Approval-Based Committee Voting:  
Axioms, Algorithms, and Applications

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Abstract

Approval-based committee (ABC) rules are voting rules that output a fixed-size subset of candidates, a so-called committee. ABC rules select committees based on dichotomous preferences, i.e., a voter either approves or disapproves a candidate. This simple type of preferences makes ABC rules widely suitable for practical use. In this survey, we summarize the current understanding of ABC rules from the viewpoint of computational social choice. The main focus is on axiomatic analysis, algorithmic results, and relevant applications.

This is an early draft meant to solicit feedback. Any feedback, criticism, and comments are very welcome. We use the following notation to signify the current state of this survey: sections marked with ■■ are largely complete and contain most of the topics we want to discuss, sections marked with □■ are missing some important topics and will be extended, and sections marked with □□ simply do not exist yet.

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1 Introduction

In committee elections we are given a set of candidates, a set of voters, the preferences that each voter has over these candidates, and an integer \( k \)—the desired size of the committee to be elected. The goal is to select—based on the voters’ preferences—a committee, that is, a subset of exactly \( k \) candidates.

Committee election rules are useful for selecting a representative body, such as a parliament\(^1\) [97, 177], a faculty council, or a board of a trade union. Moreover, selecting finalists in a competition, based on preferences of judges/experts is also an instance of committee elections. Other possible applications of committee election rules that have been identified in the literature include finding group recommendations [143, 144, 192], diversifying search results [194], locating public facilities [96, 192], and selecting validators in consensus protocols, such as the blockchain [53, 60].

The model of committee elections has been studied for different types of voters’ ballots. In this survey we are looking at the approval-based variant of the committee election model, where each voter submits his or her preferences via a subset of candidates—this subset is called the approval set of the voter. Example 1 in Section 2 illustrates the model considered in this survey. Another interesting variant of the model of committees elections is ordinal-based, where each voter orders the candidates from the most to the least preferred one. We will not consider ordinal-based committee elections in this survey—for an appropriate overview of this model we refer the reader to the recent book chapter [93].

There are several arguments for using approval ballots in committee elections. Compared to the ordinal-based model, providing approval-based preferences requires less cognitive effort from the voters, and thus this kind of voting is more practical. Brams and Herschbach [36] and Aragones et al. [6] discuss positive effects of using approval ballots on voters’ participation, and Brans and Herchef [36] further argues that using approval ballots can reduce negative campaigning; Brams and Fishburn [34] discuss other possible positive implications of using approval ballots in political elections. In fact, approval ballots are often used, but mostly in scientific societies and in the context of selecting a single winner (see, e.g., [35]). Further recent experimental studies [1, 135, 136, 206] explore the possibility of using approval ballots in political elections, and their conclusions are optimistic.

\(^1\)Currently, most countries use legislatures based on political parties for electing parliaments. However, in national-wide elections in some countries (e.g., in Belgium, Finland, Latvia, Luxembourg, Netherlands, Sweden) and in district elections in France and Switzerland, open-list systems are used—these systems allow to vote for individual candidates rather than only for political parties. Similarly, countries such as Australia and Republic of Ireland (and for some elections, India, Malta, and New Zealand) use Single Transferable Vote, a rule which also allows to vote for individual candidates. Indeed, a few important arguments for allowing the voters to vote for individual candidates have been raised. For example, when voting for individual candidates, the electorate makes the selected candidates more responsible to itself rather than to their political parties. Similarly, it allows the candidates to focus on campaigning for the citizens’ votes rather than for gaining the influence within their parties [3, 5, 62, 65]. For a more general, comparative analysis of different electoral systems we refer the reader to the book of Lijphart and Grofman [139] and to a recent review by Grofman [107].
In committee elections the submitted preferences of the voters over the candidates are separable, and the voters cannot specify relations between candidates. For example, it is not possible for a voter to indicate that she believes that a certain group of candidates would particularly well work together in the elected committee or that she thinks that some two candidates should never be elected together. We discuss several related models that allow to deal with this kind of input information in Section 7.

