

# Stable Marriage with Multi-Modal Preferences

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We thoroughly study a generalized version of the famous STABLE MARRIAGE problem, now based on multi-modal preference lists. The central twist herein is to allow each agent to rank its potentially matching counterparts based on more than one “evaluation mode” (e.g., more than one criterion); thus, each agent is equipped with multiple preference lists, each ranking the counterparts in a possibly different way. We introduce and study three natural concepts of stability, investigate their mutual relations and focus on computational complexity aspects with respect to computing stable matchings in these new scenarios. Mostly encountering computational hardness (NP-hardness), we can also spot few islands of tractability and make a surprising connection to the GRAPH ISOMORPHISM problem.

Additional Key Words and Phrases: Stable matching; concepts of stability; multi-layer (graph) models; NP-hardness; parameterized complexity analysis; exact algorithms

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## 1 INTRODUCTION

Information about the same “phenomenon” can come from different, possibly contradictory, sources. For instance, when evaluating applicants for an open position, data on their experience and professional achievements may give rise to a different ranking of the applicants than data on their formal qualifications and degrees. In other words, one has to deal with a multi-modal data scenario. Clearly, to make an objective and rational decision, it makes sense to take into account several information resources in order to achieve best possible results. In this work we systematically apply this point of view to the STABLE MARRIAGE problem [27]; a key observation here is that several natural and well-motivated “multi-modal variants” of STABLE MARRIAGE arise. We investigate the complexity of computing matchings that are stable according to the correspondingly varying definitions.

In the classic (conservative) STABLE MARRIAGE problem [27], we are given two disjoint sets  $U$  and  $W$  of  $n$  agents each, where each of the agents has a *strict preference list* that ranks *every* member of the other set. The goal is to find a bijection (which we call a *matching*) between  $U$  and  $W$  that

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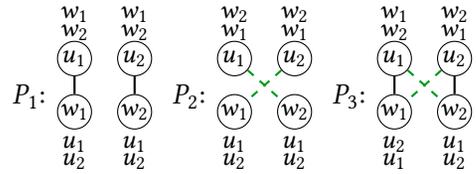
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does not contain a *blocking pair*, i.e. a pair that can endanger the stability of the matching. A pair of agents is *blocking* a matching if they are not matched to each other but prefer each other to their respective partners in the matching.

Gale and Shapley [27] introduced the STABLE MARRIAGE problem in the fields of Economics and Computer Science in the 1960s. One of their central results was that every STABLE MARRIAGE instance with  $2n$  agents admits a stable matching, which can be found by their algorithm in  $O(n^2)$  time. Since then STABLE MARRIAGE has been intensively studied in Economics, Computer Science, and Social and Political Science [29, 31, 32, 37, 39]. Practical applications of STABLE MARRIAGE (and its variants) include partnership issues in various real-world scenarios, matching graduating medical students (so-called residents) with hospitals [40], students with schools [1, 9], and organ donors with patients [42, 49], and the design of content delivery systems [38] and other distributed markets [50].

The original model of STABLE MARRIAGE assumes, roughly speaking, that there is a (subjective) criterion and that each agent has a single preference list depending on this criterion. In typically complex real-world scenarios, however, there are usually multiple aspects one takes into account when making a decision. For instance, if we consider the classical partnership scenario, then there could be different criteria such as working hours, family background, physical appearance, health, hobbies, etc. In other words, we face a much more complex multi-modal scenario. Accordingly, the agents may have multiple preference lists, each defined by a different criterion; we call each of these criteria a *layer* (also see Example 2.4 for another interpretation of layers). For an illustration, let us consider the following stable marriage example with two sets of two agents each, denoted as  $\{u_1, u_2\}$  and  $\{w_1, w_2\}$ , and three layers of preferences, denoted as  $P_1, P_2$ , and  $P_3$ .

In the diagram the preferences are depicted right above (resp. right below) the corresponding agents; preferences are represented through vertical lists where more preferred agents are put above the less preferred ones. In the first layer, all agents from the same set have the same preference list, i.e. both  $u_1$  and  $u_2$  rank  $w_1$  higher than  $w_2$  while both  $w_1$  and  $w_2$  rank  $u_1$  higher than  $u_2$ . Similarly, in the second layer,



both  $u_1$  and  $u_2$  rank  $w_2$  higher than  $w_1$  while both  $w_1$  and  $w_2$  rank  $u_1$  higher than  $u_2$ . In the last layer, the preference lists of the two agents from the same set are reverse to each other. For instance,  $u_1$  ranks  $w_1$  higher than  $w_2$ , which is opposite to  $u_2$ . In terms of the classic STABLE MARRIAGE problem, we will have three independent instances, one for each layer. The corresponding stable matching(s) for each instance are depicted through the edges between the agents (we use different styles of lines to distinguish different stable matching). For instance, the first layer admits exactly one stable matching, which matches  $u_1$  with  $w_1$ , and  $u_2$  with  $w_2$ . Yet, if we want to take all these layers jointly into account, then we need to extend the traditional concept of stability.

With multiple preference lists for each agent, there are three natural ways to extend the original stability notion, each leading to a new concept. In the following, we intuitively describe these concepts and defer the formal definitions and motivating examples to Section 2. Assume that each agent has  $\ell$  (possibly different) preferences lists. All three new concepts are defined for a certain integer threshold  $\alpha$  with  $1 \leq \alpha \leq \ell$ , which quantifies “the strength” of stability.

- The first concept, called  *$\alpha$ -layer global stability*, extends the original stability concept in a straightforward way. It assumes that the matched pairs agree on a set  $S$  of  $\alpha$  layers where no unmatched pair is blocking the matching in any layer from  $S$ . In our example, the matching  $M_1 = \{\{u_1, w_1\}, \{u_2, w_2\}\}$  is stable in the first and the last layer, and thus it is 2-layer globally stable.
- The second concept, called  *$\alpha$ -layer pair stability*, restricts the “blocking ability” of the unmatched pairs. It *forbids* each unmatched pair to block more than  $\ell - \alpha$  layers. Considering again our running

example, we can verify that the 2-layer globally stable matching  $M_1$  is also 2-layer pair stable as each unmatched pair is blocking at most one layer. Indeed, we will see that  $\alpha$ -layer pair stability is strictly weaker than  $\alpha$ -layer global stability (Proposition 3.1 and Example 3.2).

– The third concept, called  *$\alpha$ -layer individual stability*, focuses on the “willingness” of an agent to stay with its partner. It requires that for each unmatched pair, at least one of the agents prefers to stay with its partner in at least  $\alpha$  layers. In our example, matching  $M_1$  is also 2-layer individually stable. Thus, it is tempting to assume that  $\alpha$ -layer individual stability also generalizes  $\alpha$ -layer global stability. This is, however, not true as the matching  $M_2 = \{\{u_1, w_2\}, \{u_2, w_1\}\}$  is 2-layer globally stable but not 2-layer individually stable. Neither does the latter imply the former, so both concepts are incomparable. We refer to Example 3.4 for more explanations.

**Related work.** Our framework unifies and generalizes a couple of models known in the literature on stable matchings. Very recently, the STABLE MARRIAGE problem with multiple preference lists per agent has been studied by Aziz et al. [3] and Miyazaki and Okamoto [45]. Aziz et al. [3] introduced *certain stability* and *possible stability* for the case when there is a specific probability distribution over preference profiles. These two concepts are (mathematically) equivalent to our all-layer global stability and 1-layer global stability, respectively. Thus, our  $\alpha$ -layer global stability generalizes these two notions and defines a spectrum of solution concepts in-between. A 1-layer globally stable matching always exists. Aziz et al. showed that deciding all-layer global stability is NP-hard for  $\alpha = \ell = 16$  layers; we strengthen this result by requiring only two layers (Theorem 4.6). In their work, Miyazaki and Okamoto [45] aimed to find a *jointly stable* matching that is stable in all layers, which turns out to be equivalent to our all-layer global stability as well. The authors showed that for  $\alpha = \ell = 2$ , checking all-layer global stability is NP-hard. We strengthen this by showing hardness for arbitrary  $\alpha \leq \ell$  (Proposition 5.3). When one side of the preferences is single-layered (see Section 6.1), Miyazaki and Okamoto observed that finding an all-layer globally stable matching can be reduced to finding a super stable matching for an instance where one side of the preferences may be just a partial order; the latter is known to be polynomial-time solvable by using an extended variant of Irving’s algorithm [30, 39]. The same result can be achieved as a corollary of our Propositions 3.5 and 6.1 and by using Algorithm 1. Note that Algorithm 1 decides all-layer individual stability for the general case when the input preferences may be not single-layered.

Aggregating the preference lists of multiple layers into one (by comparing each pair of agents) and then searching for a “stable” matching for the agents with aggregated preferences is an intuitive approach to stable marriage with multi-modal preferences. However, the aggregated preferences may be intransitive or even cyclic. Addressing this situation, Farzadi et al. [25] considered a generalized variant of STABLE MARRIAGE where each agent  $u$  of one side, say  $U$ , has a strict preference list  $>_u$  (as in the original STABLE MARRIAGE), while each agent  $w$  of the other side, say  $W$ , may order each possible pair of partners separately, expressed by a subset  $B_w \subseteq U \times U$  of ordered pairs. They called a matching  $M$  *stable* if no unmatched pair  $\{u, w\}$  satisfies “ $w >_u M(u)$  and  $(u, M(u)) \in B_w$ ”. It turns out that our individual stability and their stability concept for a *more generalized* case where both sides of the agents may have intransitive preferences are related, and we can use one of their results as a subroutine. In a way, our analysis provides a more fine-grained view, since we consider a richer model and thus are able to discuss how certain assumptions on elements of this model (e.g., the number of layers, the threshold value  $\alpha$ , etc.) affect the computational complexity of the problem. Aziz et al. [2] considered a variant of STABLE MARRIAGE, where each agent has a probability of preferring one agent over another. Assigning a probability of 1 to either  $(x, y)$  or  $(y, x)$  for each  $x$  and  $y$ , their variant is closely related to the one of Farzadi et al..

Weems [53] considered STABLE MARRIAGE where each agent has two preference lists that are the reverse of each other; this is a special case of our  $\alpha$ -layer global stability model for  $\alpha = \ell = 2$ . He

provided a polynomial-time algorithm to find a matching that is stable in both layers. We refer to several expositions [8, 29, 34, 35, 37, 39] for a broader overview on STABLE MARRIAGE and related problems.

