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Multiwinner approval rules as apportionment methods

Markus Brill
Technische Universität Berlin, Berlin, Germany

Jean-François Laslier
Paris School of Economics, Paris, France

Piotr Skowron
University of Warsaw, Warszawa, Poland

Abstract
We establish a link between multiwinner elections and apportionment problems by showing how approval-based multiwinner election rules can be interpreted as methods of apportionment. We consider several multiwinner rules and observe that some, but not all, of them induce apportionment methods that are well-established in the literature and in the actual practice of representation, be it proportional or non-proportional. For instance, we show that proportional approval voting induces the D'Hondt method and that Monroe's rule induces the largest remainder method. Our approach also yields apportionment methods implementing degressive proportionality. Furthermore, we consider properties of apportionment methods and exhibit multiwinner rules that induce apportionment methods satisfying these properties.

Keywords
Apportionment; multiwinner elections; panachage; personalized voting

1. Introduction

In a parliamentary election, the candidates are traditionally organized in political parties and the election determines how many parliamentary seats each party is allocated. Under so-called 'closed party list' systems of representation, a voter is allowed to give her vote to one and only one party. In a sense, this forces the voter to approve all candidates from

Corresponding author:
Piotr Skowron, University of Warsaw, Warszawa, Poland.
Email: p.skowron@mimuw.edu.pl

one party and no candidate from any other party. Counting the ballots under a closed-list system amounts to deciding how many seats each party gets based on the 'size' of each party in the electorate, and is therefore formally an *apportionment* problem, identical, for instance, to the problem of allocating seats to the states of a union based on the population figures.

Though widely used (Farrell, 2001), closed-list systems have a number of drawbacks. For instance, it is a known feature of these systems that candidates tend to campaign *within* their parties (for being placed on a good position on the party list), rather than campaigning for the citizens' votes. Closed-list systems thus favor party discipline, at the potential expense of alienating the political elites from the citizens (e.g. see Ames, 1995; André et al., 2015; Chang, 2005; Colomer, 2011).

In an attempt to overcome these drawbacks, many countries use various forms of 'open-list' system, keeping the idea of party lists but giving some flexibility to the voters by allowing each voter to vote for specific candidates inside a chosen party list. In some (rare) cases, voters are given even more freedom. Under so-called 'panachage' systems, sometimes used in Luxembourg and in France, voters can vote for candidates from different parties. Personalization is complete when a rule allows voters to vote directly for individual candidates, and the outcome of the election does not depend on how candidates are grouped into parties. Such is the case for some elections in Switzerland, were variants of multiwinner approval voting are used in several cantons (see Laslier and Van der Straeten, 2016). In a recent book, Renwick and Pilet (2016) extensively examine the trend toward greater personalization, which they see as 'one of the key shifts in contemporary politics.'

This paper will take as its object such fully open voting systems. In the literature, they are called 'multiwinner' rules because they elect a fixed number of candidates, for instance a whole parliament. In a multiwinner election, we are given a set of voters who entertain preferences over a set of alternatives. Based on these preferences, the goal is to select a *committee*, i.e. a (fixed-size) subset of the alternatives. Preferences are usually specified either as rankings, i.e. linear orders over the set of all alternatives (e.g. Brams and Fishburn, 2002; Elkind et al., 2017), or as approval votes, i.e., yes/no assessments of the alternatives (e.g. Kilgour and Marshall, 2012). We are particularly interested in the latter variant, where each voter can be thought of as specifying a subset of alternatives that she finds 'acceptable'.

The decision scenario modeled by multiwinner elections—selecting a subset of objects from a potentially much larger pool of available objects—is ubiquitous, and includes picking players to form a sports team, selecting items to display in an online shop, choosing the board of directors of a company, etc. (for a more detailed discussion on the applications of multiwinner election rules, we refer the reader to the recent survey by Faliszewski et al. 2017), but we are here chiefly interested in the political context. In this context, even if the voting rule allows personal candidacy and votes, most candidates do belong to political parties. It therefore makes sense to study, as a theoretical benchmark, the behavior of a multiwinner rule in the specific cases where voter preferences perfectly reflect some underlying party structure.

To do so, we imagine that the ballots cast are such that, when a voter votes for a candidate belonging to some party, she also votes for all the candidates of that party, and for no other candidates. As we have already mentioned, counting such ballots, and

deciding how many candidates are elected from each party, is an apportionment problem. Any apportionment problem can be seen as a very simple approval voting instance: all voters approve all the candidates from their chosen party, and only those. By contrast, in the context of approval-based multiwinner elections, voters are allowed to express more fine-grained preferences over individual candidates rather than over whole political parties.

This paper formally establishes and explores this analogy between multiwinner elections and apportionment problems. We show how an apportionment problem can be phrased as an instance of an approval-based multiwinner election, thereby rendering multiwinner rules applicable to the apportionment setting. As a result, every approval-based multiwinner rule induces a method of apportionment. Exploring this link between multiwinner rules and apportionment methods is interesting for at least two reasons. First, observing what kind of apportionment method a given multiwinner rule induces yields new insights into the nature of the rule. Indeed, we argue that, in order to understand multiwinner voting rules, one should first understand what they do in the important subdomain of apportionment. Second, every multiwinner rule inducing a given apportionment method can be seen as an extension of the apportionment method to a more general setting where candidates have no party affiliations (or party affiliations are ignored in the election process).

After formally establishing the link between approval-based multiwinner rules and apportionment methods, we consider several known multiwinner rules and observe that some of them induce (and extend) apportionment methods that are well-established in the apportionment literature. For instance, *proportional approval voting (PAV)* induces the D'Hondt method (aka the Jefferson method) and *Monroe's rule* induces the largest remainder method (aka the Hamilton method). We also consider properties of apportionment methods related to proportionality (such as *lower quota*), degressive proportionality (such as the *Penrose condition*), or thresholds of representation, and exhibit multiwinner rules that induce apportionment methods satisfying these properties. Consequently, our work provides new tools for a normative comparison of various multiwinner voting rules, with respect to different forms of proportionality and non-proportionality.

The paper is organized as follows. Section 2 formally introduces both the apportionment problem and the multiwinner election setting. Section 3 shows how approval-based multiwinner rules can be employed as apportionment methods, and contains several results related to proportional representation. Section 4 is devoted to non-proportional representation in the form of degressive proportionality and thresholds, and Section 5 concludes the paper.

2. The apportionment problem and approval-based multiwinner elections

In this section, we provide the formal setting for the apportionment problem and for approval-based multiwinner elections. For a natural number $t \in \mathbb{N} = \{1, 2, \dots\}$, let $[t]$ denote the set $\{1, 2, \dots, t\}$.

2.1. Apportionment methods

In the apportionment setting, there is a finite set of voters and a finite set of p parties $P_1, \dots, P_p$. Every voter votes for exactly one party, and for each $i \in [p]$, we let v_i denote the number of votes that party P_i receives, i.e. the number of voters who voted for P_i. The goal is to allocate h (parliamentary) *seats* among the parties. Formally, an instance of the apportionment problem is given by a tuple (v, h), where $v = (v_1, \dots, v_p) \in \mathbb{N}^p$ is the vote distribution and $h \in \mathbb{N}$ is the number of seats to distribute. We use v_+ to denote the total number of votes, $v_+ = \sum_{i=1}^p v_i$. Throughout this paper, we assume that $v_i > 0$ for all $i \in [p]$ and $h > 0$. An *apportionment method* M maps every instance (v, h) to a nonempty set[1] $M(v, h)$ of *seat distributions*. A seat distribution is a vector $(x_1, \dots, x_p) \in (\mathbb{N} \cup \{0\})^p$ with $\sum_{i=1}^p x_i = h$. Here, x_i corresponds to the number of seats allocated to party P_i.

In our proofs we often argue about seat distributions that result from a given seat distribution by taking away a single seat from a party and giving it to another party. For a seat distribution $x = (x_1, \dots, x_p)$ and two parties P_i and P_j such that $x_i > 0$, let $x_{i \frown j}$ denote the seat distribution x' with $x'_i = x_i - 1$, $x'_j = x_j + 1$, and $x'_\ell = x_\ell$ for all $\ell \notin \{i, j\}$.

2.1.1. Divisor methods. A rich and very well-studied class of apportionment methods is defined via divisor sequences.

Definition 1 (Divisor method). Let $d = (d(0), d(1), d(2), \dots)$ be a sequence with $0 < d(j) \le d(j + 1)$ for all $j \in \mathbb{N} \cup \{0\}$. The *divisor method based on d* is the apportionment method that maps a given instance (v, h) to the set of all seat allocations that can result from the following procedure: Start with the empty seat allocation $(0, \dots, 0)$ and iteratively assign a seat to a party P_i maximizing $v_i/d(s_i)$, where s_i is the number of seats that have already been allocated to party P_i.

