Poincaré inequalities and Wasserstein contraction under variable curvature bounds

Arnaud GUILLIN

Université Clermont Auvergne, Laboratoire de Mathématiques Blaise Pascal

High Dimensional Probability 2020

joint work with P. Cattiaux and M. Fathi.

Introduction

Functional inequalities are now a crucial tool in many domains: probability, PDE, geometry,...

This is due to their various properties:

- concentration of masure (exponential, Gaussian,...)
- convergence of semigroup (in L^2, in entropy,...)
- tensorization property (adimensional when product of measure,...)
- ...

Poincaré inequality

One of the most famous is perhaps the Poincaré inequality (PI), or spectral gap inequality.

Let μ be a probability measure on $\mathbb{R}^n$ associated with the potential V, i.e $d\mu := e^{-V}dx$, satisfies (PI) of constant C_P if for every smooth function

$$\mathrm{Var}_\mu(f) := \int f^2 d\mu - \left(\int f d\mu\right)^2 \leq C_P \int |\nabla f|^2 d\mu$$

which entails exponential concentration of measure and L^2 exponential convergence of the associated semi-group (definition to come soon).

How to prove a Poincaré inequality?

It is of course one of the main question.

During the past decades, many techniques have emerged (absolutely non exhaustive list)

- Hardy-Muckenhoupt criterion in dimension one,
- curvature conditions of Bakry-Emery,
- Lyapunov conditions,
- ...

and a bunch of perturbation (Rothaus lemma,...) and (dependent) tensorization techniques to extend theses results.

Curvature condition and Γ-calculus of Bakry-Emery

Let consider $d\mu = e^{-V}dx$ and the reversible diffusion

$$dX_t = \sqrt{2}dB_t - \nabla V(X_t)$$

with generator

$$L = \Delta - \nabla V.\nabla$$

and semigroup P_t, i.e. $P_t f(x) := \mathbb{E}_x(f(X_t))$.

We may then define:

- Carré-du-Champ : $\Gamma(f,g) := \frac{1}{2}\left(L(fg) - fLg - gLf\right)$.
- Γ_2 : $\Gamma_2(f,g) := \frac{1}{2}\left(L\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(g,LF)\right)$.

Bakry-Emery's curvature condition $CD(\rho,\infty)$**:**

$$\forall f, \quad \Gamma_2(f) \geq \rho\,\Gamma(f).$$

It is not hard to remark that when

$$L = \Delta - \nabla V.\nabla$$

we have

$$\Gamma(f) = |\nabla f|^2$$

so that our Poincaré inequality is

$$\mathrm{Var}_\mu(f) \leq C_P \int -fLfd\mu.$$

We also have

$$\Gamma_2(f) = \|\mathrm{Hess}(f)\|_2^2 + \nabla f.\mathrm{Hess}(V)\nabla f$$

and thus the Bakry-Emery's curvature condition

$$\forall f,\ \Gamma_2(f) \geq \rho\Gamma(f) \iff \forall u,\ u^t\mathrm{Hess}(V)\,u \geq \rho|u|^2.$$

Some equivalent condition to get Poincaré

Theorem

There is equivalence between

1. *Poincaré inequality with constant* C
2. *Exponential convergence in* L^2

$$\forall f, \forall t, \qquad Var_\mu(P_t f) \leq e^{-2t/C} Var_\mu(f)$$

3. *Integrated* Γ_2 *criterion*

$$\int \Gamma_2(f) d\mu \geq \frac{1}{C} \int \Gamma(f) d\mu$$

These are quite easily proved via time differentiation and/or integration by parts.

It doesn't give however practical conditions to get a Poincaré inequality

Bakry-Emery's result

Theorem

The following two conditions are equivalent

- $CD(\rho_0, \infty)$ *condition,*
- *Gradient commutation property*

$$\Gamma(P_t f) \leq e^{-2\rho_0 t} P_t \Gamma(f)$$

and if $CD(\rho, \infty)$ *holds with* $\rho_0 > 0$ *then*

$$C_P \leq \frac{1}{\rho_0}.$$

Main question

Can we get a better estimate if we suppose that we have variable curvature bounds:

$$\forall u, \quad u.\mathrm{Hess}(V(x))\, u \geq \rho(x)|u|^2 \geq \rho_0|u|^2 > 0$$

????

Remark : we are not aware of many results in this case. Let us just mention Veysseire's result proving that

$$C_P \leq \mu(1/\rho).$$

Note that by Jensen's inequality : $\frac{1}{\mu(\rho)} \leq \mu(1/\rho)$.

