The Baum-Connes conjecture, localisation of categories and quantum groups

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Contents

Chapter 1

The Baum-Connes conjecture

1.1 Proper actions

Let G be a topological group, and H its compact subgroup. Let X be a paracompact Hausdorff topological space with a G-action, such that the quotient X/G is paracompact and Hausdorff.

Definition 1.1. The action $G \times X \to X$ is *proper* if for every point $p \in X$ there is a triple (U, H, ρ), such that U is an open set in X with $p \in U$ and $gu \in U$ for every $(g, u) \in G \times U$, $\rho\colon U \to G/H$ is a G-map.

Let $S := \rho^{-1}(eH)$, where $e \in G$ is a unit of G. Then $G \times S$ is a H-space by

$$(g, s)h = (gh^{-1}, hs),$$

and there is a map

$$G \times_H S \longrightarrow U \xrightarrow{\rho} G/J$$

$$(g, s) \longmapsto gs$$

1.2 Universal G-space for proper actions

Recall that if X, Y are G-spaces, then a G-map from X to Y is a continuous G-equivariant map $f\colon X \to Y$

$$f(gp) = gf(p), \quad a \in G,\ p \in X.$$

Two G-maps $f_0, f_1\colon X \to Y$ are G-homotopic if they are homotopic through G-maps, i.e. there exists a homotopy $\{f_t\}$, $0 \le t \le 1$ with each f_t a G-map.

Definition 1.2. A G-space X is **proper** if

- X is paracompact and Hausdorff,
- the quotient space X/G (with the quotient topology) is paracompact and Hausdorff.
- for each $p \in X$ there exists a triple (U, H, ρ) such that
  1. U is an open neighbourhood of p in X with $gu \in U$ for all $g \in G$, $u \in U$,

2. H is a compact subgroup of G,
3. $\rho\colon U \to G/H$ is a G-map from U to G/H.

Proposition 1.3 (Chabert, Echterhoff, Meyer)**.** *If X is a locally compact Hausdorff second countable G-space, then X is proper if and only if the map*

$$G \times X \to X \times X, \quad (g, x) \mapsto (gx, x)$$

is proper (i.e. the preimage of any compact set in $X \times X$ is compact).

Definition 1.4. A **universal G-space for proper actions**, denoted $\underline{\mathrm{E}}G$ is a proper G-space such that if X is any proper G-space, then there exists a G-map $f\colon X \to \underline{\mathrm{E}}G$ and any two G-maps from X to $\underline{\mathrm{E}}G$ are G-homotopic.

Lemma 1.5. *There exists universal G-space for proper actions.*

The space $\underline{\mathrm{E}}G$ is unique up to homotopy. Indeed, if $\underline{\mathrm{E}}G$ and $(\underline{\mathrm{E}}G)'$ are both universal examples for proper actions of G, then there exists G-maps

$$f\colon \underline{\mathrm{E}}G \to (\underline{\mathrm{E}}G)'$$
$$f'\colon (\underline{\mathrm{E}}G)' \to \underline{\mathrm{E}}G$$

with $f' \circ f$ and $f \circ f'$ G-homotopic to the identity maps of $\underline{\mathrm{E}}G$ and $(\underline{\mathrm{E}}G)'$ respectively. Moreover f and f' are unique up to homotopy.

There is a following set of axioms for $\underline{\mathrm{E}}G$

1. Y is a proper G-space,

2. if H is any compact subgroup of G then there exists $p \in Y$ with $hp = p$ for all $h \in H$

3. if we view $Y \times Y$ as a G-space with action

$$g(y_0, y_1) = (gy_0, gy_1),$$

$$\rho_0, \rho_1\colon Y \times Y \to Y, \quad \rho_0(y_0, y_1) = y_0,\ \rho_1(y_0, y_1) = y_1,$$

then ρ_0 and ρ_1 are G-homotopic.

Lemma 1.6. *If Y satisfies the axioms 1,2,3, then Y is an $\underline{\mathrm{E}}G$.*

Example 1.7.