Divisor methods are often defined in a procedurally different, but mathematically equivalent way.[2] Two prominent divisor methods are the D'Hondt method and the Sainte-Laguë method, but several other divisor methods, such as the *Huntington–Hill method* (aka the *method of equal proportions*), the *Adams method* (aka the *method of smallest divisors*), and the *Dean method* are studied in the literature on fair representation. We refer the reader to the books of Balinski and Young (1982) and Pukelsheim (2014) for an extensive overview of these methods, their history, and their use in various countries.

Definition 2 (D'Hondt method). The *D'Hondt method* (aka the *Jefferson method* or the *Hagenbach–Bischoff method*) is the divisor method based on the sequence $d_{\text{D'Hondt}} = (1, 2, 3, \dots)$. Therefore, in each round, a seat is allocated to a party P_i maximizing $v_i/(s_i+1)$, where s_i is the number of seats that have already been allocated to party P_i.

The D'Hondt method was first used in 1791 to apportion seats in the US House of Representatives; currently it is used as a legislative procedure in over 40 countries, often coupled with thresholds of representation (see Section 4.2 on this last point).

Definition 3 (Sainte-Laguë method). The *Sainte-Laguë method* (aka the *Webster method*, the *Schepers method*, or the *method of major fractions*) is the divisor method based on the sequence $d_{\text{SL}} = (1, 3, 5, \dots)$. Thus, in each round, a seat is allocated to a party P_i maximizing $v_i/(2s_i+1)$, where s_i is the number of seats that have been already allocated to party P_i.

Table 1. Tables illustrating the computation of (left) the D'Hondt method and (right) the Sainte-Laguë method in Example 1. A column corresponding to party P_i contains the ratios $v_i/d(0), v_i/d(1), v_i/d(2), \ldots$ with the sequences of denominators given by the respective divisor method (the ratios are rounded to two decimal places). The $h = 10$ largest numbers in each table are printed in bold and correspond to the resulting seat allocation.

	P_1	P_2	P_3	P_4
$v_i/1$	6	7	**39**	**48**
$v_i/2$	3	3.5	**19.5**	**24**
$v_i/3$	2	2.33	**13**	**16**
$v_i/4$	1.5	1.75	**9.75**	**12**
$v_i/5$	1.2	1.4	7.8	**9.6**
$v_i/6$	1	1.17	6.5	**8.0**
$v_i/7$	0.86	1	5.57	6.86

	P_1	P_2	P_3	P_4
$v_i/1$	**6**	**7**	**39**	**48**
$v_i/3$	2	2.33	**13**	**16**
$v_i/5$	1.2	1.4	**7.8**	**9.6**
$v_i/7$	0.86	1.0	**5.57**	**6.86**
$v_i/9$	0.67	0.78	4.33	5.33

The Sainte-Laguë method was first adopted in 1842 for allocating seats in the US House of Representatives; currently it is used for parliamentary election in some countries (for instance, Latvia, New Zealand, Norway, and Sweden) and for several state-level legislatures in Germany.

The following example illustrates these two methods.

Example 1. Consider the instance with four parties P_1, P_2, P_3, P_4 and 100 voters such that $(v_1, v_2, v_3, v_4) = (6, 7, 39, 48)$, and assume that there are $h = 10$ seats to be allocated. The outcomes of the D'Hondt method and the Sainte-Laguë method, which can be computed with the help of Table 1, are $(0, 0, 4, 6)$ and $(1, 1, 4, 4)$, respectively.

As we can see in this example, different divisor methods might give different results for some instances of the apportionment problem. In particular, the D'Hondt method slightly favors large parties over small ones, in comparison with the Sainte-Laguë method (see, e.g. Balinski and Young, 1982).

2.1.2. The largest remainder method. The largest remainder method is the most well-known apportionment method that is not a divisor method. Recall that $v_+ = \sum_{i=1}^{p} v_i$ denotes the total number of votes.

Definition 4 (Largest remainder method). The *largest remainder method* (aka the *Hamilton method* or the *Hare–Niemeyer method*) is defined via two steps. In the first step, each party P_i is allocated $\lfloor \frac{v_i h}{v_+} \rfloor$ seats. In the second step, the remaining seats are distributed among the parties so that each party gets at most one of them. To do so, the parties are sorted according to the remainders $\frac{v_i h}{v_+} - \lfloor \frac{v_i h}{v_+} \rfloor$ and the remaining seats are allocated to the parties with the largest remainders.

The largest remainder method was first proposed by Alexander Hamilton in 1792 and was used as a rule of distributing seats in the US House of Representatives between 1852 and 1900. Currently, it is used for parliamentary elections in Russia, Ukraine, Tunisia, Namibia, and Hong Kong.

Example 2. Consider again the instance in Example 1. According to the largest remainder method, in the first step, party P_1 is allocated $\lfloor \frac{6 \cdot 10}{100} \rfloor = 0$ seats, P_2 gets $\lfloor \frac{7 \cdot 10}{100} \rfloor = 0$ seats, P_3 gets $\lfloor \frac{39 \cdot 10}{100} \rfloor = 3$ seats, and P_4 gets $\lfloor \frac{48 \cdot 10}{100} \rfloor = 4$ seats. There are three more seats to be distributed among the parties. The three highest fractional parts are $\frac{9}{10}$ for party P_3, $\frac{8}{10}$ for party P_4 and $\frac{7}{10}$ for party P_2, thus these three parties are allocated one additional seat each. The resulting seat allocation is $(0, 1, 4, 5)$.

2.1.3. Properties of apportionment methods related to proportionality. The literature on proportional representation has identified a number of desirable properties of apportionment methods (Balinski and Young, 1982; Pukelsheim, 2014). Here, we focus on properties requiring that the proportion of seats in the resulting apportionment should reflect, as closely as possible, the proportion of the votes cast for respective parties.

Definition 5. An apportionment method *respects lower quota* if, for every instance (v, h), each party P_i gets at least $\lfloor \frac{v_i h}{v_+} \rfloor$ seats. An apportionment method *respects quota* if each party P_i gets either $\lfloor \frac{v_i h}{v_+} \rfloor$ or $\lceil \frac{v_i h}{v_+} \rceil$ seats.

Clearly, any apportionment method respecting lower quota also respects quota. It is well-known that the largest remainder method respects quota, but no divisor method does. Moreover, the D'Hondt method is the only divisor method respecting lower quota (Balinski and Young, 1982).

2.2. Approval-based multiwinner election rules

We now introduce the setting of approval-based multiwinner elections. We have a finite set $N = \{1, 2, \ldots, n\}$ of voters and a finite set $C = \{c_1, c_2, \ldots, c_m\}$ of candidates. Voters each express their preferences by approving a subset of candidates, and we want to select a committee consisting of exactly k candidates. We will refer to k-element subsets of C as *size-k committees*, and we let $A_i \subseteq C$ denote the set of candidates approved by voter $i \in N$. Formally, an instance of the approval-based multiwinner election problem is given by a tuple (A, k), where $A = (A_1, A_2, \ldots, A_n)$ is a *preference profile* and k is the desired committee size. An *approval-based multiwinner election rule* (henceforth *multiwinner rule*) $\mathcal{R}$ is a function that maps every instance (A, k) to a nonempty set[3] $\mathcal{R}(A, k)$ of size-k committees. Every element of $\mathcal{R}(A, k)$ is referred to as a *winning committee*.

2.2.1. Thiele rules. A remarkably general class of multiwinner election rules was proposed by Danish polymath Thorvald N Thiele (1895). A *weight sequence* is an infinite sequence of real numbers $w = (w_1, w_2, \ldots)$.

Definition 6 (Thiele rules). Consider a weight sequence w, a committee $S \subseteq C$, and a voter $i \in N$ with approval set $A_i \subseteq C$. The *satisfaction* of i from S given w is defined as $u_i^w(S) = \sum_{j=1}^{|A_i \cap S|} w_j$, with $u_i^w(S) = 0$ if $A_i \cap S = \emptyset$. Given an instance (A, k) of the multiwinner election problem, the rule w-*Thiele* selects all size-k committees S that maximize the total satisfaction $\sum_{i \in N} u_i^w(S)$.

Note that multiplying a weight sequence w by a positive constant does not change the way in which a rule operates. Several established multiwinner election rules can be described as Thiele rules.

Definition 7. The *Chamberlin–Courant rule* is w_{CC}-Thiele, where $w_{CC} = (1, 0, 0, \dots)$.

The Chamberlin–Courant rule is usually defined in the context of multiwinner elections where voter preferences are given by *ranked ballots*, and voters each derive satisfaction only from their most preferred members of the committee (Chamberlin and Courant, 1983). Our definition of the rule is a straightforward adaption of the approval setting: a voter is satisfied with a committee if and only if it contains at least one candidate that the voter approves.

Definition 8. *Proportional approval voting (PAV)* is w_{PAV}-Thiele, where $w_{PAV} = (1, 1/2, 1/3, \dots)$.

