SELF-STANDARDIZED CENTRAL LIMIT THEOREMS FOR TRIMMED SUBORDINATORS

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ABSTRACT

We prove under general regularity conditions that a trimmed subordinator satisfies a self-standardized central limit theorem [SSCLT]. Our basic tools are a classic representation for subordinators and a distributional approximation result of Zaitsev (1987).

A MOTIVATING TRIMMED SUM CLT

Let $X_1, X_2 \ldots,$ be i.i.d. nonnegative nondegenerate random variables and for each $n \geq 1$ let

$$X_n^{(1)} \geq X_n^{(2)} \geq \ldots X_n^{(n)}$$

denote their order statistics. A special case of results of S. Csörgő, Haeusler and Mason (1988) characterizes when for a sequence $\{r_n\}_{n\geq 1}$ of positive integers $1 \leq r_n \leq n$ satisfying $r_n \to \infty$ and $r_n/n \to 0$ as $n \to \infty$ there exist norming and centering constants $B_{r_n} > 0$ and A_{r_n} such that

$$\frac{\sum_{i=1}^{n} X_i - X_n^{(1)} - \cdots - X_n^{(r_n)} - A_{r_n}}{B_{r_n}} \stackrel{\mathrm{D}}{\to} Z \ ,$$

where Z is a standard normal random variable. We shall soon see that an analogous CLT holds for trimmed subordinators.

SUBORDINATOR

Let V_t, $t \geq 0$, be a subordinator with Lévy measure Λ and drift 0. This means that V_t is a stationary independent increment process with nonnegative jumps satisfying $V_0 = 0$ having Laplace transform

$$\mathbf{E} \exp\left(-\theta V_t\right) = \exp\left(-t\Phi\left(\theta\right)\right), \ \theta \geq 0,$$

where

$$\Phi\left(\theta\right) = \int_0^\infty \left(1 - \exp\left(-\theta v\right)\right) \Lambda\left(\mathrm{d}v\right).$$

SOME BASIC NOTATION

Put $\overline{\Lambda}(x) = \Lambda((x,\infty))$, and for $u > 0$ let

$$\varphi(u) = \sup\{x : \overline{\Lambda}(x) > u\},$$

where $\sup \varnothing := 0$. We shall assume that $\overline{\Lambda}(0+) = \infty$.

Note that for V_t to be a subordinator its Lévy measure must satisfy

$$\int_0^1 x\Lambda(\mathrm{d}x) < \infty,$$

equivalently, for all $y > 0$

$$\int_y^\infty \varphi\left(x\right)\mathrm{d}x < \infty.$$

For future use set for any $y > 0$

$$\mu\left(y\right) := \int_y^\infty \varphi\left(x\right)\mathrm{d}x \text{ and } \sigma^2\left(y\right) := \int_y^\infty \varphi^2\left(x\right)\mathrm{d}x.$$

REPRESENTATION FOR SUBORDINATOR

Let $\omega_1, \omega_2, \ldots$ be i.i.d. exponential random variables with mean 1. Put for $n \geq 1$, the partial sums,

$$\Gamma_n = \omega_1 + \ldots + \omega_n.$$

V_t has the distributional representation

$$V_t \stackrel{\mathrm{D}}{=} \sum_{i=1}^{\infty} \varphi(\Gamma_i/t).$$

TRIMMED SUBORDINATOR

Denote for $t > 0$ the ordered jump sequence $m_t^{(1)} \geq m_t^{(2)} \geq \ldots$ of V_t on the interval $[0, t]$. It turns out for any given $t > 0$

$$\left(m_t^{(k)}\right)_{k \geq 1} \stackrel{\mathrm{D}}{=} \left(\varphi\left(\frac{\Gamma_k}{t}\right)\right)_{k \geq 1}. \tag{1}$$

Set $V_t^{(0)} := V_t$ and for any integer $k \geq 1$ consider the trimmed subordinator

$$V_t^{(k)} := V_t - m_t^{(1)} - \cdots - m_t^{(k)},$$

which on account of (1) says for any integer $k \geq 0$

$$V_t^{(k)} \stackrel{\mathrm{D}}{=} \sum_{i=k+1}^{\infty} \varphi\left(\frac{\Gamma_i}{t}\right) =: \widetilde{V}_t^{(k)}.$$