- If G is compact, then $\underline{\mathrm{E}}G = \mathrm{pt}$.

- If G is a Lie group with $\pi_0(G)$ finite, then $\underline{\mathrm{E}}G = G/H$, where H is maximal compact subgroup of G.

- If G is a p-adic group then $\underline{\mathrm{E}}G$ is the affine Bruhat-Tits building for G, denoted by βG.

  Affine Bruhat-Tits building for $\mathrm{SL}(2, \mathbb{Q}_p)$ is the $(p+1)$-regular tree, that is a tree with exactly $p+1$ edges at each vertex.

- If Γ is (countable) discrette group, then

$$\underline{\mathrm{E}}\Gamma = \{f\colon \Gamma \to [0,1] \mid \{\gamma \in \Gamma \mid f(\gamma) \neq 0\} \text{ is finite, } \sum_{\gamma\in\Gamma} f(\gamma) = 1\}$$

The action is given by $(\beta f)(\gamma) = f(\beta^{-1}\gamma)$ for $\beta, \gamma \in \Gamma$, $f\colon \Gamma \to [0,1]$. The space $\underline{\mathrm{E}}\Gamma$ is topologized by the metric

$$d(f,h) = \left(\sum_{\gamma\in\Gamma} |f(\gamma) - h(\gamma)|^2\right)^{\frac{1}{2}}.$$

Definition 1.8. A subset $\Delta \subset \underline{\mathrm{E}}G$ is G-compact if

1. $gx \in \Delta$ for all $g \in G$, $x \in \Delta$,

2. the quotient space G/Δ is compact.

Set

$$\mathrm{K}_j^G(\underline{\mathrm{E}}G) = \varinjlim_{\Delta\subset\underline{E}G,\ \Delta \text{ is } G\text{-compact}} \mathrm{K}_j^G(\Delta).$$

$\mathrm{K}_j^G(\underline{E}G)$ is the equivariant K-homology of $\underline{\mathrm{E}}G$ with G-compact supports. There is a map

$$\mu\colon\ \mathrm{K}_j^G(\underline{\mathrm{E}}) \to \mathrm{K}_j(C_r^*G)$$

$$(\mathcal{H}, \psi, \pi, T) \mapsto \mathrm{Index}(T).$$

If X is a proper G-space with quotient X/G compact, then

$$\mathcal{E}_j^G(X) := \mathcal{E}_G^j(C_0(X)) = \{(\mathcal{H}, \psi, \pi, T)\}$$

and

$$\mathrm{K}_j^G(X) := \mathrm{KK}_G^j(C_0(X), \mathbb{C}) = \{(\mathcal{H}, \psi, \pi, T)\}/\sim, \quad j = 0, 1,$$

is the Kasparov equivariant K-homology of X. If X, Y are proper G-spaces with compact quotient spaces X/G, Y/G, and $f\colon X \to Y$ is a continuous G-equivariant map, then $f^*\colon C_0(Y) \to C_0(X)$, $f^*(\alpha) = \alpha \circ f$ induces a homomorphism of abelian groups $f_*\colon\ \mathrm{K}_j^G(X) \to \mathrm{K}_j^G(Y)$,

$$(\mathcal{H}, \psi, \pi, T) \mapsto (\mathcal{H}, \psi \circ f^*, \pi, T).$$

The map

$$\mu\colon\ \mathrm{K}_j^G(\underline{\mathrm{E}}) \to \mathrm{K}_j(C_r^*G)$$

is natural, that is there is commutativity in the diagram

$$\begin{array}{ccc} \mathrm{K}_j^G(X) & \xrightarrow{\ f_*\ } & \mathrm{K}_j^G(Y) \\ & \searrow \quad \swarrow & \\ & \mathrm{K}_j(C_r^*G) & \end{array}$$

1.3 The Baum-Connes Conjecture

Conjecture 1 (P. Baum, A. Connes, 1980). *Let G be a locally compact Hausdorff second countable topological group. Then*

$$\mu\colon\ \mathrm{K}_j^G(\underline{\mathrm{E}}) \to \mathrm{K}_j(C_r^*G)$$

is an isomorphism for $j = 0, 1$.