Though sometimes attributed to Forest Simmons, PAV was already proposed and discussed by Thiele in the 19th century (Janson, 2016; Thiele, 1895).[4] According to PAV, each voter cares about the whole committee, but the marginal gain of satisfaction of an already satisfied voter from an additional approved committee member is lower than the gain of a less satisfied voter. The reason for using the particular weight sequence w_{PAV} is not obvious. Aziz et al. (2017) and Sánchez-Fernández et al. (2017) provide compelling arguments by showing that w_{PAV} is the unique weight sequence w such that w-Thiele satisfies certain axiomatic properties. Theorem 2 in this paper can be viewed as an additional—though related—argument in favor of the weight sequence w_{PAV}.

Definition 9. The *top-k rule* is w_{top-k}-Thiele, where $w_{top-k} = (1, 1, 1, \dots)$.

According to the top-k rule, the winning committee contains the k candidates that have been approved by the greatest number of voters. This very natural rule is called *simple approval* by Kilgour (2010).

The following example illustrates these Thiele rules.

Example 3. Let $k = 3$ and consider the following preference profile with 12 voters:

$A_1 = \{c_1\}$	$A_5 = \{c_1, c_4, c_5, c_6\}$	$A_9 = \{c_2\}$
$A_2 = \{c_1, c_3, c_5\}$	$A_6 = \{c_1, c_4, c_5, c_6\}$	$A_{10} = \{c_3, c_5\}$
$A_3 = \{c_1, c_5, c_6\}$	$A_7 = \{c_1, c_4\}$	$A_{11} = \{c_3\}$
$A_4 = \{c_1, c_5, c_6\}$	$A_8 = \{c_2, c_4, c_6\}$	$A_{12} = \{c_3\}$

According to the Chamberlin–Courant rule, the unique winning committee is $\{c_1, c_2, c_3\}$. Indeed, for this committee each voter approves of at least one committee member (voters 1–7 approve of c_1, voters 8 and 9 approve of c_2, and voters 10–12 approve of c_3), thus each voter gets satisfaction equal to one from this committee. Clearly, this is the highest satisfaction that the voters can get from a committee. The reader might check that $\{c_1, c_2, c_3\}$ is the only committee for which each voter has at least one approved committee member.

According to PAV, $\{c_1, c_3, c_6\}$ is the unique winning committee. Let us compute the PAV satisfaction of voters from this committee. Voters 1, 7, 8, and 10–12 approve of

a single committee member, thus the satisfaction of these voters is equal to one. Voter 9 does not approve any committee member, thus her satisfaction is equal to zero. The remaining five candidates each approve of two committee members, so their satisfaction from the committee is equal to $1 + \frac{1}{2}$. Thus, the total satisfaction of the voters from $\{c_1, c_3, c_6\}$ is equal to $6 + 5 \cdot \frac{3}{2} = 13.5$ and it can be checked that the satisfaction of voters from all other committees is lower.

The top-k rule, conversely, selects $\{c_1, c_5, c_6\}$ as the unique winning committee. Indeed, c_1, c_5, and c_6 are approved by seven, six, and five voters, respectively, and any other candidate is approved by fewer than five voters.

Other appealing Thiele rules include the *t-best Thiele rule*, which is defined by the weight sequence $w = (1, \dots, 1, 0, 0 \dots)$ with t ones followed by zeros, and the *t-median Thiele rule*, which is defined by the weight sequence $w = (0, \dots, 0, 1, 0, 0, \dots)$, where the 1 appears at position t (Skowron et al., 2016).

2.2.2. Sequential Thiele rules.
Another interesting class of multiwinner rules, sometimes referred to as *reweighted approval voting (RAV)*, consists of *sequential* variants of Thiele rules.

Definition 10. The rule *sequential w-Thiele* (aka *w*-RAV) selects all committees that can result from the following procedure. Starting with the empty committee S, in k consecutive steps add to the committee S a candidate c that maximizes $\sum_{i \in N} u_i^w(S \cup \{c\}) - u_i^w(S)$.

Just like Thiele rules, sequential Thiele rules have already been considered by Thiele (1895). *Sequential PAV* (i.e. sequential w_{PAV}-Thiele) was used for a short period in Sweden during the early 1900s (Janson, 2016).

Interestingly, a sequential Thiele rule approximates the original Thiele rule whenever the weight sequence w is non-increasing (Skowron et al., 2016). Sequential Thiele rules are appealing alternatives to their original variants for a number of reasons. First, they are computationally tractable, while finding winning committees for (non-sequential) Thiele rules is often NP-hard. Second, sequential Thiele rules satisfy properties that are violated by non-sequential Thiele rules; for instance, they (trivially) satisfy committee monotonicity while some of the non-sequential versions do not (see, e.g. Kilgour and Marshall, 2012). Third, sequential methods are easier to describe to non-experts, as compared with rules formulated as non-trivial combinatorial optimization problems.

Example 4. Consider the instance from Example 3. Sequential PAV selects candidate c_1 in the first iteration. To decide which candidate is selected in the second iteration, one must compare all remaining candidates with respect to their contribution to the total satisfaction of the voters. For instance, by selecting c_3, the satisfaction of voters will increase by $\frac{1}{2} + 3$. One can check that there are two optimal choices in the second step: selecting c_3 or selecting c_5. If c_3 is selected in the second step, then our procedure will select c_6 in the third step (selecting c_6 increases the satisfaction of voters by $1 + 4 \cdot \frac{1}{2}$). If c_5 is selected in the second step, then c_3 must be chosen in the third step (by choosing c_3 we increase the satisfaction of voters by $\frac{1}{3} + 5 \cdot \frac{1}{2}$). Consequently, there are two winning committees according to sequential PAV: $\{c_1, c_3, c_5\}$ and $\{c_1, c_3, c_6\}$.

2.2.3. Monroe's rule. The optimization problem underlying the Chamberlin–Courant rule can be thought of in terms of maximizing representation: every voter is assigned to a single candidate in the committee and this 'representative' completely determines the satisfaction that the voter derives from the committee. The rule proposed by Monroe (1995) is based on the same idea; however, Monroe requires each candidate to represent the same number of voters. For the sake of simplicity, when considering the Monroe rule, we assume that the number of voters n is divisible by the size of the committee k.

Definition 11. Assume that k divides n and consider a size-k committee S. A *balanced allocation* of the voters to the candidates in S is a function $\tau_S : N \rightarrow S$ such that $|\tau_S^{-1}(c)| = n/k$ for all $c \in S$. The satisfaction $u_i(\tau_S)$ of a voter i from τ_S is equal to one if i approves of $\tau_S(i)$, and zero otherwise. The total satisfaction of voters provided by S, denoted $u(S)$, is defined as the satisfaction from the best balanced allocation, i.e. $u(S) = \max_{\tau_S} \sum_{i \in N} u_i(\tau_S)$. The *Monroe rule* selects all committees S maximizing $u(S)$.

Example 5. Consider again the instance from Example 3 and consider committee $\{c_1, c_2, c_3\}$, which is a winning committee under the Chamberlin–Courant rule. Even though every voter has an approved committee member (voters 1–7 approve of c_1, voters 8 and 9 approve of c_2, and voters 10–12 approve of c_3), according to Monroe this committee has a total satisfaction of 10 only: since only two voters approve of c_2, there is no balanced allocation $\tau_{\{c_1,c_2,c_3\}}$ mapping every voter to an approved candidate in $\{c_1, c_2, c_3\}$. The Monroe rule selects $\{c_1, c_4, c_3\}$ as the unique winning committee. For this committee, the optimal balanced allocation maps voters 1–4 to their approved candidate c_1, voters 5–8 to their approved candidate c_4, and voters 9–12 to c_3 (only three of those four voters approve of c_3). Thus, the winning committee has a total satisfaction of 11.

2.2.4. Phragmén's rules. In the late 19th century, the Swedish mathematician Lars Edvard Phragmén proposed several methods for selecting committees based on approval votes (Phragmén, 1894, 1895, 1896, 1899). Here, we formulate two particularly interesting variants (see Brill et al., 2017; Janson, 2016).

The motivation behind Phragmén's rules is to find a committee whose 'support' is distributed as evenly as possible among the electorate. For every candidate in the committee, one unit of 'load' needs to be distributed among the voters approving this candidate. The goal is to find a committee of size k for which the maximal load of a voter (or the variance of load distribution) is as small as possible.

Definition 12. Consider an instance (A, k) of the multiwinner election problem. A load distribution for (A, k) is a matrix $L = (\ell_{i,c})_{i \in N, c \in C} \in \mathbb{R}_{\geq 0}^{|N| \times |C|}$ such that:

(i) $\sum_{i \in N} \sum_{c \in C} \ell_{i,c} = k$.
(ii) $\sum_{i \in N} \ell_{i,c} \in \{0, 1\}$ for all $c \in C$.
(iii) $\ell_{i,c} = 0$ if $c \notin A_i$.