OUR GOAL

We shall prove under regularity conditions that given a sequence of positive integers $\{k_n\}_{n\geq 1}$ converging to infinity and a sequence of positive constants $\{t_n\}_{n\geq 1}$ that the following SSCLT holds for $\widetilde{V}_{t_n}^{(k_n)}$

$$\frac{\widetilde{V}_{t_n}^{(k_n)} - t_n \mu\left(\Gamma_{k_n}/t\right)}{\sqrt{t_n}\sigma\left(\Gamma_{k_n}/t\right)} \xrightarrow{\mathrm{D}} Z.$$

As a special case we get the SSCLT of Ipsen, Maller and Resnick [IMR] (2020), who consider the case when $t_n = t$ is fixed and $k_n = n$.

A **SSCLT FOR A TRIMMED SUBORDINATOR**

Theorem 1 *Assume that* $\overline{\Lambda}(0+) = \infty$*. For any sequence of positive integers* $\{k_n\}_{n\geq 1}$ *converging to infinity and sequence of positive constants* $\{t_n\}_{n\geq 1}$ *satisfying*

$$\frac{\sqrt{t_n}\sigma\left(\Gamma_{k_n}/t_n\right)}{\varphi\left(\Gamma_{k_n}/t_n\right)} \stackrel{\mathrm{P}}{\to} \infty, \text{ as } n \to \infty, \tag{2}$$

we have uniformly in x*, as* $n \to \infty$

$$\left| P\left\{ \frac{\widetilde{V}_{t_n}^{(k_n)} - t_n\mu\left(\Gamma_{k_n}/t_n\right)}{\sqrt{t_n}\sigma\left(\Gamma_{k_n}/t_n\right)} \leq x | \Gamma_{k_n} \right\} - P\left\{Z \leq x\right\} \right| \stackrel{\mathrm{P}}{\to} 0, \tag{3}$$

which implies as $n \to \infty$

$$\frac{\widetilde{V}_{t_n}^{(k_n)} - t_n\mu\left(\Gamma_{k_n}/t_n\right)}{\sqrt{t_n}\sigma\left(\Gamma_{k_n}/t_n\right)} \stackrel{\mathrm{D}}{\to} Z. \tag{4}$$

EXAMPLE 1

There always exist $k_n \to \infty$ and $t_n \to \infty$ such that (2) holds. For example for any $k_n \to \infty$, let $t_n = \rho n$ for some $\rho > 0$. Since $\Gamma_{k_n}/k_n \overset{\mathrm{P}}{\to} 1$ and $\Gamma_{k_n}/t_n \overset{\mathrm{P}}{\to} 1/\rho$, which implies that

$$P\left\{\frac{\sqrt{t_n}\sigma\left(\Gamma_{k_n}/t_n\right)}{\varphi\left(\Gamma_{k_n}/t_n\right)} > \frac{\sqrt{\rho k_n}\sigma\left(2/\rho\right)}{\varphi\left(1/\left(2\rho\right)\right)}\right\} \to 1$$

and thus (2) holds.

EXAMPLE 2

Assume the Feller class at zero condition

$$\limsup_{x\downarrow 0}\frac{x^2\overline{\Lambda}(x)}{\int_0^x u^2\Lambda(\mathrm{d}u)}<\infty. \tag{5}$$

Since $\overline{\Lambda}(\varphi(y)-)\geq y$, (5) says that

$$\limsup_{y\to\infty}\frac{\varphi^2\left(y\right)y}{\int_0^{\varphi(y)}u^2\Lambda(\mathrm{d}u)}\leq\limsup_{y\to\infty}\frac{\varphi^2\left(y\right)\overline{\Lambda}(\varphi\left(y\right)-)}{\int_0^{\varphi(y)-}u^2\Lambda(\mathrm{d}u)}<\infty,$$

which implies that

$$\liminf_{y\to\infty}\int_y^\infty\varphi^2\left(x\right)\mathrm{d}x/\left(y\varphi^2\left(y\right)\right):=\tau>0.$$