The goal of this survey is to summarize the current state of the knowledge regarding approval-based committee (ABC) elections. The survey should help the reader to understand the most important (classes of) ABC rules and to answer the following questions: (1) What are the properties of various ABC rules? What would be a good ABC rule for a given application? (The answer to this question usually depends on types of properties that the reader considers particularly important for her application.) (2) What are the practical limitations of using a particular rule, and how one can deal with this limitations?

For a more general overview of approval voting we refer the reader to the book by Laslier and Sanver [134], Kilgour [114] (a chapter in the aforementioned book) and Kilgour and Marshall [115] have provided comprehensive overviews that deal specifically with approval-based committee elections. However, since these summaries have been written, approval-based committee elections have become subject to extensive studies within economic, political, and computer science communities. The goal of this survey is to provide a up-to-date summary of the state of the art. A particular focus of this survey is put on axiomatic and algorithmic analysis; this line of work is prevalent in computational social choice [43].

Prerequisites. We assume that the reader is familiar with basic concepts from algorithms (big-$\mathcal{O}$ notation, polynomial- vs exponential-time algorithms, the concept of approximation algorithms) and from computational complexity (NP-hardness, NP-completeness, reductions).

Python implementation. This survey is closely connected with the Python library abcvoting [129]. Most ABC rules discussed in this survey are available as Python code and can be used, e.g., for numerical experiments. Moreover, all relevant examples in this survey are also available in the abcvoting library, including the counterexamples from Appendix A.

2 The Model

Let us first fix some basic notation. We write $\mathbb{N}$ to denote the set of non-negative integers. For each $t \in \mathbb{N}$, we let $[t]$ denote the set $\{1, \ldots, t\}$. For a set $X$, we write $\mathcal{P}(X)$ to denote the powerset of $X$, i.e., the set of all subsets of $X$.

Now we can define the basic ingredients of voting: alternatives, voters, preferences, and committees. Let $C$ be a finite set of available candidates (or alternatives). We
assume that voters’ preferences are available in the form of dichotomous preferences, i.e., voters distinguish between alternatives they approve and those that they disapprove—a dichotomy. Hence a voter’s preference over candidates is given by a set of approved alternatives. Let \( N \subseteq \mathbb{N} \) denote a finite set of voters. An approval profile is the collection of all voters’ preferences; formally it is a function \( A : N \rightarrow \mathcal{P}(C) \). Throughout the paper, we use \( n \) to denote the number of voters (\(|N|\)) and \( m \) to denote the number of alternatives (\(|C|\)). Further, we write \( N(c) \) to denote the subset of voters that approve candidate \( c \), i.e., \( N(c) = \{i \in N : c \in A(i)\} \).

**Example 1.** An academic society chooses a steering committee for the next time period. Such a committee consists of four persons (\( k = 4 \)) and there are seven candidates competing for these positions, \( C = \{a, b, c, d, e, f, g\} \). All members of the society are eligible to vote and may provide approval ballots to indicate their preference. In total, 12 ballots have been submitted:

\[
\begin{align*}
A(1) & : \{a, b\} & A(2) & : \{a, b\} & A(3) & : \{a, b\} & A(4) & : \{a, c\} \\
A(5) & : \{a, c\} & A(6) & : \{a, c\} & A(7) & : \{a, d\} & A(8) & : \{a, d\} \\
A(9) & : \{b, c, f\} & A(10) & : \{e\} & A(11) & : \{f\} & A(12) & : \{g\}
\end{align*}
\]

*Figure 1* shows a graphical representation of this profile. In this figure, each column correspond to one voter (one approval set) and each candidate appears in only one row—each candidate is approved by the voters that appear below the boxes that represent the candidate. Colors are used to distinguish different candidates.