It is known that the conjecture is true for

- compact groups,
- abelian groups,
- Lie groups ($\pi_0(G)$ finite),
- p-adic groups,
- adelic groups.

It is not known if the conjecture is true for all discrete groups.

Theorem 1.9 (T. Schick). *Let B_n be tha Braid group on n strands, $n \geq 2$. Then BC is true for B_n.*

Theorem 1.10 (N. Higson, G. Kasparov). *If Γ is a discrete group which is amenable (or a-t-menable), then BC is true for Γ.*

Theorem 1.11 (I. Mineyev, G. Yu, V. Lafforgue). *If Γ is a discrete group which is hyperbolic (in Gromov's sense), then BC is true for Γ.*

Theorem 1.12 (V. Lafforgue). *If Γ is any discrete co-compact subgroup of $\mathrm{SL}(3, \mathbb{R})$, then BC is true for Γ.*

Theorem 1.13 (G. Kasparov, P. Julg). *If Γ is any discrete subgroup of $\mathrm{SO}(n, 1)$, $\mathrm{SU}(n, 1)$ or $\mathrm{Sp}(n, 1)$, then BC is true for Γ.*

There are following corollaries of the Baum-Connes conjecture.

- Novikov conjecture
- Stable Gromov-Lawson-Rosenberg conjecture
- Idempotent conjecture
- Kadison-Kaplansky conjecture
- Mackey analogy
- Construction of the discrete series via Dirac induction (Parthasarathy, Atiyah, Schmidt)
- Homotopy invariance of ρ-invariants (Keswani, Piazza, Schick)

1.3.1 The conjecture with coefficients

Definition 1.14. A G-C*-algebra is a C*-algebra A with a given continuous action of G

$$G \times A \to A$$

by C*-algebra automorphisms. The continuity condition is: for each $a \in A$

$$G \to A,\ g \mapsto ga$$

is continuous map from G to A.

Let A be a G-C*-algebra. Form the reduces crossed product C*-algebra $C_r^*(G, A)$. The goal is to determine $\mathrm{K}_j(C_r^*(G, A))$. Let $\mathrm{K}_j^G(\underline{E}G, A)$ denote the equivariant K-homology of $\underline{E}G$ with G-compact supports and coefficients A, that is

$$\mathrm{K}_j^G(\underline{E}G, A) := \lim_{\longrightarrow \Delta \subset \underline{E}G,\ \Delta\ G-\text{compact}} \mathrm{KK}_G^j(C_0(\Delta), A).$$

Conjecture 2 (P. Baum, A. Connes 1980)**.** *Let G be a locally compact Hausdorff second countable topological group, and let A be any G-C*-algebra, then*

$$\mu\colon\ \mathrm{K}_j^G(\underline{E}G, A) \to \mathrm{K}_j(C_r^*(G, A))$$

is an isomorphism for $j = 0, 1$.

Let Γ be a finitely presented discrete group which contains an expander in its Cayley graph. Such a Γ is a counter-example to the conjecture with coefficients. M. Gromov outlined a proof that such a Γ exists. A number of mathematicians are now filling in the details.

Definition 1.15. We say that the group G is *exact* if for every exact sequence of C*-algebras

$$0 \to I \to A \to B \to 0$$

the sequence

$$0 \to C_r^*(G, I) \to C_r^*(G, A) \to C_r^*(G, B) \to 0$$

is exact.

Remark 1.16. It is very hard to find an example of a group which is not exact. Gromov gave an example of a group Γ which *Cayley graph* contains an *expander graph.* Such group will not be exact. Gromov's group Γ will be also a counterexample to the Baum-Connes conjecture with coefficients. Consider a Stone-Čech compactification $\beta\Gamma$. Then we can identify $C(\beta\Gamma)$ with $l^\infty(\Gamma)$, and there is an exact sequence

$$0 \to C_0(\Gamma) \to C(\beta\Gamma) \to C(\beta\Gamma - \Gamma) \to 0,$$

which after applying reduced crossed product $\rtimes_r \Gamma$ will not be exact.