**Our contributions.** We introduce and thoroughly study three general concepts of stability for STABLE MARRIAGE with multi-modal preferences. In Section 2 we formally define these concepts, *global stability*, *pair stability*, and *individual stability*, and provide motivating and illustrating examples. In Section 3, we study the relations between the three concepts and show that pair stability is the least restrictive form while global and individual stability are in general incomparable (also see Figure 1 on the right for a more refined picture). In Section 4, we consider the special case of *all-layer stability* ( $\alpha = \ell$ ) for our three stability concepts. On the one hand, we provide a polynomial-time algorithm for checking individual stability for an arbitrarily large number of preference lists. On the other hand, through an involved construction, we show that it is NP-hard to determine whether an all-layer globally stable (or all-layer pair stable) matching exists, whenever  $\alpha \geq 2$ . The hardness results demonstrate a complexity dichotomy for both global and pair stability since for  $\ell = 1$ , all three stability concepts are equivalent, and such stable matching(s) always exist and are polynomial-time computable. In Section 5, we investigate the case of finding stable matchings for less than all layers and only find NP-hardness. In particular, complementing the polynomial-time solvability result for checking individual stability when  $\ell = \alpha$ , we show that the problem becomes hard already when  $\ell = 3$  and  $\alpha = 2$ . In Section 6, we identify two special scenarios with strong but natural restrictions on the preference lists. For the first scenario we assume that one side of the agents has *single-layered preferences*, i.e. the preference list of each agent remains the same in all layers. Under such restrictions, pair stability and individual stability are equivalent, and stable matchings can be computed in polynomial time; for global stability we obtain W[1]-hardness (and also NP-hardness) and XP membership for the threshold parameter  $\alpha$ . In the second scenario, we assume that the preferences of all agents on each side are *uniform* in each layer, i.e. for each fixed layer and side all agents have the same preference list. When considering individual stability we find surprisingly tight connections to the GRAPH ISOMORPHISM problem. Table 1 gives a broad overview on our complexity results.

Due to space constraints we deferred the whole version of the proofs marked by “proof sketch” and the proofs for results marked with the asterisk  $\star$  to [16].

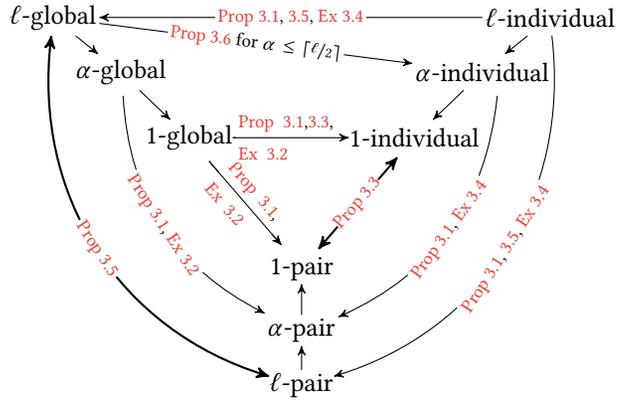


Fig. 1. Relations among the different multi-layer concepts of stability for different values of  $\alpha$ . Herein, an arc is to be read like an implication: one property implies the other.

In Section 5, we investigate the case of finding stable matchings for less than all layers and only find NP-hardness. In particular, complementing the polynomial-time solvability result for checking individual stability when  $\ell = \alpha$ , we show that the problem becomes hard already when  $\ell = 3$  and  $\alpha = 2$ . In Section 6, we identify two special scenarios with strong but natural restrictions on the preference lists. For the first scenario we assume that one side of the agents has *single-layered preferences*, i.e. the preference list of each agent remains the same in all layers. Under such restrictions, pair stability and individual stability are equivalent, and stable matchings can be computed in polynomial time; for global stability we obtain W[1]-hardness (and also NP-hardness) and XP membership for the threshold parameter  $\alpha$ . In the second scenario, we assume that the preferences of all agents on each side are *uniform* in each layer, i.e. for each fixed layer and side all agents have the same preference list. When considering individual stability we find surprisingly tight connections to the GRAPH ISOMORPHISM problem. Table 1 gives a broad overview on our complexity results.

Due to space constraints we deferred the whole version of the proofs marked by “proof sketch” and the proofs for results marked with the asterisk  $\star$  to [16].

## 2 DEFINITIONS AND NOTATIONS

For each natural number  $t$  by  $[t]$  we denote the set  $\{1, 2, \dots, t\}$ . Let  $U = \{u_1, \dots, u_n\}$  and  $W = \{w_1, \dots, w_n\}$  be two  $n$ -element disjoint sets of agents. There are  $\ell$  layers of preferences, where  $\ell$  is a positive integer. For each  $i \in [\ell]$  and each  $u \in U$ , let  $\succ_u^{(i)}$  be a linear order on  $W$  that represents

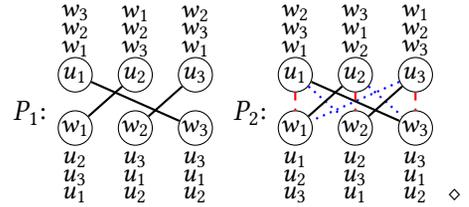
Table 1. The computational complexity of finding matchings stable according to our three introduced concepts— $\alpha$ -layer global stability,  $\alpha$ -layer pair stability, and  $\alpha$ -layer individual stability—for instances with  $2n$  agents and  $\ell$  layers. Unless stated explicitly, all results hold for each value of  $\alpha$  specified in the first column. The result marked by “ $\diamond$ ” has also been achieved by Aziz et al. [3] and Miyazaki and Okamoto [45] for two layers and 16 layers, respectively.

Parameters	global stability	pair stability	individual stability
Arbitrary			
$1 = \alpha$	$O(n^2)$ [Gale and Shapley]	$O(n^2)$ [Gale and Shapley]	$O(n^2)$ [Gale and Shapley]
$2 \leq \alpha = \ell$	NP-h $^\diamond$ [T. 4.6]	NP-h [C. 4.7]	$O(\ell \cdot n^2)$ [T. 4.5]
$\lfloor \ell/2 \rfloor < \alpha < \ell$	NP-h [P. 5.3]	NP-h [P. 5.5+P. 5.6]	NP-h for $3\alpha \leq 2\ell$ [T. 5.1]
$2 \leq \alpha \leq \lfloor \ell/2 \rfloor$	NP-h [P. 5.3]	NP-h [C. 5.4]	NP-h [T. 5.2]
Single-layered	NP-h for unbounded $\alpha$ [T. 6.3]	NP-h for $2 \leq \alpha \leq \lfloor \ell/2 \rfloor$ [C. 5.4]	NP-h for $2 \leq \alpha \leq \lfloor \ell/2 \rfloor$ [T. 5.2]
	W[1]-h & in XP for $\alpha$ [T. 6.3]		
	$O(\ell \cdot n^2)$ when $\alpha = \ell$ [T. 6.3]	$O(\ell \cdot n^3)$ when $\alpha > \lfloor \ell/2 \rfloor$ [P. 6.4]	$O(\ell \cdot n^3)$ when $\alpha > \lfloor \ell/2 \rfloor$ [P. 6.4]
Uniform			
$\alpha \geq \ell/2 + 1$	$O(\ell \cdot n)$ [P. 6.10]	?	$n^{O(\log(n))} + O(\ell \cdot n^2)$ [C. 6.9]
$\alpha = \ell/2$	$O(\ell \cdot n)$ [P. 6.10]	?	GRAPH ISOM.-hard [T. 6.8]

the ranking of agent  $u$  over all agents from  $W$  in layer  $i$ . Analogously, for each  $i \in [\ell]$  and each  $w \in W$ , the symbol  $\succ_w^{(i)}$  represents a linear order on  $U$  that encodes preferences of  $w$  in layer  $i$ . We refer to such linear orders as preference lists. A preference profile  $P_i$  of layer  $i \in [\ell]$  is a collection of the preference lists of all agents in layer  $i$ , that is,  $\{\succ_a^{(i)} \mid a \in U \cup W\}$ .

A *matching*  $M \subseteq \{\{u, w\} \mid u \in U \wedge w \in W\}$  is a set of pairwise disjoint pairs, i.e. for each two pairs  $p, p' \in M$  it holds that  $p \cap p' = \emptyset$ . If  $\{u, w\} \in M$ , then we also use  $M(u)$  to refer to  $w$  and  $M(w)$  to refer to  $u$ , and we say that  $u$  and  $w$  are their respective partners under  $M$ ; otherwise we say that  $\{u, w\}$  is an *unmatched pair*. **Example 2.1** below illustrates an example matching and recalls a graphical notation that we will use throughout the paper.

*Example 2.1.* Consider two sets of agents,  $U = \{u_1, u_2, u_3\}$  and  $W = \{w_1, w_2, w_3\}$ , and two layers,  $P_1$  and  $P_2$ . One can check that  $P_1$  has only one stable matching  $M_1 = \{\{u_1, w_3\}, \{u_2, w_1\}, \{u_3, w_2\}\}$ , and  $P_2$  has three stable matchings: (1)  $M_1$ , (2)  $M_2 = \{\{u_1, w_1\}, \{u_2, w_2\}, \{u_3, w_3\}\}$ , and (3)  $M_3 = \{\{u_1, w_2\}, \{u_2, w_3\}, \{u_3, w_1\}\}$ .



Let us now introduce two notions that we will use when defining various concepts of stability.

**Definition 2.2 (Dominating and blocking).** Let  $M$  be a matching over  $U \cup W$ . Consider an unmatched pair  $\{u, w\} \notin M$  and a layer  $i \in [\ell]$ . We say that  $\{u, w\}$  *dominates*  $\{u, v\}$  in layer  $i$  if  $w \succ_u^{(i)} v$ . We say that  $\{u, w\}$  is *blocking matching  $M$  in layer  $i$*  if (1)  $u$  is unmatched in  $M$  or  $\{u, w\}$  dominates  $\{u, M(u)\}$  in layer  $i$ , and (2)  $w$  is unmatched in  $M$  or  $\{u, w\}$  dominates  $\{w, M(w)\}$  in layer  $i$ .

For an arbitrary layer, say  $i$ , a matching  $M$  is *stable in layer  $i$*  if no unmatched pair is blocking  $M$  in layer  $i$ . We illustrate the concept of dominating and blocking pairs through **Example 2.1**. Consider the matching  $M_3 = \{\{u_1, w_2\}, \{u_2, w_3\}, \{u_3, w_1\}\}$  and profile  $P_1$  of layer 1. Here, pair  $\{u_1, w_3\}$  dominates both  $\{u_1, w_2\}$  (since  $u_1$  prefers  $w_3$  to  $w_2$ ) and  $\{u_2, w_3\}$  (since  $w_3$  prefers  $u_1$  to  $u_2$ ). Thus,  $\{u_1, w_3\}$  is a blocking pair and so it witnesses that  $M$  is not stable in profile  $P_1$ .