Observing that

$$\frac{t_n\sigma^2\left(\Gamma_{k_n}/t_n\right)}{\varphi^2\left(\Gamma_{k_n}/t_n\right)}=\Gamma_{k_n}\frac{\int_{\Gamma_{k_n}/t_n}^\infty\varphi^2\left(x\right)\mathrm{d}x}{\left(\Gamma_{k_n}/t_n\varphi^2\left(\Gamma_{k_n}/t_n\right)\right)},$$

we see that (2) holds, whenever $\Gamma_{k_n}\overset{\mathrm{P}}{\to}\infty$ and $\Gamma_{k_n}/t_n\overset{\mathrm{P}}{\to}\infty$. We note in passing that (5) is satisfied whenever Λ is regularly varying at zero with index $-\alpha$, $0<\alpha<2$.

IMR SELF-STANDARDIZED CLT

Ipsen, Maller and Resnick [IMR] (2020) have shown whenever there exist constants a_n and b_n such that for a non-degenerate random variable Δ

$$\frac{m_1^{(n)} - b_n}{a_n} \stackrel{\mathrm{D}}{=} \frac{\varphi\left(\Gamma_n\right) - b_n}{a_n} \stackrel{\mathrm{D}}{\to} \Delta \tag{6}$$

then for all $t > 0$ the following self-standardized trimmed CLT holds

$$\frac{\widetilde{V}_t^{(n)} - t\mu\left(\Gamma_n/t\right)}{\sqrt{t}\sigma\left(\Gamma_n/t\right)} \stackrel{\mathrm{D}}{\to} Z.$$

CHARACTERIZATION

IMR (2020) have shown that for (6) to hold it is necessary that there exist functions $a(r)$ and $b(r)$ of $r > 0$ such that whenever $a(r)x + b(r) > 0$

$$\lim_{r\to\infty} \frac{r - \overline{\Lambda}(a(r)x + b(r))}{\sqrt{r}} = h(x),$$

where for some $\gamma \leq 0$

$$h(x) = \begin{cases} 2x, \text{ if } \gamma = 0, \\ -\frac{2}{\gamma}\log(1-\gamma x), \text{ when } \gamma < 0 \text{ and } 1 - \gamma x > 0. \end{cases}$$

In which case $P\{\Delta \leq x\} = P\{Z \leq h(x)\}$.

IMR CASE $\gamma < 0$

In the case $\gamma < 0$, Proposition 4.1 of IMR (2020) says that

$$\int_0^x u^2 \Lambda(\mathrm{d}u) \sim \frac{2x^2\sqrt{\overline{\Lambda}(x)}}{|\gamma|}, \text{ as } x \downarrow 0, \tag{7}$$

and $\overline{\Lambda}(x)$ is slowly varying at 0. Since $\varphi(z) \searrow 0$ as $z \nearrow \infty$. This implies that as y/t converges to ∞

$$\frac{t\sigma^2(\varphi(y/t))}{\varphi^2(y/t)} = \frac{t\int_0^{\varphi(y/t)} u^2\Lambda(\mathrm{d}u)}{\varphi^2(y/t)}$$

$$\sim \frac{2t\sqrt{\overline{\Lambda}(\varphi(y/t))}}{|\gamma|}, \text{ as } y/t \to \infty.$$

Setting $\Gamma_n = y$ we see that (2) holds, when $t_n = t > 0$ fixed. A Lévy measure that satisfies (7) is not in the Feller class at zero.

IMR CASE $\gamma = 0$

Proposition 4.2 of IMR (2020) implies that in the case $\gamma = 0$,

$$\frac{\int_0^{\varphi(x)} u^2 \Lambda(\mathrm{d}u)}{\varphi^2(x)\sqrt{x}} \to \infty, \text{ as } x \to \infty.$$

This implies that as y/t converges to ∞ and ty is bounded away from 0, then

$$\frac{t\sigma^2(\varphi(y/t))}{\varphi^2(y/t)} = \frac{\sqrt{ty}\int_0^{\varphi(y/t)} u^2 \Lambda(\mathrm{d}u)}{\varphi^2(y/t)\sqrt{y/t}} \to \infty.$$

Setting $\Gamma_n = y$ we again see that (2) holds, when $t_n = t > 0$ fixed.