1.4 Assembly map

Let $\Delta \subset X$ be a proper G-space.

Definition 1.17. We say that Δ is G-compact if Δ is G-invariant and Δ/G is compact and Hausdorff.

We define an equivariant K-homology of $\underline{\mathrm{E}\,G}$ by means of Kasparov equivariant KK-theory

$$\mathrm{K}_j^G(\underline{\mathrm{E}\,G}) = \operatorname{colim}_{\substack{\Delta \in \underline{\mathrm{E}\,G} \\ G\text{-compact}}} \mathrm{KK}_G^j(C_0(\Delta), \mathbb{C}), \tag{1.1}$$

$$\mathrm{K}_j^G(\underline{\mathrm{E}\,G}, A) = \operatorname{colim}_{\substack{\Delta \in \underline{\mathrm{E}\,G} \\ G\text{-compact}}} \mathrm{KK}_G^j(C_0(\Delta), A). \tag{1.2}$$

If A, B are separable G-C*-algebras, then there is a *Kasparov descent map*

$$\mathrm{KK}_G^j(A, B) \to \mathrm{KK}^j(C_r^*(G, A), C_r^*(G, B)). \tag{1.3}$$

Let M be a C^∞-manifold, $\partial M = \emptyset$, and

$$\begin{array}{ccc} C^\infty(E^0) & \xrightarrow{D} & C^\infty(E^1) \\ \subset\downarrow & & \subset\downarrow \\ L^2(E^0) & \xrightarrow{D} & L^2(E^1) \end{array}$$

an elliptic differential operator. It may have to be normalized

$$D \mapsto D(\mathrm{Id} + D^*D)^{-\frac{1}{2}}.$$

Then

$$\mathrm{Index}(D) \in \mathrm{KK}^0(C_0(M), \mathbb{C}).$$

In the definition of assembly map

$$\mu\colon\ \mathrm{K}_j^G(\underline{\mathrm{E}}G) \to \mathrm{K}_j(C_r^*(G))$$

we use the Kasparov product and descent map. Recall that if A, B, D are separable G-C*-algebras, then there is a product

$$\mathrm{KK}_G^i(A, B) \otimes \mathrm{KK}_G^j(B, D) \to \mathrm{KK}_G^{i+j}(A, D), \quad i, j = 0, 1,$$

and descent map

$$\mathrm{KK}_G^j(A, B) \to \mathrm{KK}^j(C_r^*(G, A), C_r^*(G, B)), \quad j = 0, 1.$$

Let X be a proper G-compact G-space. We define a map

$$\mathrm{KK}_G^j(C_0(X), \mathbb{C}) \to \mathrm{K}_j(C_r^*(G))$$

as the composition of Kasparov descent map

$$\mathrm{KK}_G^j(C_0(X), \mathbb{C}) \to \mathrm{KK}^j(C_r^*(G, X), C_r^*(G))$$

and Kasparov product with

$$\mathbf{1} = X \times \mathbb{C} \in \mathrm{K}_0(C_r^*(G, X)) = \mathrm{KK}^0(\mathbb{C}, C_r^*(G, X)).$$

Recall the definition of equivariant K-homology of $\underline{\mathrm{E}}G$

$$\mathrm{K}_j^G(\underline{E}G) := \lim_{\longrightarrow \Delta\subset \underline{E}G,\ \Delta\ G\text{-compact}} \mathrm{KK}_G^j(C_0(\Delta), \mathbb{C}).$$

For each G-compact $\Delta \subset \underline{E}G$ we have

$$\mu\colon\ \mathrm{KK}_G^j(C_0(\Delta), \mathbb{C}) \to \mathrm{K}_j(C_r^*(G)).$$