We are interested in matchings which are stable in multiple layers, *i.e.* we aim at generalizing the classic STABLE MARRIAGE problem [27, 29, 37, 39] which is defined for a single layer to the case of multiple layers.

At a high level, the ideas behind all our concepts are similar: in order to call a matching *stable* for multiple layers we require it to be stable in at least a certain, given number  $\alpha$  of layers ( $\alpha$  is a number indicating the “strength” of the stability). However, the three concepts require different levels of agreement with respect to which layers are required for stability. Informally speaking, on the one end of the spectrum we have a variant of stability where we require a global agreement of the agents regarding the set of  $\alpha$  layers for which the matching must be stable. On the other end of the spectrum we have a variant where we assume that the agents act independently: an agent  $a$  would deviate if it would find another agent, say  $b$ , such that  $a$  prefers  $b$  to its matched partner in some  $\alpha$  layers, and  $b$  prefers  $a$  to its matched partner in another, possibly different, set of  $\alpha$  layers. In the intermediate case, we require that a deviating pair must agree on the subset of layers which form the reason for deviation.

We formally define the three concepts below. Let there be  $\ell$  layers of preferences.

## 2.1 $\alpha$ -layer global stability

A matching  $M$  is  *$\alpha$ -layer globally stable* if there exist  $\alpha$  layers in each of which  $M$  is stable.

*Definition 2.3.* A matching  $M$  is  *$\alpha$ -layer globally stable* if there is a subset  $S \subseteq [\ell]$  of  $\alpha$  layers, where for each  $i \in S$  and each unmatched pair  $\{u, w\} \notin M$  at least one of the two conditions holds: (1)  $\{u, M(u)\}$  dominates  $\{u, w\}$  in layer  $i$ , or (2)  $\{w, M(w)\}$  dominates  $\{w, u\}$  in layer  $i$ .

The following example describes a scenario where  $\alpha$ -layer global stability appears to be useful.

*Example 2.4.* Assume that the preferences of the agents depend on external circumstances which are not known a priori. Assume that each layer represents a different possible state of the universe. If we want to find a matching that is stable in as many states of the universe as possible, then we need to find an  $\alpha$ -layer globally stable matching for the highest possible value of  $\alpha$ .  $\diamond$

Already for  $\alpha$ -layer global stability we see substantial differences compared to the original concept of stability for a single layer. It is guaranteed that such a matching always exists for  $\alpha = 1$ ; indeed this would be a matching that is stable in an arbitrary layer. However, one can observe that as soon as  $\alpha > 1$  an  $\alpha$ -layer globally stable matching might not exist (see [Example 3.2](#)).

## 2.2 $\alpha$ -layer pair stability

While  $\alpha$ -layer global stability requires that the agents globally agree on a certain subset of  $\alpha$  layers for which the matching should be stable, pair stability forbids each unmatched pair to block more than a certain number of layers. The formal definition, using the domination concept, is as follows:

*Definition 2.5.* A matching  $M$  is  *$\alpha$ -layer pair stable* if for each unmatched pair  $\{u, w\} \notin M$  there is a subset  $S \subseteq [\ell]$  of  $\alpha$  layers such that for each  $i \in S$  at least one of the two conditions holds: (1)  $\{u, M(u)\}$  dominates  $\{u, w\}$  in layer  $i$ , or (2)  $\{w, M(w)\}$  dominates  $\{w, u\}$  in layer  $i$ .

**Definition 2.5** can be equivalently formulated using a generalization of the concept of blocking pairs. Let  $\beta \in [\ell]$ . A pair  $\{u, w\}$  is  *$\beta$ -blocking*  $M$  if there is a subset  $S$  of  $\beta$  layers such that for each  $i \in S$ , pair  $\{u, w\}$  is blocking  $M$  in layer  $i$ .

**PROPOSITION 2.6** ( $\star$ ). *A matching  $M$  is  $\alpha$ -layer pair stable if and only if no unmatched pair is  $(\ell - \alpha + 1)$ -blocking  $M$ .*

The following example motivates  $\alpha$ -layer pair stability.

*Example 2.7.* Consider the case when the preferences of the agents depend on a context, yet the context is pair-specific. For instance, in matchmaking a woman may have different preferences over men depending on which country they will decide to live in. Thus, a pair of a man and a woman is blocking if they agree on certain conditions (*i.e.* layers), and if they will find each other more attractive than their current partners according to the agreed conditions.  $\diamond$

In Section 3, we show that  $\alpha$ -layer global stability implies  $\alpha$ -layer pair stability (Proposition 3.1). This, among other things, implies that for  $\alpha = 1$  an  $\alpha$ -layer pair stable matching always exists. However, as soon as  $\alpha \geq 2$  the existence is no longer guaranteed (see Example 3.2).

### 2.3 $\alpha$ -layer individual stability

*Definition 2.8.* A matching  $M$  is  $\alpha$ -layer *individually stable* if for each unmatched pair  $\{u, w\} \notin M$  there is a subset  $S \subseteq [\ell]$  of  $\alpha$  layers such that at least one of the two conditions holds: (1)  $\{u, M(u)\}$  dominates  $\{u, w\}$  in each layer of  $S$ , or (2)  $\{w, M(w)\}$  dominates  $\{w, u\}$  in each layer of  $S$ .

The following example illustrates a potential application in the domain of partnership agencies.

*Example 2.9.* Assume that each layer describes a single criterion for preferences. The preferences of each agent may differ depending on the criterion. For instance, the two sets of agents can represent, respectively, men and women, as in the traditional STABLE MARRIAGE problem. Different criteria may correspond, for instance, to the intelligence, sense of humor, physical appearance etc. Assume that an agent  $a$  will have no incentive to break his or her relationship with  $b$ , and to have an affair with  $c$  if he or she prefers  $b$  to  $c$  according to at least  $\alpha$  criteria. In order to match men with women so that they form stable relationships, one needs to find an  $\alpha$ -layer individually stable matching.  $\diamond$

Definition 2.8 can be equivalently formulated using a generalization of the concept of dominating pairs. Let  $\beta \in [\ell]$ . We say that a pair  $\{u, w\}$  is  $\beta$ -*dominating*  $\{u, w'\}$  if there is a subset  $R$  of  $\beta$  layers such that for each  $i \in R$  the pair  $\{u, w\}$  dominates  $\{u, w'\}$  in layer  $i$ .

PROPOSITION 2.10 ( $\star$ ). *A matching  $M$  is  $\alpha$ -layer individually stable if and only if no unmatched pair  $\{u, w\}$  exists that is both  $(\ell - \alpha + 1)$ -dominating  $\{u, M(u)\}$  and  $(\ell - \alpha + 1)$ -dominating  $\{w, M(w)\}$ .*

For  $\alpha = 1$  an  $\alpha$ -layer individually stable matching always exists (see Propositions 3.1 and 3.3); however, this is no longer the case when  $\alpha \geq 2$  (see Proposition 3.1 and Example 3.2).

In this paper, we investigate how the three concepts relate to each other, and study the algorithmic complexity of finding matchings that are stable according to the above definitions. Now, we formally define the search problem of finding an  $\alpha$ -layer globally stable matching. The other two problems, PAIR STABLE MARRIAGE and INDIVIDUALLY STABLE MARRIAGE, are defined analogously.

#### GLOBALLY STABLE MARRIAGE

**Input:** Two disjoint sets  $U$  and  $W$  of  $n$  agents each,  $\ell$  preference profiles, and an integer  $\alpha \in [\ell]$ .

**Output:** Return an  $\alpha$ -layer globally stable matching if one exists, or report that there is no such stable matching.

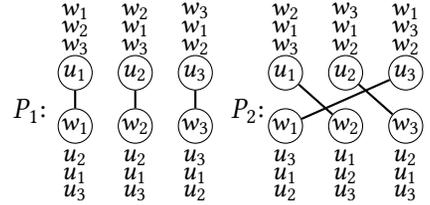
### 3 RELATIONS BETWEEN THE MULTI-LAYER CONCEPTS OF STABILITY

Below we establish relations among the three concepts. We start by showing that  $\alpha$ -layer pair stability is a weaker notion than  $\alpha$ -layer global stability and  $\alpha$ -layer individual stability.

PROPOSITION 3.1 ( $\star$ ). *An  $\alpha$ -layer globally stable matching is  $\alpha$ -layer pair stable. An  $\alpha$ -layer individually stable matching is  $\alpha$ -layer pair stable.*

**Example 3.2** shows an  $\alpha$ -layer pair stable matching which is not  $\alpha$ -layer globally stable. Using **Proposition 3.1**, this also shows that  $\alpha$ -layer global stability is strictly stronger than  $\alpha$ -layer pair stability.

*Example 3.2.* Consider six agents with two layers of preference profiles. Observe that matching  $M = \{\{u_1, w_1\}, \{u_2, w_3\}, \{u_3, w_2\}\}$  is 1-layer individually stable and, thus 1-layer pair stable. However,  $M$  is blocked by  $\{u_2, w_1\}$  in  $P_1$  and by  $\{u_1, w_2\}$  in  $P_2$ . Thus,  $M$  is not 1-layer globally stable. Indeed the only 1-layer globally stable matchings are indicated by the solid lines, which are also 1-layer individually stable. As soon as  $\alpha \geq 2$ ,  $\alpha$ -layer pair stability does not always exist, even if  $\ell > \alpha$ .



To see this we augment the instance with one more layer whose preference lists are identical to the first layer given in **Example 2.1**. One can verify that for each of all six possible matchings, there is always an unmatched pair that is blocking at least two layers.  $\diamond$

For  $\alpha = 1$ , we observe that 1-layer pair stability is equivalent to 1-layer individual stability.

**PROPOSITION 3.3** (★). *A matching is 1-layer pair stable iff. it is 1-layer individually stable.*

**Example 3.4** shows that for  $\alpha > 1$ , individual stability and pair stability are not equivalent, neither is individual stability equivalent to global stability.