Often, we are only interested in how often a specific approval set occurs in an approval profiles and thus ignore the names (identifiers) of the voters who cast the approval ballots. In such cases we do not specify the concrete mapping from \( N \) to approval sets but use the following notation:

\[
3 \times \{a, b\} \quad 3 \times \{a, c\} \quad 2 \times \{a, d\} \quad 1 \times \{b, c, f\}
\]
The reader may ponder which steering committee should be selected given this approval profile—there is certainly more than one sensible choice. In the following section, we will see how different voting rules decide in this situation.

As we have seen in this example, committees are sets of candidates. Typically, we are interested in committees of a specific size, which we denote by $k$. The input for choosing such a committee is an election instance $E = (A, k)$, i.e., a preference profile $A$ and a desired committee size $k$. Note that given $A$, we can derive $N$ and $C$ from this function: $N$ is the domain of $A$ (viewed as a function) and—under the mild assumption that all candidates are approved by at least one voter—$C$ is the union of all function values, i.e., $C = \bigcup_{i \in N} A(i)$. Thus we do not mention $N$ and $C$ in this notation.

Let us now define the key concept of this survey: approval-based committee (ABC) rules. An ABC rule is a voting rule for choosing committees, i.e., an ABC rule takes an election instance as input and outputs one or more size-$k$ subsets of candidates. We refer to these size-$k$ subsets as winning committees. If an ABC rule outputs more than one committee, we say that these committees are tied.

Finally, let us introduce the concept of representation, which will be of importance throughout the survey. Given a committee $W$, we say that candidate $c \in W$ represents voter $i$ if voter $i$ approves candidate $c$ (i.e., $c \in A(i)$). The representation of voter $i$ is the number of approved candidates in the committee (i.e., $|A(i) \cap W|$). Our general assumption is that voters desire a committee that grants them a high representation.

## Dramatis Personae: ABC Rules

We now present the main characters in this survey: approval-based committee rules, short: ABC rules. The goal of this section is to provide an overview of important ABC rules, in particular those with desirable properties. For readers who are less interested in this overview, we have marked the most relevant ABC rules with a bar on the side of the page; one can largely follow the survey with knowledge only about these rules.

Some of the ABC rules defined in the following are resolute, i.e., return always a single winning committee, and some are irresolute. Most rules can be defined either way; we have chosen the more natural definition for each rule.

For the following definitions, we assume that we are given an election instance $E = (A, k)$ for a voter set $N$ and a candidate set $C$.

### 3.1 Thiele Methods

In the single-winner setting, i.e., if $k = 1$, there is only one reasonable voting rule when presented with approval ballots: Approval Voting. Approval Voting selects those alternatives that are approved by the maximum number of voters, all of which are (co-)winners.
according to this rule. Most ABC rules introduced in this section are equivalent to Approval Voting for the case $k = 1$ (we discuss notable exceptions in Section 3.6). There is, however, one ABC rule that extends the reasoning of Approval Voting to $k > 1$ in the most natural manner; this rule is therefore called Multi-Winner Approval Voting (short: AV).

**Rule 1** (Multi-Winner Approval Voting, AV). This ABC rule selects the $k$ candidates which are approved by most voters. Formally, the AV-score of an alternative $c \in C$ is defined as $sc_{AV}(A, c) = |\{v \in N : c \in A(v)\}|$ and AV selects committees $W$ that maximize $sc_{AV}(A, W) = \sum_{c \in W} sc_{AV}(A, c)$.

**Example 2.** Let us consider the instance of Example 1. To compute AV, we count how often each alternative is approved: $a$: 8 times, $b$: 4, $c$: 4, $d$: 2, $e$: 1, $f$: 2 and $g$: 1. We want to select the four most-approved alternatives. As $d$ and $f$ both have the fourth-most approvals, AV returns two tied committees: the sets $W_1 = \{a, b, c, d\}$ and $W_2 = \{a, b, c, f\}$. It is noteworthy that $W_1$ leaves three voters completely unsatisfied with the chosen alternatives, whereas $W_2$ satisfies all but two.