If Δ, Ω are two G-compact subsets of $\underline{E}G$ with $\Delta \subset \Omega$, then the diagram

$$\begin{array}{ccc} \mathrm{KK}_G^j(C_0(\Delta), \mathbb{C}) & \longrightarrow & \mathrm{KK}_G^j(C_0(\Omega), \mathbb{C}) \\ & \searrow \quad \swarrow & \\ & \mathrm{K}_j(C_r^*(G)) & \end{array}$$

commutes, so we obtain

$$\mu\colon\ \mathrm{K}_j^G(\underline{E}G) \to \mathrm{K}_j(C_r^*(G)).$$

If A is a G-C*-algebra then we define the equivariant K-homology of $\underline{E}G$ with coefficients in A by

$$\mathrm{K}_j^G(\underline{\mathrm{E}}G; A) := \lim_{\longrightarrow \Delta\subset \underline{E}G,\ \Delta\ G\text{-compact}} \mathrm{KK}_G^j(C_0(\Delta), A).$$

We define also a map

$$\mu\colon rK_j(\underline{\mathrm{E}}G; A) \to \mathrm{K}_j(C_r^*(G, A)).$$

as the composition of Kasparov descent map

$$\mathrm{KK}_G^j(C_0(X), A) \to \mathrm{KK}^j(C_r^*(G, X), C_r^*(G, A))$$

and Kasparov product with

$$\mathbf{1} = X \times \mathbb{C} \in \mathrm{K}_0(C_r^*(G, X)) = \mathrm{KK}^0(\mathbb{C}, C_r^*(G, X)).$$

For each G-compact $\Delta \subset \underline{E}G$ we have

$$\mu\colon\ \mathrm{KK}_G^j(C_0(\Delta), A) \to \mathrm{K}_j(C_r^*(G, A)).$$

If Δ, Ω are two G-compact subsets of $\underline{E}G$ with $\Delta \subset \Omega$, then the diagram

$$\begin{array}{ccc} \mathrm{KK}_G^j(C_0(\Delta), A) & \longrightarrow & \mathrm{KK}_G^j(C_0(\Omega), A) \\ & \searrow \quad \swarrow & \\ & \mathrm{K}_j(C_r^*(G, A)) & \end{array}$$

commutes, so we obtain

$$\mu\colon\ \mathrm{K}_j^G(\underline{E}G), A) \to \mathrm{K}_j(C_r^*(G, A)).$$

Let A, B be G-C*-algebras. We denote by KK_G a category of G-C*-algebras with morphisms $\mathrm{KK}_G(A, B)$. Let $\varphi \in \mathrm{KK}_G(A, B)$. On the left side of the Baum-Connes conjecture φ induces a map

$$\mathrm{K}_j^G(\underline{\mathrm{E}\,G}, A) \to \mathrm{K}_j^G(\underline{\mathrm{E}\,G}, B) \tag{1.4}$$

by the Kasparov product with φ.

For any subgroup $H \leq G$ there is a restriction map

$$\mathrm{KK}_G^0(A, B) \to \mathrm{KK}_H^0(A, B),\ \varphi \mapsto \varphi|_H. \tag{1.5}$$

Theorem 1.18. *Suppose φ is an equivalence when restricted to any compact subgroup $H \leq G$. Then the map 1.4 is an isomorphism.*

$$\begin{array}{ccc}
\mathrm{K}_j(\underline{\mathrm{E}\,G}, A) & \longrightarrow & \mathrm{K}_j(\underline{\mathrm{E}\,G}, B) \\
\| & & \| \\
\operatorname{colim}_{\substack{\Delta \subset \underline{\mathrm{E}\,G} \\ G\text{-compact}}} \mathrm{KK}^j_G(C_0(\Delta), A) & \longrightarrow & \operatorname{colim}_{\substack{\Delta \subset \underline{\mathrm{E}\,G} \\ G\text{-compact}}} \mathrm{KK}^j_G(C_0(\Delta), B)
\end{array}$$