*Example 3.4.* Consider the example given in **Section 1**. Recall that both  $M_1$  and  $M_2$  are 2-layer globally stable (and 2-layer pair stable). However,  $M_1$  is 2-layer individually stable while  $M_2$  is not. To see why  $M_2$  is not 2-layer individually stable, we can verify that the unmatched pair  $p = \{u_1, w_1\}$  dominates  $\{u_1, w_2\}$  in the first and the third layer and it dominates  $\{u_2, w_1\}$  in the first two layers. By **Proposition 2.10**,  $M_2$  is not 2-layer individually stable since  $\ell - \alpha + 1 = 2$ . If we restrict the example to the last two layers only, then matching  $M_2$  is also evidence that an  $\ell$ -layer globally stable (which, by **Proposition 3.5**, implies  $\ell$ -layer pair stability) is not  $\ell$ -layer individually stable.  $\diamond$

For  $\alpha = \ell$  global stability and pair stability are equivalent.

**PROPOSITION 3.5** (★). *A matching is  $\ell$ -layer globally stable iff. it is  $\ell$ -layer pair stable.*

It is somehow counter-intuitive that even an  $\ell$ -layer globally stable matching (i.e. a matching that is stable in each layer) may not be  $\ell$ -layer individually stable (see **Example 3.4**). **Propositions 3.1** and **3.5** imply that  $\ell$ -layer global stability is strictly weaker than  $\ell$ -layer individual stability. By **Example 3.4**,  $\ell$ -layer global stability does not imply  $\ell$ -layer individual stability. However, it implies  $\lceil \ell/2 \rceil$ -layer individual stability.

**PROPOSITION 3.6** (★). *Every  $\ell$ -layer globally stable matching is  $\lceil \ell/2 \rceil$ -layer individually stable. There exist  $\ell$ -layer globally stable matchings which are not  $(\lceil \ell/2 \rceil + 1)$ -layer individually stable.*

The relations among the different concepts of multi-layer stability are depicted in **Figure 1**.

A 1-layer globally stable matching always exists. By **Proposition 3.1**, we obtain the following.

**PROPOSITION 3.7.** *An instance with  $\ell$  layers always admits a matching which is 1-layer globally stable, 1-layer pair stable, and 1-layer individually stable.*

#### 4 ALL-LAYERS STABILITY ( $\alpha = \ell$ )

In this section, we discuss the special case when  $\alpha = \ell$ . It turns out that deciding whether a given instance with  $2n$  agents admits an  $\ell$ -layer individually stable matching can be solved in linear time:  $O(\ell \cdot n^2)$ . Moreover, we identify a close connection between our  $\ell$ -layer individual stability and the super stability concept for preferences with partial orders [39]. Using this connection, we could

---

**ALGORITHM 1:** Algorithm for finding an  $\ell$ -layer individually stable matching.

---

**Input:** A set of agents  $U \cup W$  and  $\ell$  layers of preferences.

---

```

1 repeat
2   foreach agent  $u \in U$  do
3     foreach layer  $i = 1, 2, \dots, \ell$  do
4        $w \leftarrow$  the first not-yet-marked agent in  $u$ 's preference list in layer  $i$ 
5       repeat
6         foreach  $u'$  with  $w: u \succ_w^{(j)} u'$  for some layer  $j$  do mark  $\{u', w\}$ 
7          $w \leftarrow$  the first not-yet-marked agent in  $u$ 's preference list in layer  $i$ 
8       until  $\{u, w\}$  is not marked
9 until (some agent's preference list consists of only marked agents) or (no new pair was marked in the last iteration)
10 if some preference contains only marked agents then no  $\ell$ -layer individually stable matching exists
11 else return  $M = \{\{u, w\} \mid u \in U \text{ with } w \text{ being the first not-yet-marked agent in } u\text{'s preference list}\}$  as an  $\ell$ -layer individually stable matching

```

---

also solve our individual stability problem in a higher running time,  $O(\ell \cdot n^3)$  (see [16]). For the other two concepts of stability, however, the problem is NP-hard even when  $\ell = 2$ .

#### 4.1 Algorithm for $\ell$ -layer individual stability

The algorithm for deciding  $\ell$ -layer individual stability is based on the following simple lemma.

LEMMA 4.1 (★). *Let  $u \in U$  and  $w \in W$  be two agents such that  $w$  is the first ranked agent of  $u$  in some layer  $i \in [\ell]$ , and let  $u' \in U \setminus \{u\}$  be another agent such that  $w$  prefers  $u$  over  $u'$  in some layer  $j \in [\ell]$ . Then, no  $\ell$ -layer individually stable matching contains  $\{u', w\}$ .*

Lemma 4.1 leads to Algorithm 1 which looks quite similar to the so-called extended Gale-Shapley algorithm by Irving [30]. The crucial difference is that we loop into different layers and we cannot delete a pair  $p$  of agents that does not belong to any stable matching, as it may still serve to block certain matchings. Instead of deleting such pair, we will *mark* it. Herein, *marking a pair  $\{u, w\}$*  means marking the agent  $u$  (resp.  $w$ ) in the preference list of  $w$  (resp.  $u$ ) in every layer.

The correctness of Algorithm 1 follows from Lemmas 4.2 to 4.4.

LEMMA 4.2 (★). *If a pair  $\{u', w\}$  is marked during the execution of Algorithm 1, then no  $\ell$ -layer individually stable matching contains this pair.*

The following lemma ensures that in Line 11 if  $w$  is matched to an agent  $u$ , then it is the most preferred unmarked agent of  $u$  in *all* layers.

LEMMA 4.3 (★). *If no agent's preference list consists of only marked agents and there is an agent  $u$  and two different layers  $i, j \in [\ell]$ ,  $i \neq j$  such that the first unmarked agent in the preference list of  $u$  in layer  $i$  differs from the one in layer  $j$ , then Algorithm 1 will mark at least one more pair.*

When Algorithm 1 terminates and no agent contains a preference list that consists of only marked agents, then we can construct an  $\ell$ -layer individually stable matching by assigning to each agent  $u$  its first unmarked agent in any preference list (note that Lemma 4.3 ensures on termination that for each agent it holds that in all layers is first unmarked agent is the same).

LEMMA 4.4 (★). *If upon termination no agent's preference list consists of only marked agents, then the matching  $M$  computed by Algorithm 1 is  $\ell$ -layer individually stable.*

Finally, we prove that Algorithm 1 computes an  $\ell$ -layer individually stable matching if one exists.

**THEOREM 4.5.** For  $\alpha = \ell$ , *Algorithm 1* solves INDIVIDUALLY STABLE MARRIAGE in  $O(\ell \cdot n^2)$  time.

**PROOF SKETCH.** Let  $I = (U, W, P_1, P_2, \dots, P_\ell)$  with  $2n$  agents be the input of *Algorithm 1*. By *Lemma 4.2*, no  $\ell$ -layer individually stable matching contains a marked pair. If there is an agent whose preference list consists of only marked agents, then we can immediately conclude that the given instance is a no-instance. Otherwise, *Lemma 4.4* proves that the algorithm returns an  $\ell$ -layer individually stable matching.  $\square$

## 4.2 NP-hardness for $\ell$ -layer global stability and $\ell$ -layer pair stability

In contrast to  $\ell$ -layer individual stability, in this section we show that deciding GLOBALLY STABLE MARRIAGE is NP-hard as soon as  $\alpha = \ell = 2$ . We establish this by reducing a NP-complete variant of the 3-SAT problem [28] to the decision variant of GLOBALLY STABLE MARRIAGE. The idea behind this reduction is to introduce for each variable four agents that admit exactly two possible globally stable matchings, one for each truth value. Then, we construct a satisfaction gadget for each clause by introducing six agents. These agents will have three possible globally stable matchings. We use a layer for each literal contained in the clause to enforce that setting the literal to false will exclude exactly one of the three globally stable matchings. Hence, unless one of the literals in the clause is set to true, no globally stable matching remains. Using the above idea, we can already show hardness of deciding  $\alpha$ -layer global stability for  $\alpha = \ell = 3$ . With some tweaks and using a restricted variant of 3-SAT, we can strengthen our hardness result to hold even for  $\ell = 2$ .

Before moving on with our result, we note that, independent from our work, Aziz et al. [3] and Miyazaki and Okamoto [45] considered all-layer global stability. While the former showed hardness for  $\ell = 16$ , the latter strengthened it to the case with  $\ell = 2$ . Using a slightly different reduction, we show the same hardness result as Miyazaki and Okamoto. Indeed, our reduction can be adapted to show hardness for many other cases. For instance, by adapting the construction we show hardness not only for the two-layer case but also for arbitrary  $\alpha < \ell$  (*Proposition 5.3*). Moreover, we will see that our reduction can also be adapted to tackle a more general stability concept—pair stability (*Proposition 5.5*).

**THEOREM 4.6 (★).** For each  $\alpha = \ell \geq 2$ , GLOBALLY STABLE MARRIAGE is NP-hard.

Since  $\ell$ -layer global stability equals  $\ell$ -layer pair stability (*Proposition 3.5*), by *Theorem 4.6* we obtain the following corollary for the pair stability.

**COROLLARY 4.7.** For each  $\alpha = \ell \geq 2$ , PAIR STABLE MARRIAGE is NP-hard.

## 5 MULTI-LAYER STABILITY WITH $\alpha < \ell$

In this section we show that for each of the three concepts that we introduced in *Section 2* the problem of computing a multi-layer stable matching is computationally hard as soon as  $2 \leq \alpha < \ell$ .

### 5.1 NP-hardness for $\alpha$ -layer individual stability

For  $\alpha$ -layer individual stability with  $\alpha < \ell$  we no longer have the observation in *Lemma 4.1* (which was crucial for solving the case with  $\alpha = \ell$  in polynomial time). Indeed, we show that the problem is NP-hard even if  $\ell = 3$  and  $\alpha = 2$  by reducing from the NP-complete 3-SAT problem (where each clause has exactly three variables). The idea behind the reduction is a variable gadget that allows two  $\alpha$ -layer individually stable matchings, one for each truth value, and a clause gadget that allows exactly three  $\alpha$ -layer individually stable matchings, called *clause matchings*. One of the agents in the clause gadget, together with an agent from the variable gadget, will form an unmatched pair that would be  $(\ell - \alpha + 1)$ -dominating one clause matching if the corresponding literal in the clause is set false. Thus, unless at least one literal in the clause is set true, no  $\alpha$ -layer individually stable

matching remains for the clause gadget. Utilizing this idea, we can show hardness for arbitrary constants  $\ell$  and  $\alpha$  with  $\ell/2 < \alpha \leq 2\ell/3$ .