COROLLARY

With added regularity one can use non random norming and centering.

Corollary 1 *Assume that* V_t, $t \geq 0$, *is a subordinator with drift* 0, *whose Lévy tail function* $\overline{\Lambda}$ *is regularly varying at zero with index* $-\alpha$, *where* $0 < \alpha < 1$. *For any sequence of positive integers* $\{k_n\}_{n\geq 1}$ *converging to infinity and sequence of positive constants* $\{t_n\}_{n\geq 1}$ *such that* $k_n/t_n \to \infty$ *we have as* $n \to \infty$,

$$\frac{V_{t_n}^{(k_n)} - t_n \mu\left(k_n/t_n\right)}{\sqrt{t_n}\sigma\left(k_n/t_n\right)} \xrightarrow{\mathrm{D}} \sqrt{\frac{2}{\alpha}} Z. \tag{8}$$

CLT FOR TRIMMED SUMS

The analog of Corollary 1 for a sequence of i.i.d. positive random variables $\xi_1, \xi_2 \ldots$ in the domain of attraction of a stable law of index $0 < \alpha < 2$ says that as $n \to \infty$,

$$\frac{\sum_{i=r_n+1}^{n} \xi_n^{(i)} - nc\left(r_n/n\right)}{\sqrt{n} a\left(r_n/n\right)} \xrightarrow{\mathrm{D}} \sqrt{\frac{2}{2-\alpha}} Z, \tag{9}$$

where for each $n \geq 2$, $\xi_n^{(1)} \geq \cdots \geq \xi_n^{(n)}$ denote the order statistics of $\xi_1, \ldots, \xi_n$, $\{r_n\}_{n \geq 1}$ is a sequence of positive integers $1 \leq r_n \leq n$ satisfying $r_n \to \infty$ and $r_n/n \to 0$ as $n \to \infty$, and $c\left(r_n/n\right)$ and $a\left(r_n/n\right)$ are appropriate centering and norming constants. For details refer to S. Csörgő, Horváth and Mason (1986). The proof of Corollary 1 borrows ideas from the proof of their Theorem 1.

SPECIAL CASE OF A RESULT OF ZAITSEV

Fact (Zaitsev (1987)) *Let Y be an infinitely divisible mean 0 and variance 1 random variable with Lévy measure Λ and Z be a standard normal random variable. Assume that the support of Λ is contained in a closed ball with center 0 of radius τ. Then for universal positive constants C_1 and C_2 for any $\lambda > 0$*

$$\pi(Y, Z; \lambda) \leq C_1 \exp\left(-\frac{\lambda}{C_2 \tau}\right),$$

where for $\lambda > 0$

$$\pi(Y, Z; \lambda) := \sup_{B \in \mathcal{B}} \max\{F, G\},$$

where

$$F = P\{Y \in B\} - P\left\{Z \in B^\lambda\right\}$$

and

$$G = P\{Z \in B\} - P\left\{Y \in B^\lambda\right\}$$

with $B^\lambda = \{y \in \mathbb{R}\colon \inf_{x \in B} |x - y| < \lambda\}$ for $B \in \mathcal{B}$, the Borel sets of $\mathbb{R}$

IN PARTICULAR

A particular, the Zaitsev Fact says that for all $x \in \mathbb{R}$ and $\lambda > 0$,

$$P\{Z \leq x - \lambda\} - C_1 \exp\left(-\frac{\lambda}{C_2 \tau}\right) \leq P\{Y \leq x\}$$

$$\leq P\{Z \leq x + \lambda\} + C_1 \exp\left(-\frac{\lambda}{C_2 \tau}\right).$$

SKETCH OF PROOF OF THEOREM 1

For each $t > 0$ and $y > 0$ consider the random variable

$$T(t, y) = \sum_{i=1}^{\infty} \varphi\left(\frac{y}{t} + \frac{\Gamma_i'}{t}\right),$$

with Γ_i', $i \geq 1$, $\overset{\mathrm{D}}{=} \Gamma_i$, $i \geq 1$, which has Laplace transform

$$\Upsilon_{t,y}(\theta) := E \exp(-\theta T(t, y)) = \exp(-t\Phi_{t,y}(\theta)),$$

where $\Phi_{t,y}(\theta)$ is the Laplace exponent,

$$\Phi_{t,y}(\theta) = \int_0^{\infty} t\left(1 - \exp\left(-\theta\varphi\left(\frac{y}{t} + u\right)\right)\right) \mathrm{d}u.$$