Every load distribution L corresponds to a size-k committee $S_L = \{c \in C : \sum_{i \in N} \ell_{i,c} = 1\}$.

The multiwinner election rules max-Phragmén and var-Phragmén are defined as optimization problems over the set of all load distributions.[5]
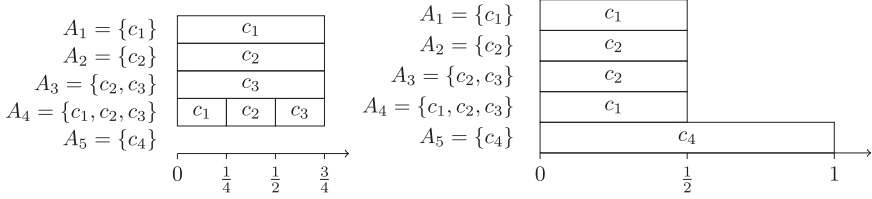
Figure 1. Illustration of Example 6. Left: a load distribution minimizing the maximal voter load. Right: the unique load distribution minimizing the variance of voter loads.

Definition 13. The rule *max-Phragmén* maps an instance (A, k) to the set of size-k committees corresponding to load distributions L minimizing $\max_{i \in N} \sum_{c \in C} \ell_{i,c}$. The rule var-Phragmén maps an instance (A, k) to the set of size-k committees corresponding to load distributions L minimizing $\sum_{i \in N} (\sum_{c \in C} \ell_{i,c})^2$.

Example 6. Let $k = 3$ and consider the following preference profile with five voters:

$$A_1 = \{c_1\} \qquad A_3 = \{c_2, c_3\} \qquad A_5 = \{c_4\}$$
$$A_2 = \{c_2\} \qquad A_4 = \{c_1, c_2, c_3\}$$

It can be checked that max-Phragmén selects the committee $\{c_1, c_2, c_3\}$ and that var-Phragmén selects the committee $\{c_1, c_2, c_4\}$. Optimal load distributions corresponding to these committees are illustrated in Figure 1. Load distributions minimizing the maximal voter load (like the one illustrated on the left in Figure 1) satisfy $\max_{i \in N} \sum_{c \in C} \ell_{i,c} = 3/4$ and $\sum_{i \in N} (\sum_{c \in C} \ell_{i,c})^2 = 4 \cdot (3/4)^2 = 9/4$, and the load distribution minimizing the variance of voter loads (illustrated on the right in Figure 1) satisfies $\max_{i \in N} \sum_{c \in C} \ell_{i,c} = 1$ and $\sum_{i \in N} (\sum_{c \in C} \ell_{i,c})^2 = 4 \cdot (1/2)^2 + 1^2 = 2$.

3. Apportionment via multiwinner election rules

In this section, we demonstrate how approval-based multiwinner rules can be employed as apportionment methods. For a given instance (v, h) of the apportionment problem, this procedure involves three steps:

1. Translate (v, h) into an instance (A, k) of the multiwinner problem.
2. Apply the multiwinner rule $\mathcal{R}$ to (A, k).
3. Translate committee(s) in $\mathcal{R}(A, k)$ into seat distribution(s) for (v, h).

We now describe each step in detail.

Step 1. Given an instance (v, h) of the apportionment problem, we construct an instance (A, k) of the multiwinner problem as follows. For each $i \in [p]$, we introduce a set C_i consisting of h candidates, and a set N_i consisting of v_i voters. Each voter in N_i approves of all candidates in C_i (and of no other candidates). Furthermore, we define $C = \bigcup_{i=1}^{p} C_i$, $N = \bigcup_{i=1}^{p} N_i$, and $k = h$. Intuitively, C_i is the set of members of party P_i and N_i is the set of voters who voted for party P_i.

Step 2.   We can now apply multiwinner rule $\mathcal{R}$ to (A, k) in order to find the set $\mathcal{R}(A, k)$ of winning committees.

Step 3.   For every winning committee $S \in \mathcal{R}(A, k)$, we can extract a seat distribution x for the instance (v, h) in the following way: the number x_i of seats of a party P_i is given by the number of candidates from C_i in the committee S, i.e. $x_i = |C_i \cap S|$.
   The next example illustrates this three-step procedure.

Example 7 (Sequential PAV as an apportionment method). Consider   the   instance $(v, h) = ((20, 40, 30, 10), 10)$ of the apportionment problem. We construct an instance of the multiwinner election setting with 40 candidates; each party P_i has a set C_i consisting of 10 candidates. Further, there are $\sum_{i=1}^{4} v_i = 100$ voters: 20 voters approve the 10 candidates in C_1, 40 voters approve the 10 candidates in C_2, 30 voters the 10 candidates in C_3, and 10 voters the 10 candidates in C_4. Applied to this instance, sequential PAV selects a winning committee consisting of two candidates from C_1, four candidates from C_2, three candidates from C_3, and one candidate from C_4. Thus, the resulting seat allocation is $(x_1, x_2, x_3, x_4) = (2, 4, 3, 1)$.

   For a given multiwinner rule $\mathcal{R}$, we let $M_{\mathcal{R}}$ denote the apportionment method defined by steps 1 to 3. We say that a multiwinner election rule $\mathcal{R}$ *satisfies (lower) quota* if the corresponding apportionment method $M_{\mathcal{R}}$ satisfies the respective property.

3.1. Apportionment methods induced by Thiele rules

In this section, we consider apportionment methods induced by Thiele rules. Fix a weight sequence w and let $\mathcal{R}$ denote the rule w-Thiele. For every instance (v, h) of the apportionment setting, $M_{\mathcal{R}}(v, h)$ contains all seat distributions x maximizing the total voter satisfaction $u(x)$, which is given by

$$u(x) = \sum_{i \in [p]} v_i u_i^w(x) = \sum_{i \in [p]} v_i \left(\sum_{j=1}^{x_i} w_j \right) \tag{1}$$

Here, $u_i^w(x) = \sum_{j=1}^{x_i} w_j$ denotes the satisfaction of a voter from party P_i.
   We let $\hat{u}(i, s)$ denote the marginal increase in $u(\cdot)$ when assigning the sth seat to party P_i. Owing to equation (1), we have $\hat{u}(i, s) = v_i w_s$. Taking away a seat from party P_i and giving it to party P_j results in a change of total voter satisfaction of

$$u(x_{i \frown j}) - u(x) = \hat{u}(j, x_j + 1) - \hat{u}(i, x_i) = v_j w_{x_j+1} - v_i w_{x_i}$$

Optimality of $x \in M_{\mathcal{R}}(v, h)$ implies that $u(x_{i \frown j}) - u(x) \leq 0$ for all $i, j \in [p]$ with $x_i > 0$.
   Whenever the weight vector w is non-increasing, w-Thiele induces the same apportionment method as its sequential variant.[6]

Proposition 1. *Let w be a weight sequence with $w_j \geq w_{j+1} \geq 0$ for all $j \in \mathbb{N}$. The apportionment method induced by w-Thiele coincides with the apportionment method induced by sequential w-Thiele.*

Proof. Fix a weight sequence $w = (w_1, w_2, \dots)$ satisfying $w_j \geq w_{j+1} \geq 0$ for all $j \in \mathbb{N}$. Let $\mathcal{R}$ be w-Thiele and let $\mathcal{R}'$ be sequential w-Thiele. We show that the apportionment methods induced by $\mathcal{R}$ and $\mathcal{R}'$ coincide, i.e. $M_{\mathcal{R}'} = M_{\mathcal{R}}$.

Consider an instance (v, h) of the apportionment setting. Then $M_{\mathcal{R}}(v, h)$ contains all seat distributions x maximizing the total voter satisfaction $u(x)$, which is given by equation (1).

Now consider the apportionment method $M_{\mathcal{R}'}$ induced by sequential w-Thiele. This method starts with the empty seat allocation $x^{(0)} = (0, \dots, 0)$ and iteratively assigns a seat to a party P_i such that the *marginal increase* in total voter satisfaction is maximized. Recall that $\hat{u}(i, s)$ denotes the marginal increase in $u(\cdot)$ when assigning the sth seat to party P_i. Owing to equation (1), this quantity is independent of x_j for $j \neq i$ and equals $\hat{u}(i, s) = v_i w_s$. For $\ell = 1, \dots, h$, method $M_{\mathcal{R}'}$ iteratively chooses a party P_{i^*} with $i^* \in \arg\max_{i \in [p]} \hat{u}(i, x_i^{(\ell-1)} + 1)$ and sets $x^{(\ell)} = (x_1^{(\ell-1)}, \dots, x_{i^*}^{(\ell-1)} + 1, \dots, x_p^{(\ell-1)})$.