**THEOREM 5.1.** INDIVIDUALLY STABLE MARRIAGE is NP-hard for each  $\ell/2 < \alpha \leq 2\ell/3$ .

**PROOF SKETCH.** We reduce from the NP-complete 3-SAT problem [28], where each clause has exactly three variables, to the decision version of INDIVIDUALLY STABLE MARRIAGE for  $\alpha = 2$  and  $\ell = 3$ . The proof for the more general case with  $\ell/2 < \alpha \leq 2\ell/3$  can be found in [16].

Let  $I = (X, C)$  be an instance of 3-SAT, where  $X = \{x_1, x_2, \dots, x_n\}$  is the set of  $n$  variables and  $C = \{C_1, C_2, \dots, C_m\}$  is the set of  $m$  clauses. For each variable  $x_i \in X$ , we create four *variable agents*  $x_i, \bar{x}_i, y_i, \bar{y}_i$ . We will construct three layers of preference lists for the variable agents so that each 2-layer individually stable matching contains either  $M_i^{\text{true}} := \{\{x_i, y_i\}, \{\bar{x}_i, \bar{y}_i\}\}$  or  $M_i^{\text{false}} := \{\{x_i, \bar{y}_i\}, \{\bar{x}_i, y_i\}\}$ . Briefly put, using  $M_i^{\text{true}}$  and  $M_i^{\text{false}}$  will correspond to setting the variable  $x_i$  to true or false, respectively. For each clause  $C_j \in C$ , we create six *clause agents*  $a_j, b_j, c_j, d_j, e_j, f_j$ . We will construct preference lists for these clause agents so that for each clause, there are exactly three different ways in which these agents are matched in a 2-layer individually stable matching. We use two layers to enforce that setting a different literal contained in the clause  $C_j$  to false excludes exactly one of these ways. In total, we have  $4n + 6m$  agents and we divide them into two groups  $U$  and  $W$  with  $U = \{x_i, \bar{x}_i \mid i \in [n]\} \cup \{a_i, b_i, c_i \mid i \in [m]\}$  and  $W = \{y_i, \bar{y}_i \mid i \in [n]\} \cup \{d_j, e_j, f_j \mid j \in [m]\}$ .

**Preference lists of the variable agents.** The preference lists of the variable agents restricted to the variable agents have the same pattern. We remark that here and in the description for the clause gadget, agents which were not explicitly mentioned in the preference list will be ordered behind the ones mentioned, in an arbitrary order.

$$\begin{aligned} \forall i \in [n]: & \text{Layer (1): } x_i: y_i > \bar{y}_i, & \bar{x}_i: \bar{y}_i > y_i, & y_i: \bar{x}_i > x_i, & \bar{y}_i: x_i > \bar{x}_i. \\ & \text{Layer (2): } x_i: y_i > \bar{y}_i, & \bar{x}_i: y_i > \bar{y}_i, & y_i: \bar{A}_i > x_i > \bar{A}_i > \bar{x}_i, & \bar{y}_i: x_i > \bar{x}_i. \\ & \text{Layer (3): } x_i: \bar{y}_i > y_i, & \bar{x}_i: \bar{y}_i > y_i, & y_i: \bar{A}_i > \bar{x}_i > \bar{A}_i > x_i, & \bar{y}_i: \bar{x}_i > x_i. \end{aligned}$$

Herein,  $\bar{A}_i$  denotes a list (in an arbitrary order) of all clause agents  $a_j$  such that the corresponding clause  $C_j$  contains the positive literal  $x_i$ .  $\bar{A}_i$  denotes a list (in an arbitrary order) of all clause agents  $a_j$  such that the corresponding clause  $C_j$  contains the negative literal  $\bar{x}_i$ .

**Preference lists of the clause agents.** When restricted to the clause agents only, the preference lists for the clause agents are “uniform”: In each layer, the preference lists of all agents corresponding to the same clause are the same (see Section 6.2). To unify the expression, for each clause  $C_j = \ell_j^1 \vee \ell_j^2 \vee \ell_j^3$  and each integer  $t \in \{1, 2, 3\}$  by  $Y_j^{(t)}$  we denote the variable agent  $y_i$  that corresponds to the literal  $\ell_j^t$ .

$$\begin{aligned} \forall j \in [m]: & \text{Layer (1): } a_j: d_j > Y_j^{(2)} > e_j > Y_j^{(3)} > f_j; & b_j \text{ and } c_j: d_j > e_j > f_j; & d_j \text{ and } e_j \text{ and } f_j: a_j > b_j > c_j. \\ & \text{Layer (2): } a_j: e_j > Y_j^{(3)} > f_j > Y_j^{(1)} > d_j; & b_j \text{ and } c_j: e_j > f_j > d_j; & d_j \text{ and } e_j \text{ and } f_j: b_j > c_j > a_j. \\ & \text{Layer (3): } a_j: f_j > Y_j^{(1)} > d_j > Y_j^{(2)} > e_j; & b_j \text{ and } c_j: f_j > d_j > e_j; & d_j \text{ and } e_j \text{ and } f_j: c_j > a_j > b_j. \end{aligned}$$

Before we show the correctness of our construction, we discuss some properties that each 2-layer individually stable matching  $M$  must satisfy.

**CLAIM 1 (★).** Let  $M$  be a 2-layer individually stable matching for our three-layer preference profiles. For each variable  $x_i \in X$ , it holds that either  $M_i^{\text{true}} \subseteq M$  or  $M_i^{\text{false}} \subseteq M$ .

We obtain a similar result for the clause agents. For each clause  $C_j \in C$  let  $N_j^1 = \{\{a_j, d_j\}, \{b_j, e_j\}, \{c_j, f_j\}\}$ ,  $N_j^2 = \{\{a_j, e_j\}, \{b_j, f_j\}, \{c_j, d_j\}\}$ ,  $N_j^3 = \{\{a_j, f_j\}, \{b_j, d_j\}, \{c_j, e_j\}\}$ .

CLAIM 2 (★). Let  $M$  be a 2-layer individually stable matching  $M$ . For each clause  $C_j \in C$  the following holds. (1) One of  $\{N_j^1, N_j^2, N_j^3\}$  belongs to  $M$ . (2) Let  $\ell_j^{(t)}$  be the  $t^{\text{th}}$  literal in  $C_j$ ,  $t \in \{1, 2, 3\}$  such that  $N_j^t \subseteq M$ . If  $\ell_j^{(t)}$  is positive, then  $M_i^{\text{true}} \subseteq M$ . If  $\ell_j^{(t)}$  is negative, then  $M_i^{\text{false}} \subseteq M$ .

Using [Claims 1](#) and [2](#), we can show that  $(X, C)$  admits a satisfying truth assignment if and only if there exists a 2-layer individually stable matching for the thus-constructed instance. The full proof can be found in [\[16\]](#).  $\square$

For each  $\alpha < \ell/2$ , we also obtain hardness results by reducing from the NP-hard PERFECT SMTI problem, which asks to find a perfect *SMTI-stable* matching with (possibly) incomplete preference lists and ties [\[33, 41\]](#), defined as follows. A preference list is *incomplete* if not all agents from one side are considered acceptable to an agent from the other side. A preference list has a *tie* if there are two agents in the list which are considered to be equally good. As a result, a preference list of an agent  $u$  from one side can be considered as a weak (i.e. transitive and complete) order  $\succeq_u$  on a subset of the agents on the other side. We use  $>_u$  and  $\sim_u$  to denote the asymmetric and symmetric part of the preference list, respectively. Equivalently, two agents  $x$  and  $y$  are said to be tied by  $u$  if  $x \succeq_u y$  and  $y \succeq_u x$ , denoted as  $x \sim_u y$ . Formally, we say that a matching  $M$  for a PERFECT SMTI with two disjoint sets  $U$  and  $W$  of agents is *SMTI-stable* if there are no SMTI-blocking pairs for  $M$ . A pair  $\{u, w\}$  is *SMTI-blocking*  $M$  if all of the following three conditions are satisfied: (i)  $u$  and  $w$  appear in the preference lists of each other, (ii)  $w >_u M(u)$  or  $u$  is not matched to any agent from  $W$ , and (iii)  $u >_w M(w)$  or  $w$  is not matched to any agent from  $U$ .

The reduction is based on the following ideas: In a PERFECT SMTI instance  $I$  the preference list of an agent  $z$  may have ties. To encode ties, we first “linearize” the preference list of  $z$  in  $I$  to obtain two linear preference lists such that the resulting lists restricted to the tied agents are reverse to each other. Then, we let half of the  $\ell$  layers have one of the preference lists and let the remaining half to have the other list. Since  $\alpha < \ell/2$ , it is always possible to find half layers which fulfill our  $\alpha$ -layer individual stability constraint. In  $I$ , two agents, say  $x$  and  $y$ , may not be acceptable to each other. To encode this, we add to the source instance  $\ell$  pairs of dummy agents with  $\ell$  layers of preferences that preclude  $x$  and  $y$  from being matched together. However, to make sure that an agent  $x$  is not matched to any dummy agent, we have to require that  $\ell \geq 4$ .

THEOREM 5.2 (★). For each fixed number  $\ell$  of layers with  $\ell \geq 4$  and for each fixed value  $\alpha$  with  $2 \leq \alpha \leq \lfloor \ell/2 \rfloor$ , INDIVIDUALLY STABLE MARRIAGE is NP-hard even if on one side the preference list of each agent is the same in all layers.

## 5.2 NP-hardness for $\alpha$ -layer global stability and $\alpha$ -layer pair stability

To find a matching  $M$  that is  $\alpha$ -layer globally stable, even if  $\alpha < \ell$ , the main difficulty is not just to determine  $\alpha$  layers where  $M$  should be stable. In fact, we sometimes need to find a matching that is stable in some specific layers. This requirement allows us to adapt the construction in the proof of [Theorem 4.6](#) to show hardness for deciding  $\alpha$ -layer global stability for the case when  $2 \leq \alpha < \ell$ .

PROPOSITION 5.3 (★). For each  $\alpha$  with  $\alpha \geq 2$ , GLOBALLY STABLE MARRIAGE is NP-hard.

We remark that our proof for [Proposition 5.3](#) also implies hardness for  $\alpha = \ell$  for arbitrary  $\ell \geq 2$ . Observe that in the instance constructed in the proof of [Theorem 5.2](#) every agent from the side  $W \cup R$  has the same preference list in all layers. Later on, in [Proposition 6.1](#) we will show that in such a case the concepts of  $\alpha$ -layer individual stability and  $\alpha$ -layer pair stability are equivalent. Thus, we obtain the same hardness result for pair stability when  $\alpha \leq \lfloor \ell/2 \rfloor$ .