We will try to go towards $\frac{1}{\mu(\rho)}$.

To do so, let us look at the proof of the previous theorem:

It is quite simple to remark that (suppose $\mu(f)=0$)

$$\frac{d}{dt}\int (P_tf)^2 d\mu = 2\int P_tfLP_tfd\mu = -2\int \Gamma(P_tf)d\mu$$

so that

$$\begin{aligned} \mathrm{Var}_\mu(f) &= -\int_0 \infty \frac{d}{dt}\int (P_tf)^2 d\mu dt \\ &= 2\int_0^\infty \int \Gamma(P_tf)d\mu dt \\ &\le 2\int_0^\infty e^{-2\rho_0 t}dt \int \Gamma(f)d\mu \\ &= \frac{1}{\rho_0}\int \Gamma(f)d\mu \end{aligned}$$

and it is thus crucial to establish the gradient commutation property!

The proof of the gradient commutation property (by Bakry-Emery) is quite elegant.

Consider $\Psi(s) = P_s\Gamma(P_{t-s}f)$ then

$$\begin{aligned} \Psi'(s) &= P_s[L\Gamma(P_{t-s}f) - 2\Gamma(P_{t-s}f, LP_{t-s}f)] \\ &= 2P_s\Gamma_2(P_{t-s}f) \\ &\geq 2\rho_0 P_s\Gamma(P_{t-s}f) \\ &= 2\rho_0\Psi(s) \end{aligned}$$

and Gronwall's Lemma gives the conclusion!!!

But difficult to extend to variable curvature bounds!!!

Proposition

Assume variable curvature lower bounds with ρ continuous, and V C^2, then

$$|\nabla P_t f|(x) \leq \mathbb{E}_x \left(e^{-\int_0^t \rho(X_s)ds} |\nabla f|(X_t) \right)$$

Remarks:

- find an alternative proof in abstract measure space in Braun-Habermann-Sturm;
- formally we may hope that

$$\int_0^t \rho(X_s)ds \sim t\mu(\rho)$$

  and thus C_p could be close $\frac{1}{\mu(\rho)}$.
  However the control is not so direct and thus the previous argument cannot be done so simply!

Proof of the proposition: coupling

Recall that

$$\begin{aligned}(\nabla V(z) - \nabla V(z')).(z - z') &= \int_0^1 (z - z').\mathrm{Hess}V(sz + (1-s)z')(z - z')d\\ &\geq \int_0^1 \rho(sz + (1-s)z')|z - z'|^2 ds\end{aligned}$$

and by considering the same brownian motions for X_t starting from different points we have

$$\begin{aligned}d|X_t^x - X_t^y|^2 &= -2(X_t^x - X_t^y).(\nabla V(X_t^x) - \nabla V(X_t^Y))\\ &\leq -2\int_0^1 \rho(sX_t^x) + (1-s)X_t^y)ds\,|X_t^x - X_t^y|^2\end{aligned}$$

so that by Gronwall's Lemma

$$|X_t^x - X_t^y| \leq |x - y|\, e^{-\int_0^t \int_0^1 \rho(sX_u^x + (1-s)X_u^Y)dsdu}$$

It is then simple

$$\begin{aligned} |P_t f(x) - P_t f(y)| &\leq \mathbb{E}\left(|f(X_t^x - X_t^y|\right) \\ &\leq |x-y|\mathbb{E}\left(|\nabla f(z_t)|\, e^{-\int_0^t \int_0^1 \rho(sX_u^x + (1-s)X_u^Y) ds du}\right) \end{aligned}$$

for some z_t sandwiched by X_t^x and X_t^y.

It remains then to let y tends to x, remarking that $x \to X_t^x$ is continuous, and ρ also.

Some tools

We will need also the following lemma

Lemma

Let C be a dense subset of $L^2(\mu)$. Suppose that there exists $c > 0$ such that $\forall f \in \mathcal{C}$, a constant c_f

$$forall t, \qquad Var_\mu(P_t f) \leq c_f e^{-ct}$$

then

$$C_p \leq \frac{1}{c}.$$

This is in fact a simple consequence of the fact that $t \to \log \int (P_t f)^2 d\mu + ct$ is a bounded convex and thus necessarily non increasing function.