Proof. It suffices to assume that X is G-compact proper G-space, because Kasparov product commutes with colim. For an H-space Y we have an induced G-space $G \times_H Y$, which is a quotient $G \times Y/H$ with respect to the following H-action

$$(g, y)h = (gh^{-1}, hy).$$

By the Mayer-Vietoris sequence, it suffices to prove the theorem for $X = G \times_H S$ with S compact. Indeed, any pullback diagram

$$\begin{array}{ccc}
A_0 \otimes_B A_1 & \longrightarrow & A_0 \\
\downarrow & & \downarrow \\
A_1 & \longrightarrow & B
\end{array}$$

gives rise to the six-term exact sequence

$$\begin{array}{ccccc}
\mathrm{KK}^0(A_0 \otimes_B A_1, \mathbb{C}) & \longleftarrow & \mathrm{KK}^0(A_0, \mathbb{C}) \oplus \mathrm{KK}^0(A_1, \mathbb{C}) & \longleftarrow & \mathrm{KK}^0(B, \mathbb{C}) \\
\downarrow & & & & \uparrow \\
\mathrm{KK}^1(B, \mathbb{C}) & \longrightarrow & \mathrm{KK}^1(A_0, \mathbb{C}) \oplus \mathrm{KK}^1(A_1, \mathbb{C}) & \longrightarrow & \mathrm{KK}^1(A_0 \otimes_B A_1, \mathbb{C})
\end{array}$$

Now replace $\mathbb{C}$ by D, and consider an equivariant case to get a six-term exact sequence for $\mathrm{KK}_G(-, D)$.

$$\begin{array}{ccccc}
\mathrm{KK}^0_G(A_0 \otimes_B A_1, D) & \longleftarrow & \mathrm{KK}^0_G(A_0, D) \oplus \mathrm{KK}^0_G(A_1, D) & \longleftarrow & \mathrm{KK}^0_G(B, D) \\
\downarrow & & & & \uparrow \\
\mathrm{KK}^1_G(B, D) & \longrightarrow & \mathrm{KK}^1_G(A_0, D) \oplus \mathrm{KK}^1_G(A_1, D) & \longrightarrow & \mathrm{KK}^1_G(A_0 \otimes_B A_1, D)
\end{array}$$

Using

$$\begin{array}{ccc}
C_0(U \cup V) & \longrightarrow & C_0(U) \\
\downarrow & & \downarrow \\
C_0(V) & \longrightarrow & C_0(U \cap V)
\end{array}$$

we reduce by the five-lemma to the case $X = G \times_H S$, with S compact.

Frobenius reciprocity:

$$\begin{array}{cccc}
G & \mathrm{Ind}^G_H(\psi) & \varphi & \mathrm{Hom}_G(\mathrm{Ind}^G_H(\psi), \varphi) \\
 & & & \downarrow \simeq \\
H & \psi & \varphi|_H & \mathrm{Hom}_H(\psi, \varphi|_H)
\end{array}$$

Frobenius reciprocity in KK_G:

$$\begin{array}{cccc} G & \mathrm{Ind}_H^G(A) & B & \mathrm{KK}_G^j(\mathrm{Ind}_H^G(A), B) \\ & & & \downarrow \simeq \\ H & A & B & \mathrm{KK}_H^j(A, B) \end{array}$$

Induction of H-C*-algebra is a G-C*-algebra $G \times_H A$.

$$\begin{array}{ccc} \mathrm{KK}_G^j(C_0(G \times_H S), A) & \longrightarrow & \mathrm{KK}_G^j(C_0(G \times_H S), B) \\ \simeq \downarrow & & \simeq \downarrow \\ \mathrm{KK}_H^j(C(S), A) & \longrightarrow & \mathrm{KK}_H^j(C(S), B) \end{array}$$

□

Nest-Meyer reformulation of the Baum-Connes conjecture with coefficients:

If $\varphi \in \mathrm{KK}_G^0(A, B)$ is an equivalence when restricted to any compact subgroup $H \leq G$, then

$$\varphi_* \colon \mathrm{K}_j(C_r^*(G, A)) \to \mathrm{K}_j(C_r^*(G, B))$$

is an isomorphism for $j = 0, 1$. Meyer and Nest proved that for each C*-algebra there exists a projective object P and a weak equivalence $P \to A$. The Baum-Connes conjecture is tru for projectives.