LEVY MEASURE OF $T\,(t, y)$

Introducing the Lévy measure $\Lambda_{y/t}$ defined on $(0, \infty)$ by

$$\overline{\Lambda}_{y/t}(u) = \begin{cases} \overline{\Lambda}(u) - \frac{y}{t}, \text{ for } 0 < u < \varphi\left(\frac{y}{t}\right) \\ 0, \text{ for } u \geq \varphi\left(\frac{y}{t}\right) \end{cases},$$

we see that

$$\Phi_{t,y}\,(\theta) = \int_0^\infty t\,(1 - \exp\,(-\theta v))\,\Lambda_{y/t}\,(\mathrm{d}v)\,.$$

Clearly $T\,(t, y)$ is an infinitely divisible random variable and the support of $\Lambda_{y/t}$ is contained in $[0, \varphi(y/t)]$.

STANDARDIZED VERSION OF $T(t, y)$

One finds that

$$ET(t, y) = t \int_{y/t}^{\infty} \varphi(u) \, \mathrm{d}u =: t\mu\left(\frac{y}{t}\right)$$

and

$$VarT(t, y) = t \int_{y/t}^{\infty} \varphi^2(u) \, \mathrm{d}u =: t\sigma^2\left(\frac{y}{t}\right).$$

For each $t > 0$ and $y > 0$ consider the standardized version of $T(t, y)$

$$S(t, y) = \frac{T(t, y) - t\mu\left(\frac{y}{t}\right)}{\sqrt{t}\sigma\left(\frac{y}{t}\right)}.$$

By definition $ES(t, y) = 0$ and $VarS(t, y) = 1$.

INFINITELY DIVISIBLE RANDOM VARIABLE

The random variable $S\left(t, y\right)$ is an infinitely divisible random variable with mean zero, variance one whose Lévy measure is contained in

$$\left[0, \frac{\varphi(y/t)}{\sqrt{t}\sigma\left(\frac{y}{t}\right)}\right].$$

This allows us to use the Zaitsev fact.

APPLICATION OF ZAITSEV FACT

Applying the Zaitsev Fact to the infinitely divisible random variable $S(t, y)$ we get for any $t > 0$, $y > 0$ and $\lambda > 0$ and for universal positive constants C_1 and C_2

$$\pi(S(t, y), Z; \lambda) \leq C_1 \exp\left(-\frac{\lambda\sqrt{t}\sigma\left(\frac{y}{t}\right)}{C_2\varphi(y/t)}\right).$$

Since $\varphi(z) \to 0$, as $z \to \infty$, this implies that whenever $\{t_n\}_{n\geq1}$ is a sequence of positive constants and Y_{k_n} is a sequence of random variables such that each Y_{k_n} is independent of Γ_i', $i \geq 1$, and as $n \to \infty$

$$\frac{\sqrt{t_n}\sigma(Y_{k_n}/t_n)}{\varphi(Y_{k_n}/t_n)} \overset{\mathrm{P}}{\to} \infty, \tag{10}$$

then uniformly in x as $n \to \infty$

$$|P\{S(t_n, Y_{k_n}) \leq x | Y_{k_n}\} - P\{Z \leq x\}| \overset{\mathrm{P}}{\to} 0, \tag{11}$$

and thus as $n \to \infty$

$$|P\{S(t_n, Y_{k_n}) \leq x\} - P\{Z \leq x\}| \to 0. \tag{12}$$

PARTICULAR CASE

In particular by choosing $Y_{k_n} = \Gamma_{k_n}$ and independent of $\{\Gamma_i'\}_{i\geq 1} \stackrel{\mathrm{D}}{=} \{\Gamma_i\}_{i\geq 1}$, we get