We have thus to control $\mathrm{Var}_\mu(P_t f)$, but by our proposition on gradient commutation property we have for all 1-lipschitz function

$$\begin{aligned} \mathrm{Var}_\mu(P_t f) &= 2\int_t \infty \int |\nabla P_s f|^2 d\mu ds \\ &\leq 2\int_t \infty \mathbb{E}_\mu \left(e^{-2\int_0^s \rho(X_u)du} \right) ds \end{aligned}$$

so that if we are able to show

$$\mathbb{E}_\mu \left(e^{-2\int_0^s \rho(X_u)du} \right) \leq Ce^{-cs}$$

we have

$$\mathrm{Var}_\mu(P_t f) \leq \frac{2C}{c} e^{-ct}$$

and the previous lemma gives us Poincaré inequality!

Transportation-Information inequalities (WI)

The following result of G-Léonard-Wu-Yang is crucial

Theorem

1. *If a Poincaré inequality holds with constant C then for all bounded function u we have*

$$\mathbb{E}_\mu\left(e^{\int_0^t (u(X_s)ds-\mu(u))ds}\right) \leq e^{\frac{Ct}{2Osc(u)^2}}$$

2. *If $\rho(x) \geq \rho_0 > 0$ then for all Lipschitz function u*

$$\mathbb{E}_\mu\left(e^{\int_0^t (-u(X_s)+\mu(u))ds}\right) \leq e^{\frac{t}{2\rho_0\|u\|^2_{Lip}}}$$

it leads to an improved inequality if we suppose moreover that $u \geq u_0$ and that $\mu(u) > 0$: forall $0 < \varepsilon < 1$

$$\mathbb{E}_\mu \left(e^{-\int_0^t u(X_s)ds} \right) \leq 2e^{-C(\varepsilon)t}$$

where

$$C(\varepsilon) = \min \left(\varepsilon\mu(u), u_0 + ((1-\varepsilon)\tilde{c}\mu(u)/\|u\|_c)^2 \right)$$

where $\|u\|_c$ denotes the oscillation or the lipschitzian norm, and $\tilde{c}$ the Poincaré constant or the inverse of the ρ_0.

It leads us to the following main results

main results

Theorem

Suppose that $\rho(x) \geq \rho_0 > 0$ *then*

$$C_P \leq \frac{1}{\rho_0 + \varepsilon'}$$

where

$$\varepsilon' = \left((\mu(\rho) - \rho_0) + \frac{1}{\rho_0} \|\rho\|^2 \right) \left(1 - \sqrt{1 - \frac{(\mu(\rho) - \rho_0)^2}{((\mu(\rho) - \rho_0) + \frac{1}{\rho_0} \|\rho\|^2)^2}} \right).$$

Remarks:

1. it leads in fact to a self-improvement of the Bakry-Emery's result!
2. it can even work in the logconcave situation, as in this case, using Bobkov's result a Poincaré inequality (with a bad constant is available).
3. comparison with Veyserre's result relating the Poincaré constant to $\mu(1/\rho)$ is not so easy, however in his case, $1/\rho$ has to be integrable!

Wasserstein contraction results

Let us recall the definition of the Wasserstein distance

$$W_p^p(\nu,\mu) := \inf_{\|X\sim\nu, Y\sim\mu\}} \mathbb{E}(|X-Y|^p).$$

Recall also Von Renesse-Sturm's result

Theorem

The following are equivalent

1. $CD(\rho_0,\infty)$
2. *for all* x, y,
$$W_p(P_t^*\delta_x, P_t^*\delta_y) \le e^{-\rho_0 t}|x-y|.$$

so that if we wish to get a better rate with variable curvature bounds, we may only get

$$W_p(P_t^*\delta_x, P_t^*\delta_y) \le Ce^{-(\rho_0+\varepsilon')t}|x-y|.$$

We will in fact follow the same approach but we are faced to slightly different problems. We have at first to reinforce slightly the curvature condition to

$$(\nabla V(z) - \nabla V(z')).(z - z') \geq (\kappa(z) + \kappa(z'))|z - z'|^2$$

which implies $2\kappa(x) \leq \rho(x)$ but will be efficient for the coupling technique as we get

$$|X_t^x - X_t^y| \leq |x - y| \, e^{-\int_0^t (\kappa(X_s^x) + \kappa(X_s^y)) ds}$$

and thus

$$W_1(P_t^* \delta_x, P_t^* \delta_y) \leq |x - y| \mathbb{E} \left(e^{-\int_0^t (\kappa(X_s^x) + \kappa(X_s^y)) ds} \right)$$

so that the main difficulties are the dependence in x and y. It needs ultracontractive bounds to get a result.