For every seat allocation x, the total voter satisfaction $u(x)$ can be written as $u(x) = \sum_{i \in [p]} \sum_{s=1}^{x_i} \hat{u}(i, s)$. In particular, $u(x)$ is independent of the order in which seats are allocated to parties. Furthermore, our assumption $w_1 \geq w_2 \geq \dots$ implies that the marginal satisfaction $\hat{u}(i, s)$ is monotonically decreasing in s for every party P_i. Therefore, any seat distribution maximizing $u(\cdot)$ (i.e. any $x \in M_{\mathcal{R}}(v, h)$) can be iteratively constructed by applying method $M_{\mathcal{R}'}$. □

An immediate corollary is that Thiele rules with a non-increasing weight sequence, such as PAV, can be computed efficiently in the apportionment setting.

Further, we can use Proposition 1 to show that every Thiele rule with a positive non-increasing weight sequence induces a divisor method. For a weight sequence $w = (w_1, w_2, \dots)$ with $w_j > 0$ for all $j \in \mathbb{N}$, let $d_w = (d_w(0), d_w(1), \dots)$ be the sequence defined by $d_w(s) = 1/w_{s+1}$ for all $s \in \mathbb{N} \cup \{0\}$. For example, $w_{\text{PAV}} = (1, 1/2, 1/3, \dots)$ yields $d_{w_{\text{PAV}}} = (1, 2, 3, \dots)$.

Theorem 1. *Let w be a weight sequence with $w_j \geq w_{j+1} > 0$ for all $j \in \mathbb{N}$. The apportionment method induced by w-Thiele coincides with the divisor method based on $d_w = (1/w_1, 1/w_2, 1/w_3, \dots)$.*

Proof. Fix a weight sequence w with $w_j \geq w_{j+1} > 0$ for all $j \in \mathbb{N}$. Let $\mathcal{R}(w)$ denote the rule w-Thiele, and let $\mathcal{R}'(w)$ denote sequential w-Thiele. It follows from Proposition 1 that $M_{\mathcal{R}(w)}$ coincides with $M_{\mathcal{R}'(w)}$. Furthermore, let $M(d_w)$ denote the divisor method based on d_w. We are going to show that $M_{\mathcal{R}'(w)}$ coincides with $M(d_w)$.

Both $M(d_w)$ and $M_{\mathcal{R}'(w)}$ work iteratively. At each iteration, $M(d_w)$ assigns a seat to a party P_i maximizing $v_i/d_w(s_i)$, where v_i is the number of votes for party P_i and s_i is the number of seats allocated to party P_i in previous iterations. Method $M_{\mathcal{R}'(w)}$, conversely, assigns a seat to a party maximizing the marginal increase in total voter satisfaction. The marginal increase in total voter satisfaction when giving an additional seat to party P_i equals $\hat{u}(i, s_i + 1) = v_i w_{s_i+1}$.

Using $d_w(s) = 1/w_{s+1}$, we can observe that the quantities $(\frac{v_i}{d_w(s_i)})_{i \in [p]}$ used in divisor method $M(d_w)$ exactly coincide with the quantities $(v_i w_{s_i+1})_{i \in [p]}$ used in the method $M_{\mathcal{R}'(w)}$. Thus, both methods assign seats in exactly the same way. □

3.2. Proportional approval voting as an extension of the D'Hondt method

A particularly interesting consequence of Theorem 1 is that PAV, the Thiele rule with $w_{\text{PAV}} = (1, 1/2, 1/3, \dots)$, induces the D'Hondt method.

Corollary 1. *The apportionment method induced by PAV coincides with the D'Hondt method.*

The observation that PAV reduces to the D'Hondt method in the party-list setting occasionally occurs (without proof) in the literature (e.g. see Brams et al., 2017; Pereira, 2016; Sánchez-Fernández et al., 2016).[7] Theorem 1 shows that this is just one special case of the general relationship between Thiele rules and divisor methods (see also Janson, 2016: Remark 11.4).

Since the D'Hondt method satisfies lower quota, the same holds for PAV (and sequential PAV) in the apportionment setting.

Corollary 2. *PAV and sequential PAV satisfy lower quota.*

The fact that PAV satisfies lower quota can also be established by observing that every multiwinner rule satisfying *extended justified representation* (Aziz et al., 2017) or *proportional justified representation* (Sánchez-Fernández et al., 2017) also satisfies lower quota. See the appendix for details.

Further, we show that PAV is the *only* Thiele rule satisfying lower quota.[8]

Theorem 2. *PAV is the only Thiele rule satisfying lower quota.*

Proof. Let $\mathcal{R}$ be a Thiele rule such that $M_{\mathcal{R}}$ satisfies lower quota. We will show that $\mathcal{R}$ is based on a weight sequence w with $w_j = w_1/j$ for all $j \in \mathbb{N}$, from which we will infer that $M_{\mathcal{R}}$ is equivalent to PAV.

Fix $j \in \mathbb{N}$. We will show $w_j = w_1/j$ in two steps.

Step 1. Given a natural number $Z > 1$, we define an instance (v, h) of the apportionment problem as follows. There are $Z + 1$ parties and the vote distribution is given by $v = (Z, jZ - 1, \dots, jZ - 1)$. That is, party P_1 gets Z votes and all other parties get $jZ - 1$ votes. Thus, the total number of votes is $v_+ = Z + Z(jZ - 1) = jZ^2$. Furthermore, we set $h = v_+/Z = jZ$.

Consider a seat allocation $x \in M_{\mathcal{R}}(v, h)$. Since $M_{\mathcal{R}}$ satisfies lower quota, we have

$$x_1 \geq \left\lfloor \frac{Z \cdot jZ}{Z^2 j} \right\rfloor = 1$$

and

$$x_\ell \geq \left\lfloor \frac{(jZ - 1) \cdot jZ}{Z^2 j} \right\rfloor = \left\lfloor \frac{(jZ - 1)}{Z} \right\rfloor = j - 1$$

for all $\ell \neq 1$. Thus, P_1 is allocated at least one seat, and each of the other Z parties is allocated at least $j - 1$ seats. From the pigeonhole principle, we infer that at least one of

the parties $P_2, \ldots, P_{Z+1}$ gets *exactly* $j - 1$ seats. Let P_ℓ be such a party and consider the seat allocation $x_{1 \frown \ell}$. Since x maximizes $u(\cdot)$, we have

$$0 \geq u(x_{1 \frown \ell}) - u(x) = \hat{u}(\ell, j) - \hat{u}(1, 1) = (jZ - 1)w_j - zw_1$$

It follows that

$$w_j \leq \frac{w_1}{j} \cdot \frac{Z}{Z - 1}$$

Since this inequality has to hold for all natural numbers $Z > 1$, we infer that $w_j \leq w_1/j$.

Step 2. Next, we construct another instance (v', h') of the apportionment problem, again parameterized by a natural number $Z > 1$. There are $Z+1$ parties and the vote distribution v' is given by $v' = (jZ, Z - 1, \ldots, Z - 1)$. Thus, $v'_+ = Z^2 - Z + jZ$. The number of seats is given by $h' = v'_+/z = Z + j - 1$.

Consider a seat allocation $x \in M_\mathcal{R}(v', h')$. Since $M_\mathcal{R}$ satisfies lower quota, we infer that

$$x_1 \geq \left\lfloor \frac{jZ(Z + j - 1)}{Z(Z + j - 1)} \right\rfloor = j$$

The pigeonhole principle implies that at least one of parties $P_2, \ldots, P_{Z+1}$ gets no seat. Let P_ℓ be such a party and consider $x_{1 \frown \ell}$. We have

$$0 \geq u(x_{1 \frown \ell}) - u(x) = \hat{u}(\ell, 1) - \hat{u}(1, j) = (Z - 1)w_1 - jZ \cdot w_j$$

and thus $w_j \geq (w_1/j) \cdot (Z-1/z)$. Again, this inequality holds for all Z, and thus $w_j \geq w_1/j$.

We have therefore shown that $w_j = w_1/j$ for all $j \in \mathbb{N}$. It follows that $w = c \cdot w_{\text{PAV}}$ for some constant $c > 0$. As a consequence, $\mathcal{R}$ is equivalent to PAV. $\square$

Theorem 2 characterizes PAV as the only Thiele rule respecting lower quota. Since PAV does not respect (exact) quota, it follows that no Thiele rule respects quota. Owing to Proposition 1, the same is true for sequential Thiele rules.

The load-balancing rule max-Phragmén also induces the D'Hondt method. Indeed, Phragmén formulated his rule as a generalization of the D'Hondt method (Janson, 2016; Mora and Oliver, 2015).

Theorem 3. *The apportionment method induced by max-Phragmén coincides with the D'Hondt method.*

Proof. In the apportionment setting, optimal load distributions have a very simple structure. Given a seat distribution x, it is clearly optimal to distribute the load of x_i (1 for each seat that is allocated to party P_i) uniformly among the v_i voters of the party. Therefore, the maximal voter load for x is given by $\max_{i \in [p]} x_i/v_i$. By definition, the D'Hondt method selects seat distributions minimizing this quantity (see Balinski and Young, 1982: Proposition 3.10). $\square$

There is also a *sequential* variant of Phragmén's rule.[9] It is straightforward to verify that, in the apportionment setting, this variant coincides with max-Phragmén and thus also induces the D'Hondt method.