COROLLARY 5.4. For each  $\ell$  with  $\ell \geq 4$  and each  $\alpha$  with  $2 \leq \alpha \leq \lfloor \ell/2 \rfloor$ , PAIR STABLE MARRIAGE is NP-hard even if on one side the preference lists of each agent are the same in all layers.

For  $\alpha > \lceil \ell/2 \rceil$ , we use an idea similar to the one used for showing [Proposition 5.3](#).

**PROPOSITION 5.5 (★).** *For each  $\alpha$  with  $\lceil \ell/2 \rceil + 1 \leq \alpha \leq \ell$ , PAIR STABLE MARRIAGE is NP-hard.*

[Corollary 5.4](#) and [Proposition 5.5](#) cover the whole range of the potential values of  $\alpha$  except for  $\alpha = \lfloor \ell/2 \rfloor + 1$  when  $\ell$  is odd. As we will see in the next section [Theorem 5.2](#) cannot be strengthened to cover the value  $\alpha = \lfloor \ell/2 \rfloor + 1$  (see [Proposition 6.4](#)), and so, also [Corollary 5.4](#) cannot be directly strengthened. However, we can tweak the construction from [Theorem 5.2](#), breaking the restriction that on one side the preference list of each agent is the same in all layers, and obtain hardness for  $\alpha$ -layer pair stability for  $\alpha = \lfloor \ell/2 \rfloor + 1$ .

**PROPOSITION 5.6 (★).** *For each  $\alpha \geq 3$  with  $\ell = 2\alpha - 1$ , PAIR STABLE MARRIAGE is NP-hard.*

## 6 TWO SPECIAL CASES OF PREFERENCES

In this section, we consider two well-motivated special cases of our general multi-layer framework for stable matchings. We will discuss how the corresponding simplifying assumptions affect the computational complexity of finding multi-layer stable matchings. Surprisingly, even under seemingly strong assumptions some variants of our problem remain computationally hard.

### 6.1 Single-layered preferences on one side

Consider the special case where the preferences of the agents from  $U$  can be expressed through a single layer. Formally, we model this by assuming that for each agent from  $U$ , its preference list is the same in all layers. In this case we say that the agents from  $U$  have *single-layered preferences*.

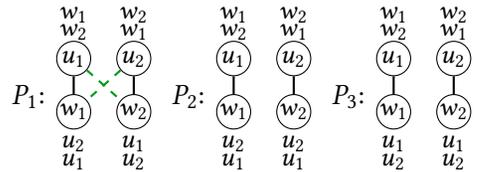
Single-layered preferences on one side can arise in many real-life scenarios. For instance, consider the standard example of matching residents with hospitals and, similarly as in [Example 2.9](#), assume that each layer corresponds to a certain criterion. It is reasonable to assume that the hospitals evaluate their potential employees with respect to the level of their qualifications only (thus, having a single layer of preferences), while the residents take into account a number of factors such as how far is a given hospital from the place they live, the level of compensation, the reputation of the hospital, etc.

First, we observe that in such a case two out of our three stability concepts are equivalent.

**PROPOSITION 6.1 (★).** *If each agent from  $U$  has single-layered preferences, then  $\alpha$ -layer pair stability and  $\alpha$ -layer individual stability are equivalent for each  $\alpha$ .*

For profiles with single-layered preferences of the agents on one side,  $\alpha$ -layer global stability is strictly stronger than the other two concepts:

*Example 6.2.* Consider four agents with three layers of profiles and let  $M = \{\{u_1, w_2\}, \{u_2, w_1\}\}$ . We show that  $M$  is 2-layer pair stable but not 2-layer globally stable. Indeed,  $M$  is stable only in the first layer, so it cannot be 2-layer globally stable. To see why  $M$  is 2-layer pair stable, consider the unmatched pairs  $\{u_1, w_1\}$  and  $\{u_2, w_2\}$ . The first one only blocks the third layer, and the latter one only blocks the second layer.



◇

**Global stability.** By [Propositions 3.5](#) and [6.1](#), we know that for the case that the input instance has single-layered preferences, all three all-layer stability concepts are the same. Thus, when  $\alpha = \ell$ , we can use [Algorithm 1](#) to decide  $\alpha$ -layer global stability in polynomial time for the single-layered preferences case. Moreover, for  $\alpha < \ell$ , we can also decide  $\alpha$ -layer global stability in time  $O(\ell^\alpha \cdot \alpha \cdot n^2)$

by guessing a subset of  $\alpha$  layers and using [Algorithm 1](#). However, the following result shows that fixed-parameter tractability (FPT) for the parameter  $\alpha$ , *i.e.*, the existence of an algorithm running in  $f(\alpha) \cdot (\ell \cdot n)^{O(1)}$  time for some computable function  $f$ , is unlikely (for details on parameterized complexity we refer to the books of Cygan et al. [19], Downey and Fellows [21], Flum and Grohe [26], and Niedermeier [46]).

**THEOREM 6.3 (★).** *Even if the preferences of the agents from  $U$  are single-layered, GLOBALLY STABLE MARRIAGE is NP-hard and is  $W[1]$ -hard for the threshold parameter  $\alpha$ . For  $\alpha = \ell$ , the problem is solvable in  $O(\ell \cdot n^2)$  time, while for  $\alpha < \ell$  it is solvable in  $O(\ell^\alpha \cdot \alpha \cdot n^2)$  time.*

**Individual stability and pair stability.** By [Theorem 5.2](#), checking  $\alpha$ -layer individual stability for preferences single-layered on one side is NP-hard if  $\ell \geq 4$  and  $\alpha \leq \lfloor \ell/2 \rfloor$ . For the case of  $\alpha \geq \lfloor \ell/2 \rfloor + 1$ , we establish a relation between the individual stability for preferences single-layered on one side and the stability concept in the traditional single-layer setting, but with general incomplete and possibly intransitive preferences as studied by Farzadi et al. [25]. This relation allows us to construct a polynomial-time algorithm for checking  $\alpha$ -layer individual stability.

**PROPOSITION 6.4 (★).** *If the preferences of the agents on one side are single-layered and  $\alpha \geq \lfloor \ell/2 \rfloor + 1$ , then PAIR STABLE MARRIAGE and INDIVIDUALLY STABLE MARRIAGE can be solved in  $O(\ell \cdot n^3)$  time.*

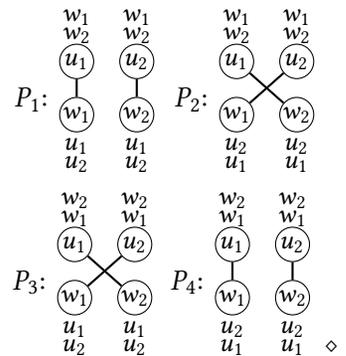
## 6.2 Uniform preferences in each layer

In this section we consider the case when in each layer the preferences of all agents from  $U$  (resp.  $W$ ) are the same—we call such preferences *uniform in each layer*. This special case is motivated with the following observation pertaining to [Example 2.9](#): preferences uniform in a layer can arise if the criterion corresponding to the layer is not subjective. For instance, if a layer corresponds to the preferences regarding the wealth of potential partners, then it seems natural that everyone prefers to be matched with a wealthier partner (and so the preferences of all agents for this criterion are the same); similarly it seems reasonable that the preferences of all hospitals are the same: candidates with higher grades are ranked higher by each hospital.

First, we observe that for uniform preferences, no two of the three concepts are equivalent.

*Example 6.5.* Consider four agents with four layers of uniform preferences. Let  $M_1 = \{\{u_1, w_1\}, \{u_2, w_2\}\}$  and  $M_2 = \{\{u_1, w_2\}, \{u_2, w_1\}\}$ . One can check that both  $M_1$  and  $M_2$  are *not* 3-layer globally stable and *not* 3-layer individually stable.

Both  $M_1$  and  $M_2$  are 3-layer pair stable. For instance, for  $M_1$ , the unmatched pairs  $\{u_1, w_2\}$  and  $\{u_2, w_1\}$  are only blocking layer (2) and layer (3), respectively. If we restrict the instance to only the first three layers, then we have the following results. (1)  $M_1$  is not even 2-layer globally stable but it is 2-layer individually stable as for each unmatched pair  $p$  with respect to  $M_1$ , at least one agent in  $p$  obtains a partner which is its most preferred agent in at least two layers. (2)  $M_2$  is 2-layer globally stable but it is not 2-layer individually stable. To see why it is not 2-layer individually stable, we consider the unmatched pair  $\{u_1, w_1\}$ . There are  $3 - 2 + 1 = 2$  layers where  $u_1$  prefers  $w_1$  to its partner  $M_2(u_1) = w_2$  and there are two layers where  $w_1$  prefers  $u_1$  to its partner  $M_2(w_1) = u_2$ .



**Individual stability.** For preferences that are uniform in each layer, there is a close relation between INDIVIDUALLY STABLE MARRIAGE and GRAPH ISOMORPHISM, the problem of finding an isomorphism between graphs or deciding that there exists none. Herein, an instance of GRAPH ISOMORPHISM

consists of two undirected graphs  $G$  and  $H$  with the same number of vertices and the same number of edges. We want to decide whether there is an edge-preserving bijection  $f: V(G) \rightarrow V(H)$  between the vertices, *i.e.* for each two vertices  $u, v \in V(G)$  it holds that  $\{u, v\} \in E(G)$  if and only if  $\{f(u), f(v)\} \in E(H)$ . We call such bijection an isomorphism between  $G$  and  $H$ .

We explore this relation through the following construction. For an instance  $I$  of INDIVIDUALLY STABLE MARRIAGE we construct two directed graphs  $G_I$  and  $H_I$  as follows. The agents from  $U$  and  $W$  will correspond to the vertices in  $G_I$  and  $H_I$ , respectively. For each two vertices  $u, u' \in U$  we add to  $G_I$  an arc  $(u, u')$  if the agents from  $W$  prefer  $u$  to  $u'$  in at least  $(\ell - \alpha + 1)$  layers. Analogously, for each two agents  $w$  and  $w'$ , we add to  $H_I$  an arc  $(w, w')$  if the agents from  $U$  prefer  $w$  to  $w'$  in at least  $(\ell - \alpha + 1)$  layers. Let  $E(G_I)$  and  $E(H_I)$  denote the arc sets of  $G_I$  and  $H_I$ , respectively.