We continue with an ABC rule that can be seen as the exact opposite of AV. Whereas AV disregards whether some voters completely disagree with a committee, the Approval Chamberlin–Courant rule grants as many voters as possible at least one approved alternative in the committee. This rule was first mentioned by Thiele\(^3\) [203], and then independently introduced in a more general context by Chamberlin and Courant [61].

**Rule 2** (Approval Chamberlin–Courant, CC). The CC rule outputs all committees $W$ that maximize $sc_{CC} = |\{v \in N : A(v) \cap W \neq \emptyset\}|$.

**Example 3.** Consider again the instance of Example 1:

$$3 \times \{a, b\} \quad 3 \times \{a, c\} \quad 2 \times \{a, d\} \quad 1 \times \{b, c, f\} \quad 1 \times \{e\} \quad 1 \times \{f\} \quad 1 \times \{g\}.$$  

There is exactly one committee that grants each voter (at least) one approved candidate: $W = \{a, e, f, g\}$. While this committee indeed provides some satisfaction for every voter, it includes alternatives ($e$ and $g$) that are approved only by single voters.

The two ABC rules we discussed so far—AV and CC—can not only be seen as extreme points in the spectrum of ABC rules, they also belong to the same general class of

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\(^2\)Let us briefly mention a few restricted forms of AV that are widely used in political settings: Block Voting, where voters may not approve more than $k$ candidates, Limited Voting, where voters may approve at most $s$ candidates with $s < k$, and Single Non-Transferable Vote (SNTV), which is Limited Voting for $s = 1$. For our purposes none of these rules is of particular interest as they are all special cases of AV.

\(^3\)Thorvald Nicolai Thiele (1838–1910) was a Danish astronomer and mathematician. He was professor of astronomy at the University of Copenhagen and director of the Copenhagen University Observatory. He is most known for his work in mathematics, in particular in statistics [137, 156, 204, 205]. The contributions of Thiele to voting theory are discussed in detail by Janson [112].
ABC rules. Thiele methods, a class of ABC rules introduced by Thiele in the late 19th century [203], encompass all rules that maximize the total (the sum of) satisfaction of voters, or, in other words, maximize the total welfare. By differing the definition of how voters are satisfied with a given committee, a very broad spectrum of ABC rules can be covered.

**Rule 3** (Thiele methods, $w$-Thiele\textsuperscript{4}). A Thiele method is parameterized by a non-decreasing function $w : \mathbb{N} \rightarrow \mathbb{R}$ with $w(0) = 0$. The score of a committee $W$ given a profile $A$ is defined as

$$s_{cw}(A, W) = \sum_{i \in N} w(|W \cap A(i)|);$$

the $w$-Thiele method returns committees with maximum score.

Indeed, AV is the $w$-Thiele method with $w(x) = x$; CC is the $w$-Thiele method with $w(x) = \min(1, x)$. This follows immediately from the respective definitions. The following Thiele method is arguably one of the most important: Proportional Approval Voting, in short PAV. Also this rule was defined in Thiele’s original paper [203]. The definition (and properties) of PAV crucially depend on the harmonic function.

**Rule 4** (Proportional Approval Voting, PAV). Let $h(x) = \sum_{i=1}^{x} 1/i$ denote the harmonic function. PAV is $h$-Thiele, i.e., it is the $w$-Thiele with $w(x) = h(x)$. In other words, PAV assigns to each committee $W$ the PAV-score $s_{cPAV}(A, W) = \sum_{i \in N} h(|W \cap A(i)|)$ and returns all committees with maximum score.

By using the harmonic function $h(\cdot)$, we introduce a flattening satisfaction function for voters, akin to the law of diminishing returns. As a consequence, PAV balances the (justified) demands of large groups with the conflicting goal of satisfying small groups. Indeed, as we will see in Section 5, Proportional Approval Voting achieves this balance in a proportional fashion. Figure 2 shows a visualization of the defining $w$-functions of different Thiele methods:

$$w_{AV}(x) = x \quad w_{PAV} = \sum_{i=1}^{x} 1/i \quad w_{CC} = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \geq 1. \end{cases}$$

Note that also visually the function defining PAV is “in between” AV and CC.