3.3. Monroe's rule as an extension of the largest remainder method

We now turn to Monroe's multiwinner rule. It turns out that it induces the largest remainder method.

Theorem 4. *Assume that the number of seats divides the total number of voters. Then Monroe's rule induces the largest remainder method.*

Proof. Let $\mathcal{R}$ denote Monroe's rule. Recall that Monroe's rule assigns voters to candidates in a balanced way, so as to maximize the total voter satisfaction. In the apportionment setting, a voter in N_i is satisfied if and only if she is assigned to a candidate in C_i. The apportionment method $M_{\mathcal{R}}$ selects seat distributions maximizing the total voter satisfaction.

For a seat allocation x, let $u(x)$ denote the total voter satisfaction provided by x. We can write $u(x) = \sum_{i \in [p]} u(i, x)$, where $u(i, x)$ is the total satisfaction that voters in N_i derive from x. For a given instance (v, h) of the apportionment problem, $u(i, x)$ can be expressed as

$$u(i, x) = \begin{cases} v_i & \text{if } x_i \geq v_i \frac{h}{v_+} \\ x_i \frac{v_+}{h} & \text{if } x_i < v_i \frac{h}{v_+} \end{cases}$$

Since the case $u(i, x) = v_i$ occurs if and only if $v_i \leq x_i v_+/h$, we have $u(i, x) = \min(v_i, x_i v_+/h)$ and

$$u(x) = \sum_{i \in [p]} u(i, x) = \sum_{i \in [p]} \min\left(v_i, x_i \frac{v_+}{h}\right)$$

We show that $M_{\mathcal{R}}$ coincides with the largest remainder method for all instances (v, h) such that v_+ divides h. Fix such an instance (v, h) and let $x \in M_{\mathcal{R}}(v, h)$. The proof consists of three steps.

Step 1. We first show that $x_i \geq \lfloor \frac{v_i h}{v_+} \rfloor$ for all parties P_i. Assume for contradiction that this is not the case and let P_i be a party with $x_i < \lfloor \frac{v_i h}{v_+} \rfloor$. We infer that $x_i \leq \frac{v_i h}{v_+} - 1$ and, by the pigeonhole principle, that there exists a party P_j such that $x_j > v_j h/v_+$. Thus, $u(i, x) = x_i v_+/h$ and $u(j, x) = v_j$. Therefore

$$u(x_{j \frown i}) - u(x) = u(i, x_{j \frown i}) + u(j, x_{j \frown i}) - u(i, x) - u(j, x)$$
$$= (x_i + 1)\frac{v_+}{h} + \min\left(v_j, (x_j - 1)\frac{v_+}{h}\right) - v_j - x_i \frac{v_+}{h}$$
$$= \frac{v_+}{h} + \min\left(v_j, (x_j - 1)\frac{v_+}{h}\right) - v_j \geq \frac{v_+}{h} > 0$$

contradicting our assumption that $x \in M_{\mathcal{R}}(v, h)$.

Step 2. We next show that $x_i \leq \lfloor \frac{v_i h}{v_+} \rfloor$ for all parties P_i. Assume for contradiction that this is not the case and let P_i be a party with $x_i > \lceil \frac{v_i h}{v_+} \rceil$. Similarly as before, we infer

that $x_i \geq v_i h/v_+ + 1$ and that there exists a party P_j with $x_j < v_j h/v_+$. Thus, $u(i, x) = v_i$ and $u(j, x) = x_j v_+/h$. Therefore

$$u(x_{i \frown j}) - u(x) = u(i, x_{i \frown j}) + u(j, x_{i \frown j}) - u(i, x) - u(j, x)$$

$$= v_i + \min\left(v_j, (x_j + 1)\frac{v_+}{h}\right) - v_i - x_j \frac{v_+}{h}$$

$$= \min\left(v_j, (x_j + 1)\frac{v_+}{h}\right) - x_j \frac{v_+}{h} \geq \frac{v_+}{h} > 0$$

As in step 1, we have reached a contradiction.

Step 3. In the first two steps, we have shown that $\lfloor \frac{v_\ell h}{v_+} \rfloor \leq x_\ell \leq \lceil \frac{v_\ell h}{v_+} \rceil$ for each party P_ℓ. Now we show that $M_\mathcal{R}$ coincides with the largest remainder method. For the sake of contradiction, assume that this is not the case. Then, there exist two parties P_i and P_j such that $x_i = \lfloor \frac{v_i h}{v_+} \rfloor + 1, x_j = \lfloor \frac{v_j h}{v_+} \rfloor$, but the remainder of party P_i is strictly smaller than the remainder of party P_j, i.e., $\frac{v_i h}{v_+} - \lfloor \frac{v_i h}{v_+} \rfloor < \frac{v_j h}{v_+} - \lfloor \frac{v_j h}{v_+} \rfloor$. It follows that

$$\frac{v_i h}{v_+} - (x_i - 1) < \frac{v_j h}{v_+} - x_j \tag{2}$$

Therefore

$$u(x_{i \frown j}) - u(x) = u(i, x_{i \frown j}) + u(j, x_{i \frown j}) - u(i, x) - u(j, x)$$

$$= (x_i - 1)\frac{v_+}{h} + v_j - v_i - x_j \frac{v_+}{h}$$

$$= \frac{v_+}{h}\left(\frac{v_j h}{v_+} - x_j - \frac{v_i h}{v_+} + x_i - 1\right) > 0$$

where the inequality is due to equation (2). We again obtain a contradiction, completing the proof. □

An immediate consequence of Theorem 4 is that Monroe's rule respects quota.

3.4. Phragmén's variance-minimizing rule as an extension of the Sainte-Laguë method

We now turn to the Sainte-Laguë method. Since the Sainte-Laguë method is the divisor method based on $(1, 3, 5, \dots)$, Theorem 1 implies that it is induced by the Thiele rule with weight sequence $(1, 1/3, 1/5, \dots)$.[10] However, this is not the only multiwinner rule inducing the Sainte-Laguë method. Recall that var-Phragmén is the variant of Phragmén's load-balancing methods that minimizes the variance of the loads.

Theorem 5. *The apportionment method induced by var-Phragmén coincides with the Sainte-Laguë method.*

Proof. Sainte-Laguë (1910) and Owens (1921) have shown that the Sainte-Laguë method selects exactly those seat distributions x minimizing

$$e(x) = \sum_{i \in [p]} v_i \left(\frac{x_i}{v_i} - \frac{h}{v_+} \right)^2 = \sum_{i \in [p]} \frac{x_i^2}{v_i} - \frac{h^2}{v_+}$$

(see also Balinski and Young, 1982). By ignoring constants, one can see that minimizing $e(x)$ is equivalent to minimizing $\sum_{i \in [p]} (x_i^2/v_i)$.

The multiwinner rule var-Phragmén chooses a committee corresponding to a load distribution minimizing $\sum_{j \in N} (\sum_{c \in C} \ell_{j,c})^2$ (cf. Definitions 12 and 13). For a given seat distribution x, we can, without loss of generality, assume that the load of x_i is distributed equally among all v_i voters in N_i. Thus, the total load assigned to a voter $j \in N_i$ equals $\sum_{c \in C} \ell_{j,c} = x_i/v_i$. Therefore

$$\sum_{j \in N} \left(\sum_{c \in C} \ell_{j,c} \right)^2 = \sum_{i \in [p]} v_i \left(\frac{x_i}{v_i} \right)^2 = \sum_{i \in [p]} \frac{x_i^2}{v_i}$$

It follows that both methods solve identical minimization problems, and thus coincide. □

3.5. Other multiwinner rules in the context of apportionment

In this section, we examine two further Thiele rules, the Chamberlin–Courant rule and the top-k rule, in the context of apportionment. Since $w_{\text{CC}} = (1, 0, 0, \dots)$ and $w_{\text{top-}k} = (1, \dots, 1, 0, 0, \dots)$, Proposition 1 applies to both rules.

The Chamberlin–Courant apportionment method produces lots of ties, because it does not strive to give more than one seat even to parties with large support. In the special case where the number p of parties exceeds the number h of seats, the Chamberlin–Courant rules assigns one seat to each of the h largest parties.[11]

Proposition 2. *Consider an instance (v, h) of the apportionment problem. If $p > h$, the apportionment method induced by the Chamberlin–Courant rule assigns one seat to each of the h parties with the greatest number of votes.*

Proposition 2 gives an interesting insight into the nature of the Chamberlin–Courant apportionment method: it selects an assembly having one representative from each of the h largest groups of a given society.

The top-k rule is at the other end of the spectrum of apportionment methods.