$$\begin{array}{ccc} \mathrm{KK}_G^j(\underline{\mathrm{E}\,G}, P) & \xrightarrow{\varphi_*} & \mathrm{KK}_G^j(\underline{\mathrm{E}\,G}, A) \\ \simeq \downarrow & & \downarrow \text{B-C map} \\ \mathrm{K}_j(C_r^*(G, P)) & \xrightarrow{\simeq} & \mathrm{K}_j(C_r^*(G, A)) \end{array}$$

1.5 Reduced crossed product algebra

Let G be a locally compact, second countable group, and A a G-C*-algebra. The goal is to compute the K-theory of the reduced crossed product algebra, $\mathrm{K}_j(C_r^*(G, A))$.

For a discrete group Γ define an agebra

$$\mathrm{F}\,\Gamma = \{\sum_{\gamma \in \Gamma} \lambda_\gamma[\gamma] \mid \operatorname{ord}\gamma < \infty,\ \lambda_\gamma \in \mathbb{C}\}$$

with addition

$$\left(\sum_{\gamma \in \Gamma} \lambda_\gamma[\gamma]\right) + \left(\sum_{\gamma \in \Gamma} \mu_\gamma[\gamma]\right) = \left(\sum_{\gamma \in \Gamma} (\lambda_\gamma + \mu_\gamma)[\gamma]\right)$$

and multiplication by $g \in \Gamma$

$$g\left(\sum_{\gamma \in \Gamma} \lambda_\gamma[\gamma]\right) = \left(\sum_{\gamma \in \Gamma} \lambda_\gamma[g\gamma g^{-1}]\right).$$

The Baum-Connes conjecture with coefficients says that there is an isomorphism

$$\underbrace{\mathrm{K}_j(\underline{\mathrm{E}\,G}, A)}_{\operatorname{colim}_{\substack{\Delta \subset \underline{\mathrm{E}\,G} \\ G\text{-compact}}} \mathrm{KK}_G^j(C_0(\Delta), A)} \xrightarrow{\simeq} \mathrm{K}_j(C_r^*(G, A))$$

Meyer-Nest conjecture says that a weak equivalence $\varphi \in \mathrm{KK}^0_G(A, B)$ gives an isomorphism

$$\varphi_*\colon\ \mathrm{K}_j(C^*_r(G, A)) \to \mathrm{K}_j(C^*_r(G, B)), \quad j = 0, 1.$$

For each G-C*-algebra A there is a weak equivalence $\varphi\colon P \to A$ with P projective, for which the Baum-Connes conjecture is true. Hence we have a diagram

$$\begin{CD}
\mathrm{K}^G_j(\underline{\mathrm{E}\,G}, P) @>{\varphi_*}>{\simeq}> \mathrm{K}^G_j(\underline{\mathrm{E}\,G}, A) \\
@V{\simeq}VV @VV{\text{BC conjecture} \Longrightarrow \simeq}V \\
\mathrm{K}_j(C^*_r(G, P)) @>{\varphi_*}>{\text{MN-conjecture} \Longrightarrow \simeq}> \mathrm{K}_j(C^*_r(G, A))
\end{CD}$$

Denote

$$L^2(G, A) = \{f\colon G \to A \mid \int_g g^{-1}(f(g)^* f(g))dg\}.$$

If we take a dicrete group Γ, then there is a following notational convention for $\gamma \in \Gamma$, $a \in A$

$$\begin{aligned}
a[\gamma] &= f\colon \Gamma \to A,\ f(\gamma) = a,\ f(g) = 0 \text{ for } g \neq \gamma, \\
(a[\gamma])^* &= [\gamma^{-1}]a^*, \\
(a_1[\gamma_1])(a_2[\gamma_2]) &= a_1(\gamma_1 a_2)[\gamma_1\gamma_2].
\end{aligned}$$