**Example 4.** Given the instance of Example 1:

$$3 \times \{a, b\} \quad 3 \times \{a, c\} \quad 2 \times \{a, d\} \quad 1 \times \{b, c, f\} \quad 1 \times \{e\} \quad 1 \times \{f\} \quad 1 \times \{g\},$$

PAV selects the committee $W = \{a, b, c, f\}$. For one voter ($\{b, c, f\}$) this committee contains three approved alternatives, for 6 voters this committee contains two approved

\textsuperscript{4}The class of Thiele methods is sometimes also referred to as weighted PAV rules [14]; we prefer the term Thiele methods as only few rules in this class are actually proportional. Kilgour and Marshall [115] refer to this class as generalized approval procedures.
alternatives, for three voters $W$ contains one approved alternative, and two voters are not at all satisfied with $W$. Thus, we have $s_{PAV}(A, W) = (1 + \frac{1}{2} + \frac{1}{3}) + 7 \cdot (1 + \frac{1}{2}) + 3 \cdot 1 = \frac{83}{6}$ and this value is optimal. Coincidentally, $W$ is one of the two committees produced by AV, namely the one with fewer dissatisfied voters. It appears that PAV strives for a compromise between AV and CC—this is an intuition that we will discuss in more detail later (Section 5.5).

Other Thiele methods that have been studied in the literature are the class of $p$-Geometric rules [192], threshold procedures [98, 115], and Sainte-Laguè Approval Voting (SLAV) [128].

Thiele methods pick committees that maximize a certain welfare of the voters, thus, they belong to a broader class of welfarist rules.

**Definition 1.** A welfare vector induced by a committee $W$ specifies for each voter, her satisfaction from $W$ (measured as the number of candidates she approves in $W$):

$$\text{welf}(W) = (|A(1) \cap W|, |A(2) \cap W|, \ldots, |A(n) \cap W|).$$

An ABC rule $\mathcal{R}$ is welfarist if there is a function $f : \mathbb{N}^N \to \mathbb{R}$, mapping welfare vectors to scores, such that for each instance $(A, k)$ we have

$$\mathcal{R}(A, k) = \arg\max_{W \subseteq C : |W| = k} f(\text{welf}(W)).$$

In this definition, $f(\text{welf}(W))$ can be viewed as the welfare that voters gain from $W$. For Thiele methods, the welfare of voters is the sum of the satisfaction of voters (determined by the $w$-function). For the class of welfarist rules, also types of aggregations other
than the sum can be used. For example, one can define \( f(\text{welf}(W)) \) as the satisfaction of the least-satisfied voter—akin to egalitarian aggregation [154]. Another example of a welfarist rule is a dictatorial rule which compares welfare vectors lexicographically given a fixed order of voters: the first voter in this order is a dictator and only if the dictator is indifferent between two outcomes, the second-in-place may decide.

These other types of aggregations have been studied in the context of committee elections with ranking based preferences (for the egalitarian aggregation see [16, 191]; for the OWA-based aggregation see [81]). For approval ballots, we are aware only of two works that consider such aggregations. Computation properties of CC an Monroe rules based on the egalitarian aggregation are considered by Betzler et al. [29]. Amanatidis et al. [2] consider OWA-based aggregation, but for other types of welfare of individual voters—we will discuss that in more detail in Section 3.6.

### 3.2 Sequential Variants of Thiele Methods

Thiele methods are defined as optimization routines: given an objective function, they return all committees that maximize this function. We define two classes of sequential procedures: sequential and reverse sequential Thiele methods. Both classes have been introduced in Thiele’s original paper [203] (see Janson’s survey for further historical remarks [112]). Furthermore, both classes can be seen as approximation algorithms for Thiele methods; we return to this analogy in Section 6.1.2.

Let us begin with sequential Thiele methods: starting with an empty committee, they add committee members one by one, in each step the one that increases the objective function the most.