We first explain how the so constructed graphs can be used to find an  $\alpha$ -layer individually stable matching in the initial instance  $I$ , or to claim that there is no such a matching.

**PROPOSITION 6.6 (★).** *A matching  $M$  for instance  $I$  is  $\alpha$ -layer individually stable if and only if for each two  $u, u' \in U$  and each two  $w, w' \in W$  the following two properties hold. (1)  $(u, u') \in E(G_I)$  implies  $(M(u'), M(u)) \notin E(H_I)$ . (2)  $(w, w') \in E(H_I)$  implies  $(M(w'), M(w)) \notin E(G_I)$ .*

Using a construction by McGarvey [43], given an arbitrary directed graph, we can indeed construct multi-layer preferences that induce this graph.

**LEMMA 6.7 (★).** *For each two directed graphs  $G_I$  and  $H_I$  with  $m$  arcs each there exists an instance with  $\ell = 2m$  layers of uniform preferences that induces  $G_I$  and  $H_I$  via McGarvey's construction, where  $m$  denotes the number of arcs in  $G_I$  (and thus in  $H_I$ ).*

Using **Proposition 6.6** and **Lemma 6.7** as tools, we can show that for uniform preferences, INDIVIDUALLY STABLE MARRIAGE is at least as hard as GRAPH ISOMORPHISM.

**THEOREM 6.8.** *GRAPH ISOMORPHISM is polynomial-time reducible to INDIVIDUALLY STABLE MARRIAGE, where the preferences of the agents are uniform in all  $\ell$  layers,  $\ell$  is even, and  $\alpha = \ell/2$ .*

**PROOF SKETCH.** Let  $G, H$  be two undirected graphs that form an instance of GRAPH ISOMORPHISM. Without loss of generality, assume that  $n \geq 4$  (number of vertices) and  $m \geq 3$  (number of edges). From  $G$  and  $H$  we construct an instance  $I$  of the problem of deciding whether there exists an  $\alpha$ -layer individually stable matching for preferences being uniform in all  $\ell$  layers, where  $\alpha = \ell/2$ . By **Lemma 6.7** we can describe this instance by providing the corresponding directed graphs  $G_I$  and  $H_I$ . We show how to construct  $G_I$  from  $G$ . The graph  $H_I$  is constructed from  $H$  analogously.

Let  $V$  and  $E$  denote the sets of vertices and edges in  $G$ . We copy all vertices from  $G$  to  $G_I$  (we will refer to these vertices as *non-special*) and we additionally introduce five types of "special" vertices, which we call  $\text{prime}^G$ ,  $\text{deputy}^G$ ,  $\binom{n}{2} + 1$  connectors,  $m$  sinks, and  $\binom{n}{2} - m$  sources. We specify the connectors, the sinks, and the sources formally: With each pair of vertices  $p = \{v, v'\} \in \binom{V}{2}$  we associate one connector, denoted as  $\text{connector}_p^G$ . One remaining connector is distinguished and denoted as  $\text{connector}_*^G$ . Further, for each edge  $e \in E$  we have one sink, denoted as  $\text{sink}_e^G$ , and for each non-edge  $e = \{v, v'\} \notin E$  we have a source, denoted as  $\text{source}_e^G$ . The arcs in  $G_I$  are constructed as follows. Let  $\text{SOURCE}^G = \{\text{source}_e^G \mid e \in \binom{V}{2} \setminus E\}$ ,  $\text{SINK}^G = \{\text{sink}_e^G \mid e \in E\}$ , and  $\text{CONNECTOR}^G = \{\text{connector}_e^G \mid e \in \binom{V}{2}\}$  denote the set of all sources, the set of all sinks, and the set of non-special connectors, respectively:

- (1) For each pair  $e = \{v, v'\} \subseteq V$  of vertices, we do the following: (i) If  $\{v, v'\} \in E$ , then we add to  $G_I$  the following five arcs:  $(v, \text{sink}_e^G)$ ,  $(v', \text{sink}_e^G)$ ,  $(v, \text{connector}_e^G)$ ,  $(v', \text{connector}_e^G)$ , and  $(\text{sink}_e^G, \text{connector}_e^G)$ . Otherwise, we add to  $G_I$  the following five arcs:  $(\text{source}_e^G, v)$ ,  $(\text{source}_e^G, v')$ ,  $(v, \text{connector}_e^G)$ ,  $(v', \text{connector}_e^G)$ , and  $(\text{source}_e^G, \text{connector}_e^G)$ . (ii) For each other non-special

- vertex  $u \in V \setminus \{v, v'\}$ , we add to  $G_I$  the arc  $(\text{connector}_e^G, u)$ . (iii) For each other source or sink  $x \in \text{SOURCE}^G \cup \text{SINK}^G \setminus \{\text{sink}_e^G, \text{source}_e^G\}$ , we add to  $G_I$  the arc  $(\text{connector}_e^G, x)$ .
- (2) For each source and each sink  $x \in \text{SOURCE}^G \cup \text{SINK}^G$  we add to  $G_I$  an arc  $(x, \text{connector}_*^G)$ . For each non-special vertex  $v \in V$ , we additionally add to  $G$  the arc  $(\text{connector}_*^G, v)$ .
- (3) For each vertex  $x \in V \cup \text{SOURCE}^G \cup \text{SINK}^G \cup \text{CONNECTOR}^G \cup \{\text{connector}_*^G\}$  except for the deputy, we add to  $G_I$  an arc  $(\text{prime}^G, x)$ .
- (4) For each vertex  $x \in \text{SOURCE}^G \cup \text{SINK}^G \cup \{\text{prime}^G\}$  we add to  $G_I$  the arc  $(\text{deputy}^G, x)$ . For each connector  $y \in \text{CONNECTOR}^G \cup \{\text{connector}_*^G\}$  we add to  $G_I$  the arc  $(y, \text{deputy}^G)$ .

This completes the construction of  $G_I$ . The arcs in  $H_I$  are constructed analogously. The correctness proof that “there is an  $\alpha$ -layer individually stable matching in the instance  $(G_I, H_I)$  with  $\alpha = \ell/2$  if and only if there is an isomorphism between  $G$  and  $H$ ” can be found in [16].  $\square$

**Theorem 6.8** implies that developing a polynomial-time algorithm for our problem is currently out of scope since the question of whether GRAPH ISOMORPHISM is solvable in polynomial time is still open. Besides **Theorem 6.8** there are other interesting implications of **Proposition 6.6** and **Lemma 6.7**. For  $\alpha \geq \ell/2 + 1$  our problem can be reduced to the TOURNAMENT ISOMORPHISM problem, which, given two tournament graphs, asks whether there is an arc-preserving bijection between the vertices of the two tournaments. TOURNAMENT ISOMORPHISM has been studied extensively in the literature (for a more detailed discussion see, e.g., [4, 51, 52]), but to the best of our knowledge it is still open whether it is solvable in polynomial time [4]. The best known algorithm solving TOURNAMENT ISOMORPHISM runs in  $n^{O(\log n)}$  time, where  $n$  denotes the number of vertices.

**COROLLARY 6.9.** *If the preferences of the agents are uniform in all  $\ell$  layers and  $\alpha \geq \ell/2 + 1$ , then INDIVIDUALLY STABLE MARRIAGE can be solved in  $n^{O(\log n)} + O(\ell \cdot n^2)$  time, where  $n$  denotes the number of agents.*

**PROOF.** Assume that  $\alpha \geq \ell/2 + 1$  and construct the two directed graphs as we did for **Proposition 6.6**. Since  $\ell - \alpha + 1 \leq \ell/2$ , for each two vertices  $x$  and  $y$  in  $G_I$  (resp.  $H_I$ ) at least one of the arcs  $(x, y)$  and  $(y, x)$  exists. Note that such graphs are not necessarily tournaments. But we can deal with the case when one of the graphs is not a tournament. If both  $(x, y)$  and  $(y, x)$  exist in  $G_I$  or in  $H_I$ , then by **Proposition 6.6** no  $\alpha$ -layer individually stable matchings exists. Thus, the only non-trivial case is when the graphs  $G_I$  and  $H_I$  are tournaments. Moreover, in such a case, the condition from **Proposition 6.6** can be reformulated as “ $(x, y) \in G_I$  if and only if  $(M(x), M(y)) \in H_I$ ” and this is the condition that  $M$  is a tournament isomorphism between  $G_I$  and  $H_I$ . Consequently, for  $\alpha \geq \ell/2 + 1$  INDIVIDUALLY STABLE MARRIAGE can be reduced to TOURNAMENT ISOMORPHISM. Note that the number of vertices in the constructed graphs equals the number of agents in our problem. Following Babai and Luks [4], we obtain an algorithm with the desired running time.  $\square$

**Global stability.** There exists a fairly straightforward polynomial-time algorithm for checking  $\alpha$ -layer global stability.

**PROPOSITION 6.10** ( $\star$ ). *If the preferences of the agents are uniform in all  $\ell$  layers, then GLOBALLY STABLE MARRIAGE can be solved in  $O(\ell \cdot n)$  time, where  $n$  denotes the number of agents.*

## 7 OPEN PROBLEMS AND CONCLUSIONS

While our work provides a rich and almost complete structure for analyzing the computational properties of deciding our proposed stability concepts for multi-layer preferences, it directly leads to the following open questions: (1) How hard is it to find an  $\alpha$ -layer individually stable matching for  $\lceil 2\ell/3 \rceil < \alpha < \ell$ ? (2) When the preferences of the agents are uniform in each layer and  $\alpha \geq \ell/2 + 1$ , we showed that the decision variant of INDIVIDUALLY STABLE MARRIAGE is solvable in quasi-polynomial

time  $n^{O(\log n)} + O(\ell \cdot n^2)$  which implies that the problem is in the complexity class LOGSNP [47]. It would be interesting to know whether it is also complete for LOGSNP.<sup>1</sup> However, LOGSNP-hardness for our problem would also imply LOGSNP-hardness for GRAPH ISOMORPHISM (see [Theorem 6.8](#)).

(3) When the preferences of the agents are uniform in each layer, how hard is it to search for an  $\alpha$ -layer pair stable matching for arbitrary  $\alpha > 1$  or an  $\alpha$ -layer individually stable matching when  $\alpha < \ell/2$ , or in general when the number of layers is constant.