$$\frac{\widetilde{V}_{t_n}^{(k_n)} - t_n\mu\left(\frac{\Gamma_{k_n}}{t_n}\right)}{\sqrt{t_n}\sigma\left(\frac{\Gamma_{k_n}}{t_n}\right)} \stackrel{\mathrm{D}}{=} \frac{\sum_{i=1}^{\infty}\varphi((Y_{k_n}+\Gamma_i')/t_n) - t_n\mu\left(\frac{Y_{k_n}}{t_n}\right)}{\sqrt{t_n}\sigma\left(\frac{Y_{k_n}}{t_n}\right)}$$

$$= \frac{T(t_n, Y_{k_n}) - t_n\mu\left(\frac{Y_{k_n}}{t_n}\right)}{\sqrt{t_n}\sigma\left(\frac{Y_{k_n}}{t_n}\right)} = S(t_n, Y_{k_n}).$$

Keeping (2) in mind, (3) and (4) follow from (11) and (12), respectively.

A POLAR LEVY MEASURE

Write $\Delta = S^{d-1}$, the unit vectors in $\mathbb{R}^d$, and let κ be a probability measure on Δ. Further let Λ be a Lévy measure on $(0,\infty)$ such that

$$\int_{(0,1]} r\Lambda\,(\mathrm{d}r) < \infty.$$

Define the *tail function* $\overline{\Lambda}(x) = \Lambda((x,\infty))$, for $x > 0$. We assume that $\overline{\Lambda}(0+) = \infty$. Define the σ-finite measure on Borel subsets B of $\mathbb{R}^d_*$, by

$$\Pi\,(B) = \int_\Delta \int_{(0,\infty)} \mathbf{1}_B\,(r\mathbf{u})\,\kappa\,(\mathrm{d}\mathbf{u})\,\Lambda\,(\mathrm{d}r)\,.$$

One finds that

$$\int_{0<|\mathbf{x}|\le 1} |\mathbf{x}|\,\Pi\,(\mathrm{d}\mathbf{x}) = \int_{(0,1]} r\Lambda\,(\mathrm{d}r) < \infty.$$

For $u > 0$ let

$$\varphi(u) = \sup\{x : \overline{\Lambda}(x) > u\},$$

where $\sup\varnothing := 0$.

AN $\mathbb{R}^d$ VALUED POLAR SUBORDINATOR

Consider the $\mathbb{R}^d$ valued Lévy process $(\mathbf{V}_t)_{t\geq 0}$ having characteristic function

$$E\exp\left(\mathrm{i}\theta'\mathbf{V}_t\right) = \exp\left(t\Psi_{\mathbf{V}}\left(\theta\right)\right),$$

with characteristic exponent defined for $\theta \in \mathbb{R}^d$, by

$$\Psi_{\mathbf{V}}\left(\theta\right) = \int_{\Delta}\int_{(0,\infty)}\left(\exp\left(\mathrm{i}r\theta'\mathbf{u}\right) - 1\right)\Lambda\left(\mathrm{d}r\right)\kappa\left(\mathrm{d}\mathbf{u}\right)$$

$$= \int_{\Delta}\int_{(0,\infty)}\left(\exp\left(\mathrm{i}\varphi\left(v\right)\theta'\mathbf{u}\right) - 1\right)\mathrm{d}v\kappa\left(\mathrm{d}\mathbf{u}\right).$$

$$= \int_{\mathbb{R}^d_*}\left(\exp\left(\mathrm{i}\theta'\mathbf{x}\right) - 1\right)\Pi\left(\mathrm{d}\mathbf{x}\right).$$

We shall call $(\mathbf{V}_t)_{t\geq 0}$ an $\mathbb{R}^d$ valued polar subordinator.

A TRIMMED POLAR SUBORDINATOR

Let $\omega_1, \omega_2, \ldots$ be i.i.d. exponential random variables with parameter 1 and for each $n \geq 1$ let $\Gamma_n = \omega_1 + \ldots + \omega_n$. Independent of $(\Gamma_i)_{i \geq 1}$, let U, $(U_i)_{i \geq 1}$ be i.i.d. Δ valued random variables with common distribution κ.