**Rule 5** (Sequential \( w \)-Thiele, seq-\( w \)-Thiele). For each \( w \)-Thiele method, we define its sequential variant, seq-\( w \)-Thiele, as follows. We start with an empty committee \( W_0 = \emptyset \). In each round \( r \in \{1, \ldots, k\} \), we compute \( W_r = W_{r-1} \cup \{c\} \), where \( c \) is a candidate that maximizes \( \text{sc}_w(A, W_{r-1} \cup \{c\}) \), i.e., the candidate that improves the committee’s score the most. If more than one candidate yields a maximum score, we break ties according to some given tie-breaking order. The seq-\( w \)-Thiele rule returns \( W_k \).

Two sequential Thiele methods will be of particular interest in this survey: sequential \( w_{\text{PAV}} \)-Thiele and sequential \( w_{\text{CC}} \)-Thiele. We refer to these two rules as seq-PAV and seq-CC. In contrast, the sequential variant of AV (seq-\( w_{\text{AV}} \)-Thiele) is not relevant to us as it is equivalent to AV. This is because the AV-score (\( \text{sc}_{\text{AV}} \)) of candidates is not influenced by the other candidates in the committee.

**Example 5.** Since the instance of Example 1 yields the same result for PAV and seq-PAV (and also for CC and seq-CC), we take a look at a different profile:

\[
3 \times \{a, b\} \quad 6 \times \{a, d\} \quad 4 \times \{b\} \quad 5 \times \{c\} \quad 5 \times \{c, d\}
\]

For \( k = 2 \), PAV selects the committee \( \{a, c\} \) with a PAV-score of 19. (Each voter except those that approve only candidate \( b \) has exactly one approved candidate in the committee.)
Let us contrast this result with seq-PAV. All sequential Thiele methods with \( w(1) > 0 \), including seq-PAV, select the candidate with the largest number of approvals in the first round—the winner according to (single-winner) Approval Voting. Thus, \( d \) is selected in the first round as it gives an AV-score of 11. In the second round, we choose between \( a \) (increasing the score by 6) and \( b \) (increasing the score by 7) and \( c \) (increasing the score by 7.5). Hence, seq-PAV returns the committee \( \{c,d\} \) with a PAV-score of 18.5.

Similar to sequential Thiele methods, reverse sequential Thiele methods build committees sequentially, but here one starts with the set of all candidates and sequentially removes the candidate that contributes the least to the committee’s score.

**Rule 6 (Sequential \( w \)-Thiele, rev-seq-\( w \)-Thiele).** For each \( w \)-Thiele method, we define its reverse sequential variant, rev-seq-\( w \)-Thiele, as follows. We start with \( W_m = C \), the set of all candidates. Each round, the candidate with the least marginal contribution the score is removed. To be precise, in each round \( r \) from \( m - 1 \) down to \( k \), we compute \( W_r = W_{r+1} \setminus \{c\} \), where \( c \) is a candidate that maximizes \( sc_w(A,W_{r+1} \setminus \{c\}) \), i.e., the candidate whose removal decreases the committee’s score the least. If more than one candidate does that, we break ties according to some given tie-breaking order. The rev-seq-\( w \)-Thiele rule returns \( W_k \).

In the remainder of the survey, we will only encounter reverse sequential PAV (rev-seq-PAV).

**Example 5 (continued).** For rev-seq-PAV, we start with the full set of candidates \( W_4 = \{a,b,c,d\} \) and remove the candidate with the least marginal contribution: removing \( a \) decreases the score by 4.5, removing \( b \) decreases the score by 5.5, \( c \) by 7.5, \( ad \) by 6.5. Thus, \( a \) is removed and \( W_3 = \{b,c,d\} \). Now, we again compute the marginal contributions: for \( b \) 7, for \( c \) 7.5, and for \( d \) 8.5. Thus, \( W_2 = \{c,d\} \), which is the winning committee. We see that for this instance seq-PAV and rev-seq-PAV yield the same winning committee. This does not hold in general.