Proposition 3. *The apportionment method induced by the top-k rule assigns all seats to the party (or parties) receiving the highest number (or numbers) of votes.*

Thus, the top-k rule induces the apportionment method M^* in Proposition 4.5 of Balinski and Young (1982). This apportionment method is often used to select (minority) governments. A minority cabinet is usually formed by the party that receives the greatest number of votes, even if it does not have a majority of the seats.

We conclude this section by mentioning two further rules that are often studied in the context of multiwinner elections. *Satisfaction approval voting* (Brams and Kilgour, 2014)

induces the same apportionment methods as the top-k rule,[12] while *minimax approval voting* (Brams et al., 2007) induces the apportionment method that assigns the seats in a way that maximizes the number of seats of the party with the fewest seats. In particular, if there are more parties than seats, all committees are winning.

4. Degressive proportionality and election thresholds

In this section, we show that Thiele rules can also be used to induce apportionment methods with appealing properties other than lower quota. We do this by exploring *degressive proportionality*, the concept that smaller populations should be allocated more representatives in comparison with the quota-based allocation methods, and *election thresholds*, the concept that a party should only be represented in parliament if it receives at least a certain fraction of votes. To the best of our knowledge, the Thiele rules considered in this section have not been studied before.

4.1. Degressive proportionality

Despite its mathematical elegance and apparent simplicity, the proportionality principle is not always desirable. For instance, in an assembly where decisions are taken under simple majority rule, a cohesive group of 51% obviously has more than a 'fair' share. Principles of justice indicate that, for decision bodies governed by majority rule, fair apportionment should follow a norm of degressive proportionality (Koriyama et al., 2013; Laslier, 2012; Macé and Treibich, 2012). In fact, degressive proportional apportionments can often be observed in parliaments that gather districts, regions, or states of very different sizes, such as the European Parliament.[13]

The *Penrose apportionment method* (aka the *square-root method* and devised by Penrose (1946); see also the work of Słomczyński and Życzkowski, 2006) allocates seats in such a way that the number of seats allocated to a party is proportional to the square root of the votes for that party. It has been proposed for a United Nations Parliamentary Assembly (Bummel, 2010) and for allocating voting weights[14] in the Council of the European Union (BBC News, 2004).

Let $\zeta : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be an increasing function. We say that an apportionment method M satisfies ζ-*proportionality* if, for every instance (v, h) and for every $x \in M(v, h)$, it holds that $x_i \geq \left\lfloor h \frac{\zeta(v_i)}{\sum_\ell \zeta(v_\ell)} \right\rfloor$ for all $i \in [p]$. Intuitively, degressively proportional apportionment methods satisfy ζ-proportionality for some concave function ζ. In particular, we say that an apportionment method satisfies the *Penrose condition* if it satisfies ζ_P-proportionality, where $\zeta_P(x) = \sqrt{x}$.

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is *multiplicative* if, for each $x, y \in \mathbb{R}$, it holds that $f(xy) = f(x)f(y)$.[15] We have the following theorem.

Theorem 6. *Let $\zeta : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be an increasing multiplicative bijection, and let $\alpha = \zeta^{-1}$ be the inverse of ζ. Then the apportionment method induced by the Thiele rule with weight sequence $w = (1/\alpha(1), 1/\alpha(2), 1/\alpha(3), \dots)$ satisfies ζ-proportionality.*

Proof. Let $\mathcal{R}$ be w-Thiele with $w_j = 1/\alpha(j)$ for all $j \in \mathbb{N}$. We show that $M_{\mathcal{R}}$ satisfies ζ-proportionality. Assume for contradiction that for some instance (v, h) of the apportionment problem, there is a seat distribution $x \in M_{\mathcal{R}}(v, h)$ with $x_i < \left\lfloor h \frac{\zeta(v_i)}{\sum_\ell \zeta(v_\ell)} \right\rfloor$ for some $i \in [p]$. We infer that $x_i \leq (h\zeta(v_i)/\sum_\ell \zeta(v_\ell)) - 1$, and, using the pigeonhole principle, that there exists a party P_j such that $x_j > h\zeta(v_j)/\sum_\ell \zeta(v_\ell)$. We conclude that $(x_i+1)/\zeta(v_i) < x_j/\zeta(v_j)$. Since $\alpha = \zeta^{-1}$ is multiplicative, we get

$$\frac{\alpha(x_i + 1)}{v_i} = \frac{\alpha(x_i + 1)}{\alpha(\zeta(v_i))} = \alpha\left(\frac{x_i + 1}{\zeta(v_i)}\right) < \alpha\left(\frac{x_j}{\zeta(v_j)}\right) = \frac{\alpha(x_j)}{\alpha(\zeta(v_j))} = \frac{\alpha(x_j)}{v_j}$$

and thus $v_i/\alpha(x_i+1) > v_j/\alpha(x_j)$. Therefore

$$u(x_{j \curvearrowright i}) - u(x) = \hat{u}(i, x_i + 1) - \hat{u}(j, x_j) = \frac{v_i}{\alpha(x_i + 1)} - \frac{v_j}{\alpha(x_j)} > 0$$

contradicting the assumption that $x \in M_{\mathcal{R}}(v, h)$. This completes the proof. $\qquad\square$

An interesting corollary of Theorem 6 is that the Penrose condition can be satisfied by using a Thiele rule. The weight sequence $w = (w_1, w_2, \ldots)$ achieving this is given by $w_j = 1/j^2$.

Corollary 3. *The apportionment method induced by the Thiele rule with weight sequence $w = (1, 1/4, 1/9, \ldots)$ satisfies the Penrose condition.*

Degressive proportionality is also an important feature of the so-called *Cambridge Compromise*, which proposes an apportionment method for the European Parliament based on an affine formula: each member state should be endowed with a fixed number (five) of delegates plus a variable number, proportional to the population of the state (Grimmett et al., 2011).

We show that such an apportionment method can be implemented via a Thiele rule.

Proposition 4. *Consider an instance (v, h) of the apportionment setting with $h \geq 5p$ and let Z be a constant with $Z > 5v_+$. Let $\mathcal{R}$ denote w-Thiele with weight sequence $w = (0, 0, 0, 0, Z, 1, 1/2, 1/3, \ldots)$. Then, for every $x \in M_{\mathcal{R}}(v, h)$ and for every $i \in [p]$*

$$x_i \geq 5 \quad and \quad x_i - 5 \geq \left\lfloor \frac{v_i(h - 5p)}{v_+} \right\rfloor$$

Proof. We first prove that each party gets at least five seats. Indeed, for the sake of contradiction let us assume that party P_i gets fewer than five seats. By transferring some seats from other parties (having more than five representatives) to P_i, we increase the satisfaction of supporters of P_i by at least $v_i Z > 5v_+$. At the same time, transferring a seat from some party P_j (having more than five seats) to P_i reduces the satisfaction of the supporters of P_j by at most v_+. Thus, such a transfer of, at most, five seats improves the total satisfaction of the voters, a contradiction.

We now show that $x_i - 5 \geq \lfloor \frac{v_i(h-5p)}{v_+} \rfloor$ for all $x \in M_{\mathcal{R}}(v, h)$ and for all $i \in [p]$. For the sake of contradiction, assume that there is an instance (v, h) of the apportionment problem and a seat distribution $x \in M_{\mathcal{R}}(v, h)$ with $x_i - 5 < \lfloor \frac{v_i(h-5p)}{v_+} \rfloor$ for some $i \in [p]$.

Thus, $x_i - 5 \leq (v_i(h-5p)/v_+) - 1$, and, by the pigeonhole principle, there exists a party P_j such that $x_j - 5 > v_j(h-5p)/v_+$. Consequently, $(x_i-4)/v_i < (x_j-5)/v_j$, and thus $v_i/(x_i-4) > v_j/(x_j-5)$. Therefore

$$u(x_{j \frown i}) - u(x) = \hat{u}(i, x_i + 1) - \hat{u}(j, x_j) = \frac{v_i}{x_i - 5 + 1} - \frac{v_j}{x_j - 5} > 0$$

contradicting the assumption that $x \in M_{\mathcal{R}}(v, h)$. This completes the proof. □

4.2. Election thresholds

Thiele rules also provide an interesting method for implementing election thresholds. Thresholds in the form of *percentage hurdles* are often encountered in parliamentary elections (see Pukelsheim, 2014: Section 7.6). For a weight sequence $w = (w_1, w_2, \dots)$, let $w[t]$ be the weight sequence obtained from w by replacing the first t elements of w with zeros. The rule $w[t]$-Thiele induces an apportionment method that allocates seats to a party only if the number of votes the party receives exceeds a certain fraction of the total number of votes *received by parties with allocated seats*.[16] Proposition 5 formalizes this behavior for the weight sequence w_{PAV}.

