$$\begin{aligned} X_1 + \dots + X_n &\sim \mathcal{N}\left(0, \frac{1_2}{\beta}\right) \\ |X_1|^2 + \dots + |X_n|^2 &\sim \text{Gamma}\left(n + \beta \frac{n(n-1)}{4}, \beta \frac{n}{2}\right) \end{aligned}$$

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- Lack of useful tridiagonal model? Special eigenfunctions!
Bolley–C.–Fontbona, C.–Lehec

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- Quadratic conditioning gives perturbation of g instead of V

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- Main technical difficulty: regularity sets of \mathcal{E}_V
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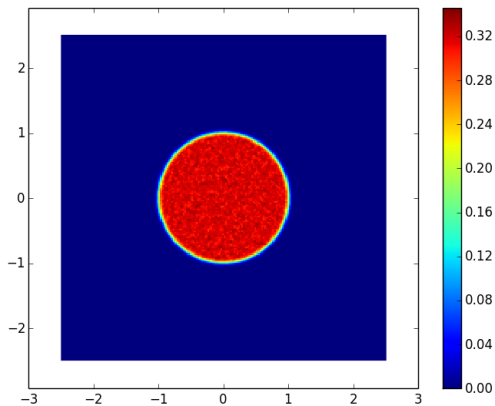
Conditioned Coulomb gas

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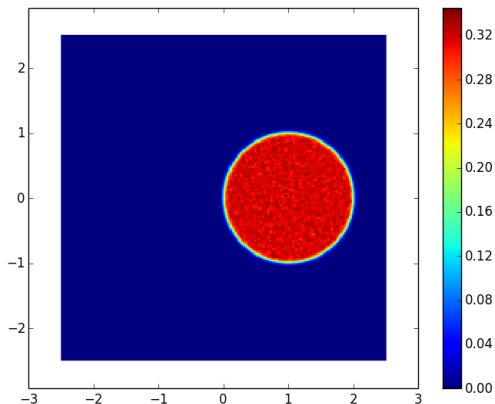
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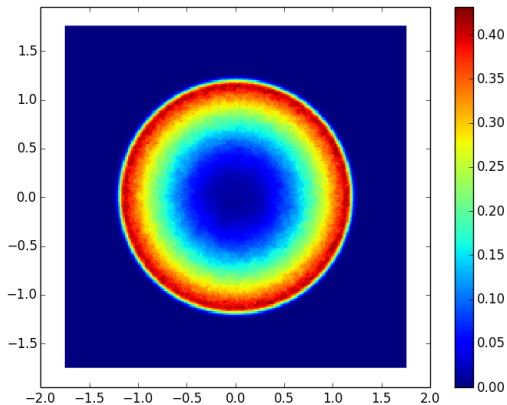
$$V(x) = |x|^2$$

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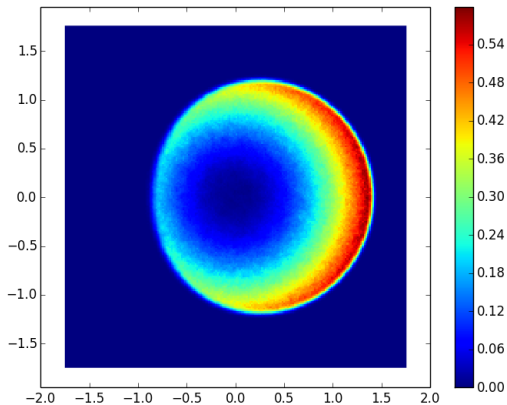
$$V(x) = |x|^2, \quad \varphi(x) = a \cdot x + b$$

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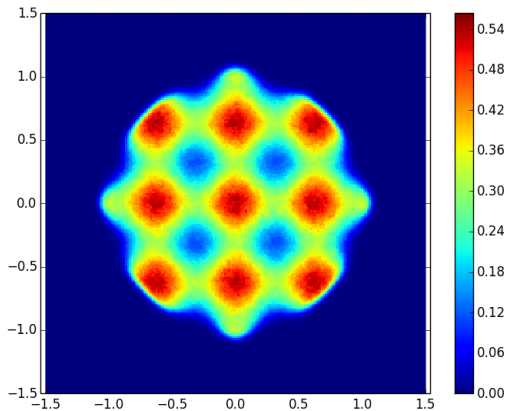
$$V(x) = |x|^4$$

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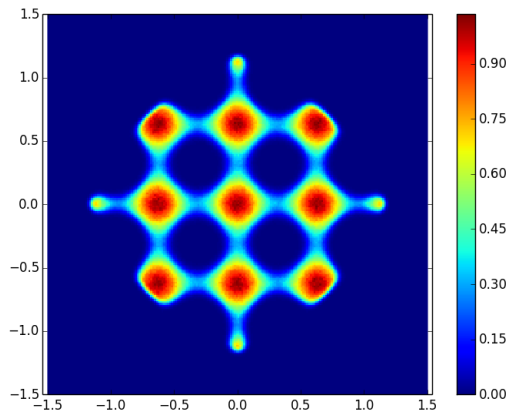
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Conditioned Coulomb gases – HMC/Julia



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Conditioned Coulomb gases – HMC/Julia



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Outline

Electrostatics

Gases

Dynamics for planar case

Conditioning

Jellium

A bit of chronology

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- Coulomb gas is a Jellium with $\rho = \frac{n}{\alpha c_d} \Delta V$ on $S = \mathbb{R}^d$

Two dimensional jellium with uniform background

- $d = 2$ and $\rho = \text{Uniform}(D(0, R))$

$$\begin{aligned} V(x) &= -\frac{\alpha}{n} U_\rho(x) \\ &= \frac{\alpha}{n} \left(\frac{|x|^2}{2R} - 1 + \log R \right) \mathbf{1}_{|x| \leq R} + \frac{\alpha}{n} \log|x| \mathbf{1}_{|x| > R}. \end{aligned}$$

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- Transition for edge fluctuations when $\beta = 2$ and $\alpha_n \sim \lambda n$

Gumbel if $\lambda > 1$ and Heavy-tailed if $\lambda = 1$.

Thank you very much for your attention!