We also believe that a number of other parameters and special cases can be motivated naturally in the context of our model, in particular parameters quantifying the degree to which the preferences of the agents differ. Analogous parameterizations have been studied in computational social choice, for instance for the NP-hard KEMENY SCORE problem [6, 7]. Continuing our research on special cases of input preferences ([Section 6](#)), it might be interesting to study STABLE MARRIAGE with multi-layer structured preferences, such as *single-peaked* [10], *single-crossing* [44, 48], and *1-Euclidean* [17, 18, 36] preferences. We note that it can be detected in polynomial time whether a preference profile has any of these structure [5, 14, 20, 22, 24, 36] and we refer the reader to Brederick et al. [15] and Elkind et al. [23] for an overview of the literature on single-peakedness and single-crossingness. We also note that Bartholdi III and Trick [5] worked on stable roommates for narcissistic and single-peaked preferences, while Brederick et al. [12] extended this line by also studying other structured preferences and including preferences with ties and incompleteness.

Finally, our multi-modal view on the bipartite variant (STABLE MARRIAGE) can be generalized to the non-bipartite variant (STABLE ROOMMATES), the case with incomplete preferences with ties, and the case where each side may have a different number of layers. It would be interesting to see whether our computational tractability results transfer to these cases.

## REFERENCES

- [1] A. Abdulkadiroğlu, P.A. Pathak, and A.E. Roth. 2005. The Boston public school match. *Amer. Econ. Rev.* 95, 2 (2005), 368–371.
- [2] H. Aziz, P. Biró, T. Fleiner, S. Gaspers, R. de Haan, N. Mattei, and B. Rastegari. 2017. Stable Matching with Uncertain Pairwise Preferences. In *Proceedings of AAMAS-17*. 344–352.
- [3] H. Aziz, P. Biró, S. Gaspers, R. de Haan, N. Mattei, and B. Rastegari. 2016. Stable Matching with Uncertain Linear Preferences. In *Proceedings of SAGT-16*. 195–206.
- [4] L. Babai and E. Luks. 1983. Canonical Labeling of Graphs. In *Proceedings of STOC-83*. 171–183.
- [5] J.J. Bartholdi III and M. Trick. 1986. Stable Matching with Preferences Derived from a Psychological Model. *Oper. Res. Lett.* 5, 4 (1986), 165–169.
- [6] N. Betzler, R. Brederick, and R. Niedermeier. 2014. Theoretical and empirical evaluation of data reduction for exact Kemeny Rank Aggregation. *Autonomous Agents and Multi-Agent Systems* 28, 5 (2014), 721–748.
- [7] N. Betzler, J. Guo, C. Komusiewicz, and R. Niedermeier. 2011. Average parameterization and partial kernelization for computing medians. *J. Comput. System Sci.* 77, 4 (2011), 774–789.
- [8] P. Biró. 2017. Applications of Matching under Preferences. In *Trends in Computational Social Choice*. 345–373.
- [9] P. Biró and S. Kiselgof. 2015. College admissions with stable score-limits. *Cent. Eur. J. Oper. Res.* 23, 4 (2015), 727–741.
- [10] D. Black. 1958. *The Theory of Committees and Elections*. Cambridge University Press.
- [11] R. Brederick, J. Chen, P. Faliszewski, J. Guo, R. Niedermeier, and G.J. Woeginger. 2014. Parameterized Algorithmics for Computational Social Choice: Nine Research Challenges. *Tsinghua Science and Technology* 19, 4 (2014), 358–373.
- [12] R. Brederick, J. Chen, U.P. Finnendahl, and R. Niedermeier. 2017. Stable Roommate with Narcissistic, Single-Peaked, and Single-Crossing Preferences. In *Proceedings of ADT-17*. 315–330.
- [13] R. Brederick, J. Chen, S. Hartung, S. Kratsch, R. Niedermeier, O. Suchý, and G.J. Woeginger. 2014. A Multivariate Complexity Analysis of Lobbying in Multiple Referenda. *JAIR* 50 (2014), 409–446.
- [14] R. Brederick, J. Chen, and G.J. Woeginger. 2013. A Characterization of the Single-Crossing Domain. *Soc. Choice Welf.* 41, 4 (2013), 989–998.
- [15] R. Brederick, J. Chen, and G.J. Woeginger. 2016. Are There Any Nicely Structured Preference Profiles Nearby? *Math. Soc. Sci.* 79 (2016), 61–73.

<sup>1</sup>LOGSNP-hardness has been encountered and discussed for natural problems in Computational Social Choice [11, 13].

- [16] J. Chen, R. Niedermeier, and P. Skowron. 2018. *Stable Marriage with Multi-Modal Preferences*. Technical Report. arXiv:1801.02693 [cs.MA,cs.DS].
- [17] J. Chen, K. Pruhs, and G.J. Woeginger. 2017. The One-Dimensional Euclidean Domain: Finitely Many Obstructions Are Not Enough. *Soc. Choice Welf.* 48, 2 (2017), 409–432.
- [18] C.H. Coombs. 1964. *A Theory of Data*. John Wiley and Sons.
- [19] M. Cygan, F.V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. 2015. *Parameterized Algorithms*. Springer.
- [20] J.P. Doignon and J.C. Falmagne. 1994. A Polynomial Time Algorithm for Unidimensional Unfolding Representations. *J. Algorithms* 16, 2 (1994), 218–233.
- [21] R.G. Downey and M.R. Fellows. 2013. *Fundamentals of Parameterized Complexity*. Springer.
- [22] E. Elkind, P. Faliszewski, and A. Slinko. 2012. Clone Structures in Voters' Preferences. In *Proceedings of EC-12*. 496–513.
- [23] E. Elkind, M. Lackner, and D. Peters. 2017. Structured Preferences. In *Trends in Computational Social Choice*. 187–207.
- [24] B. Escoffier, J. Lang, and M. Öztürk. 2008. Single-Peaked Consistency and its Complexity. In *Proceedings of ECAI-08*. 366–370.
- [25] L. Farzadi, K. Georgiou, and J. Könemann. 2016. Stable Marriage with General Preferences. *Theory Comput. Syst.* 59, 4 (2016), 683–699.
- [26] J. Flum and M. Grohe. 2006. *Parameterized Complexity Theory*. Springer.
- [27] D. Gale and L. S. Shapley. 1962. College Admissions and the Stability of Marriage. *The American Mathematical Monthly* 120, 5 (1962), 386–391.
- [28] M.R. Garey and D.S. Johnson. 1979. *Computers and Intractability—A Guide to the Theory of NP-Completeness*. W. H. Freeman and Company.
- [29] D. Gusfield and R.W. Irving. 1989. *The Stable Marriage Problem—Structure and Algorithms*. MIT Press.
- [30] R.W. Irving. 1994. Stable marriage and indifference. *Discrete Applied Mathematics* 48, 3 (1994), 261–272.
- [31] R.W. Irving. 2016. Optimal Stable Marriage. In *Encyclopedia of Algorithms*, Ming-Yang Kao (Ed.). 1470–1473.
- [32] R.W. Irving. 2016. Stable Marriage. In *Encyclopedia of Algorithms*, Ming-Yang Kao (Ed.). 2060–2064.
- [33] K. Iwama, D. Manlove, S. Miyazaki, and Y. Morita. 1999. Stable Marriage with Incomplete Lists and Ties. In *Proceedings of ICALP-99*. 443–452.
- [34] K. Iwama and S. Miyazaki. 2008. A Survey of the Stable Marriage Problem and Its Variants. In *International Conference on Informatics Education and Research for Knowledge-Circulating Society*. 131–136.
- [35] B. Klaus, D.F. Manlove, and F. Rossi. 2016. Matching under Preferences. In *Handbook of Computational Social Choice*.
- [36] V. Knoblauch. 2010. Recognizing one-dimensional Euclidean preference profiles. *J. Math. Econ.* 46, 1 (2010), 1–5.
- [37] D. Knuth. 1976. *Mariages Stables*. Les Presses de L'Université de Montréal.
- [38] B.M. Maggs and R.K. Sitaraman. 2015. Algorithmic Nuggets in Content Delivery. *SIGCOMM Computer Communication Review* 45, 3 (2015), 52–66.
- [39] D.F. Manlove. 2013. *Algorithmics of Matching Under Preferences*. Vol. 2. WorldScientific.
- [40] D. Manlove. 2016. Hospitals/Residents Problem. In *Encyclopedia of Algorithms*, Ming-Yang Kao (Ed.).
- [41] D. Manlove, R.W. Irving, K. Iwama, S. Miyazaki, and Y. Morita. 2002. Hard variants of stable marriage. *Theor. Comput. Sci.* 276, 1-2 (2002), 261–279.
- [42] D.F. Manlove and G. O'Malley. 2014. Paired and Altruistic Kidney Donation in the UK: Algorithms and Experimentation. *ACM Journal of Experimental Algorithmics* 19, 1 (2014), 2.6:1–2.6:21.
- [43] D.C. McGarvey. 1953. A Theorem on the Construction of Voting Paradoxes. *Econometrica* 21, 4 (1953), 608–610.
- [44] J.A. Mirrlees. 1971. An Exploration in the Theory of Optimal Income Taxation. *Rev. Econ. Studies* 38 (1971), 175–208.
- [45] S. Miyazaki and K. Okamoto. 2017. Jointly Stable Matchings. In *Proc. ISAAC-17*. 56:1–56:12.
- [46] R. Niedermeier. 2006. *Invitation to Fixed-Parameter Algorithms*. Oxford University Press.
- [47] C.H. Papadimitriou and M. Yannakakis. 1996. On Limited Nondeterminism and the Complexity of the V-C Dimension. *J. Comput. System Sci.* 53, 2 (1996), 161–170.
- [48] K.W.S. Roberts. 1977. Voting over income tax schedules. *J. Public Econ.* 8, 3 (1977), 329–340.
- [49] A.E. Roth, T. Sönmez, and M. Utku Ünver. 2007. Efficient Kidney Exchange: Coincidence of Wants in Markets with Compatibility-Based Preferences. *Amer. Econ. Rev.* 97, 3 (2007), 828–851.
- [50] A.E. Roth and M.A.O. Sotomayor. 1992. *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*. Part of Econometric Society Monographs.
- [51] P. Schweitzer. 2017. A Polynomial-Time Randomized Reduction from Tournament Isomorphism to Tournament Asymmetry. In *Proceedings of ICALP-17*. 66:1–66:14.
- [52] F. Wagner. 2007. Hardness Results for Tournament Isomorphism and Automorphism. In *Proceedings of MFCS-07*. 572–583.
- [53] B.P. Weems. 1999. Bistable Versions of the Marriages and Roommates Problems. *J. Comput. System Sci.* 59, 3 (1999), 504–520.