For each $t > 0$ and integer $k \geq 0$, let $\mathbf{V}_t^{(k)}$ denote $\mathbf{V}_t$ with the largest k jumps in Euclidean norm up to time t removed.

Guided by a general Lévy process representation of Rosiński (2001). we can show that for each $t > 0$ and $k \geq 0$

$$\mathbf{V}_t^{(k)} \stackrel{\mathrm{D}}{=} \sum_{i=k+1}^{\infty} \varphi\left(\frac{\Gamma_i}{t}\right) U_i =: \widetilde{\mathbf{V}}_t^{(k)},$$

where $\mathbf{V}_t^{(0)} = \mathbf{V}_t$.

A SSCLT FOR THE TRIMMED POLAR SUBORDINATOR

Let $\mathbf{u}$ denote the mean vector $(u_1, \ldots, u_d)$ and $\mathbf{U}$ the covariance matrix of U. Further, let $\mathbf{Z_U}$ denote a mean zero d–dimensional normal random vector with covariance matrix $\mathbf{U}$.

Theorem 2 *Assume that $\overline{\Lambda}(0+) = \infty$. For any sequence of positive integers $\{k_n\}_{n\geq 1}$ converging to infinity and sequence of positive constants $\{t_n\}_{n\geq 1}$ satisfying*

$$\frac{\sqrt{t_n}\sigma\left(\Gamma_{k_n}/t_n\right)}{\varphi\left(\Gamma_{k_n}/t_n\right)} \xrightarrow{\mathrm{P}} \infty, \text{ as } n \to \infty,$$

we have

$$\frac{\widetilde{V}_{t_n}^{(k_n)} - t_n\mu\left(\Gamma_{k_n}/t_n\right)\mathbf{u}}{\sqrt{t_n}\sigma\left(\Gamma_{k_n}/t_n\right)} \xrightarrow{\mathrm{D}} \mathbf{Z_U},$$

where μ and σ are exactly as in the Theorem 1.

TRIMMING GENERAL LEVY PROCESSES

Let $(X_t)_{t\geq 0}$, be a Lévy process with a nontrivial Lévy measure Λ. For any $t > 0$ denote the ordered positive jump sequence

$$m_t^{(1)} \geq m_t^{(2)} \geq \dots$$

of X_t on the interval $[0, t]$ and let

$$n_t^{(1)} \leq n_t^{(2)} \leq \dots$$

denote the corresponding ordered negative jump sequence of X_t. Consider for any positive integers k and l, the trimmed subordinator

$$X_t^{(k,l)} := X_t - m_t^{(1)} - \cdots - m_t^{(k)} - n_t^{(1)} - \cdots - n_t^{(l)}.$$

IT TURNS OUT

It turns out that a suitably self-standardized, $X_t^{(k,l)}$, say

$$S^{(k,l)}\left(t, m_t^{(k)}, n_t^{(l)}\right),$$

satisfies a CLT. In this situation one has to be careful about the centering.

Once again using the Zaitsev (1987) approximation one gets subject to regularity conditions, that uniformly in x, as $n \to \infty$,

$$\left|P\left\{S \le x | m_{t_n}^{(k_n)}, n_{t_n}^{(l_n)}\right\} - P\left\{Z \le x\right\}\right| \xrightarrow{\mathrm{P}} 0,$$

where

$$S = S^{(k_n,l_n)}\left(t_n, m_{t_n}^{(k_n)}, n_{t_n}^{(l_n)}\right).$$

THE KEY TO THE PROOF

The key to the proof is to decompose the non Gaussian component of X_t into its independent positive and negative jump parts X_t^+ and X_t^-, whose Lévy measures are determined by

$$\overline{\Lambda}_+(x) = \Lambda((x, \infty)) \text{ and } \overline{\Lambda}_-(x) = \Lambda((-\infty, -x)), x > 0,$$

represent X_t^+ and X_t^- using two independent rate 1 Poisson processes and the inverse functions defined for $u > 0$

$$\varphi_+(u) = \sup\{x : \overline{\Lambda}_+(x) > u\}, \text{ and}$$
$$\varphi_-(u) = \sup\{x : \overline{\Lambda}_-(x) > u\},$$

and then proceed analogously to the subordinator case.