Proposition 5. *Consider an instance (v, h) of the apportionment problem and fix an integer t with $1 \leq t < h$. Let $\mathcal{R}$ denote the Thiele rule with weight sequence $w_{PAV}[t]$. Then, for all seat distributions $x \in M_{\mathcal{R}}(v, h)$*

$$x_i > 0 \quad only\ if \quad \frac{v_i}{\sum_{\ell \in [p]:x_\ell > 0} v_\ell} > \frac{t}{h}$$

Proof. Consider a seat distribution $x \in M_{\mathcal{R}}(v, h)$. First, observe that there is at least one party that is allocated strictly more than t seats. (Otherwise, the total voter satisfaction $u(x)$ would be zero and could be improved by giving all seats to an arbitrary party.) Without loss of generality, let us assume that party P_1 is such a party, i.e. $x_1 > t$.

Second, observe that *every* party that is allocated *some* seat under x, is allocated strictly more than t seats. In other words, $x_\ell > 0$ implies $x_\ell > t$. (Otherwise, the total voter satisfaction provided by x could be improved by $x_{\ell \frown 1}$.)

Now consider a party P_i with $x_i > 0$. The preceding argument implies that $x_i > t$. Let v_+^x denote the total number of votes for parties with allocated seats under x, i.e. $v_+^x = \sum_{\ell \in [p]:x_\ell > 0} v_\ell$. If $v_i/x_i \geq v_+^x/h$, then $v_i/v_+^x \geq x_i/h > t/h$ and we are done. So let us assume that $v_i/x_i < v_+^x/h$. From the pigeonhole principle, we infer that there exists a party P_j with $x_j > 0$ such that $v_j/x_j \geq v_+^x/h$. Since $x_j > 0$, the previous observation implies that $x_j > t$. Therefore

$$0 \geq u(x_{i \frown j}) - u(x) = \hat{u}(j, x_j + 1) - \hat{u}(i, x_i) = \frac{v_j}{x_j + 1} - \frac{v_i}{x_i}$$

and thus

$$v_i \geq x_i \frac{v_j}{x_j + 1} = x_i \frac{v_j}{x_j} \cdot \frac{x_j}{x_j + 1} > x_i \frac{v_j}{x_j} \cdot \frac{t}{t + 1} \geq t \frac{v_j}{x_j} \geq t \frac{v_+^x}{h}$$

This completes the proof. □

An interesting open question is whether it is possible to naturally modify the definition of the Monroe system to implement election thresholds.

5. Conclusion

In legislative procedures, proportional representation is typically achieved by employing party-list apportionment methods, such as the D'Hondt method or the largest remainder method. These methods impose proportionality by assuring that each party in a representative body is allocated a number of representatives that is proportional to the number of received votes. Nevertheless, party-list systems have many drawbacks—for instance, they provide very weak links between elected legislators and their constituents.

In this paper, we have proposed a simple and natural formal framework that allows us to view approval-based multiwinner election rules as apportionment methods. Some multiwinner rules (such as PAV, Monroe's rule, Phragmén's rules, and the rules considered in Section 4) induce apportionment methods that are used—or have been proposed—for parliamentary apportionment as variants of the notions of proportional, or degressive proportional, representation. The apportionment methods that are induced by other multiwinner rules (such as the Chamberlin–Courant rule, the top-k rule, satisfaction approval voting, and minimax approval voting) are, however, less appealing. These results give interesting insights into the nature of the analyzed multiwinner rules, and they show how traditional apportionment methods can be extended to settings where voters can vote for individual candidates. Further, our methodology allowed us to discover a new interesting multiwinner election rule, which, if applied to the apportionment setting, satisfies Penrose's (square-root proportionality) condition.

Appendix

Proportional justified representation implies lower quota

In this appendix, we prove that the every multiwinner rule satisfying proportional justified representation induces an apportionment method satisfying quota. The following definition is from Sánchez-Fernández et al. (2017).

Definition 14. Consider an instance (A, k) of the multiwinner election setting and a size-k committee S. The committee S *provides proportional justified representation for* (A, k) if there does not exist a subset $N^* \subseteq N$ of voters and an integer $\ell > 0$ with $|N^*| \geq \ell n/k$ such that $|\bigcap_{i \in N^*} A_i| \geq \ell$ but $|S \cap (\bigcup_{i \in N^*} A_i)| < \ell$. A multiwinner rule $\mathcal{R}$ *satisfies proportional justified representation* if, for every instance (A, k), every committee in $\mathcal{R}(A, k)$ provides proportional justified representation for (A, k).

Proposition 6. *If a multiwinner rule satisfies proportional justified representation, then it satisfies lower quota.*

Proof. Let $\mathcal{R}$ be a multiwinner election rule satisfying proportional justified representation, and consider an instance (v, h) of the apportionment problem. Consider an arbitrary party P_i with $i \in [p]$. Translated to the multiwinner election setting, there is a group N_i of voters of size $|N_i| = v_i$ and every voter in N_i approves all h candidates in C_i. Define $\ell = \lfloor \frac{v_i h}{v_+} \rfloor$. We have $\ell \leq v_i h / v_+$ and thus $v_i \geq \ell v_+ / h$.

Let $N^* = N_i$. Since $|N^*| = v_i \geq \ell v_+ / h$ and $|\bigcap_{i \in N^*} A_i| = |C_i| = h \geq \ell$, proportional justified representation implies that every committee S that is chosen by $\mathcal{R}$ satisfies $|S \cap (\bigcup_{i \in N^*} A_i)| \geq \ell$. And since $S \cap (\bigcup_{i \in N^*} A_i) = S \cap C_i$, we know that $x_i = |S \cap C_i| \geq \ell$. Thus, party P_i is allocated at least $\ell = \lfloor \frac{v_i h}{v_+} \rfloor$ seats, as desired. $\square$

Since PAV satisfies proportional justified representation—and in fact even the stronger property *extended justified representation* (Aziz et al., 2017)—it also satisfies lower quota.

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Notes

1. Most apportionment methods allow for ties. In this paper, we do not consider any specific tie-breaking rule but rather assume that an apportionment method can return several tied outcomes.
2. See Balinski and Young (1982), Proposition 3.3. Divisor methods with $d(0) = 0$ can also be defined. For such methods, which Pukelsheim (2014) calls *impervious*, the conventions $v_i / 0 = \infty$ and $v_i / 0 \geq v_j / 0 \Leftrightarrow v_i \geq v_j$ are used. Examples of impervious divisor methods are the methods of Huntington and Hill, Adams, and Dean.
3. To accommodate tied outcomes, several committees might be winning.
4. We are grateful to Xavier Mora and Svante Janson for pointing this out to us.
5. Minimizing the variance of voter loads is equivalent to minimizing $\sum_{i \in N} (\sum_{c \in C} \ell_{i,c})^2$.
6. Thiele rules with non-increasing weight sequences are very natural, especially when viewed in the context of apportionment. Thiele rules with *increasing* sequences induce apportionment methods that allocate all seats to the single party that receives the most votes. The same seat distribution can be also obtained by using the Thiele rule with a constant weight vector (see Proposition 3).
7. In fact, Thiele (1895) already stated this equivalence, but only in the special case when perfect proportionality is possible; see the survey by Janson (2016).

8. This result also follows from Theorem 1, together with the characterization of the D'Hondt method as the only divisor method satisfying lower quota (Balinski and Young, 1982: Proposition 6.4). We give a direct proof for completeness.

9. In fact, the sequential variant is the main method that Phragmén proposed for use in actual elections; see the survey by Janson (2016) for details.

10. The fact that the Thiele rule based on the weight sequence $(1, 1/3, 1/5, \dots)$ induces the Sainte-Laguë method in the party-list setting is also stated (without proof) by Pereira (2016) and Brams et al. (2017).

11. In this special case, the apportionment method induced by the Chamberlin–Courant rule coincides with impervious divisor methods (see note 2).

12. Brams and Kilgour (2014) also propose an apportionment method based on maximizing the satisfaction of voters. This method, called *satisfaction approval voting apportionment*, is different from the apportionment method M_{SAV} that is induced by satisfaction approval voting via our generic three-step procedure defined at the beginning of Section 3.

13. Even though the composition of the European Parliament is not the result of the application of a well-defined rule, it is interesting to note that the history of successive negotiations produced such a result (see Rose, 2013).

14. The question of allocating voting weights to representatives is formally equivalent to the apportionment problem.

15. If a multiplicative function f is bijective, then its inverse f^{-1} is also multiplicative.

16. The thresholds implemented by Thiele-based apportionment methods differ from real-world election thresholds in one important aspect: Rather than requiring a certain fraction (say, 5%) of *all* votes, the thresholds we consider here require a certain fraction of those votes that are received by parties *with allocated seats*. An advantage of the latter kind of threshold requirement is that it can always be satisfied. By contrast, requiring a certain fraction of all votes can lead to situations in which *no* party reaches the required number of votes and, therefore, no seat allocation satisfies the threshold requirement.

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