An election instance where PAV, seq-PAV, and rev-seq-PAV all yield different winning committees can be found in Janson’s survey [112, Example 13.3]. The example is due to Thiele [203] and significantly larger than the examples presented here (9,321 voters). \( \top \)

### 3.3 Monroe’s Rule

Monroe’s rule [152] is an ABC rule\(^5\) related to the Chamberlin–Courant rule. It also aims at maximizing the number of voters who are represented in the elected committee. The main difference is that each committee member can represent at most \( \frac{1}{k} \)-th of the voters.

\(^5\)Although Monroe defined his rule in the original paper primarily for linear preference orders [152], he considered the modified version based on approval ballots the “most promising option” for actual (political) use. If the distinction between these two rules is necessary, the approval-based version is often denoted as \( \alpha \)-Monroe; we do not need this distinction in this survey as we focus solely on approval ballots.
Rule 7 (Monroe). Given a committee \( W \), a Monroe assignment for \( W \) is a function \( \phi: N \to W \) such that each candidate \( c \in W \) is assigned roughly the same number of voters, i.e., for all \( c \in W \) it holds that \( \lfloor n/k \rfloor \leq |\phi^{-1}(c)| \leq \lceil n/k \rceil \). The candidate \( \phi(i) \) can be viewed as the representative of voter \( i \). Let \( \Phi(W) \) be the set of all possible Monroe assignments for \( W \). The Monroe-score of a committee \( W \) is defined as the number of voters that have a representative assigned that they approve (given an optimal Monroe assignment), i.e., \( sc_{\text{Monroe}}(A, W) = \max_{\phi \in \Phi(W)} |\{ i \in N : \phi(i) \in A(i) \}| \). Monroe returns all committees with a maximum Monroe score.

Example 6. Considering Example 1:

\[
3 \times \{a, b\} \quad 3 \times \{a, c\} \quad 2 \times \{a, d\} \quad 1 \times \{b, c, f\} \quad 1 \times \{e\} \quad 1 \times \{f\} \quad 1 \times \{g\},
\]

we first note that the desired committee size \( k = 4 \) divides the number of voters \( n = 12 \) and hence a Monroe assignment assigns exactly 3 voters to each candidate. One optimal Monroe assignment (among many) is shown in Figure 3 and given by \( \phi^{-1}(a) = \{3, 7, 8\}, \phi^{-1}(b) = \{1, 2, 9\}, \phi^{-1}(c) = \{4, 5, 6\}, \phi^{-1}(e) = \{10, 11, 12\} \). The Monroe score of this assignment is \( sc_{\text{Monroe}}(A, W) = 10 \), since only voters 11 and 12 are assigned to a representative (candidate \( e \)) that they do not approve. In total there are six winning committees; committee \( \{a, b, c, e\} \) is one of them.

Monroe’s rule has also a natural sequential version, called Greedy Monroe\(^6\), introduced by Skowron et al. [191]. We present Greedy Monroe here in a slightly simpler, more practical fashion, where dissatisfied voters are not assigned to groups.

Rule 8 (Greedy Monroe). This ABC rule proceeds in \( k \) rounds: In each round \( r \in \{1, \ldots, k\} \) Greedy Monroe assigns a candidate to a group of voters \( G_r \) of size at most \( n_r \) (defined below); this candidate is added to the committee. The maximum size of a group, \( n_r \), is defined as follows: if \( n = k \cdot \lceil n/k \rceil + c \), then \( n_1 = \cdots = n_c = \lceil n/k \rceil \) and \( n_{c+1} = \cdots = n_k = \lfloor n/k \rfloor \). In round \( r + 1 \), let \( N_{r+1} \) denote the voters that have not yet

\(^6\)Greedy Monroe is called Algorithm A in the original paper [191] and is defined therein only for instances where \( k \) divides \( n \). The first general definition was given in [84].
an assigned committee member, i.e., \( N_{r+1} = N \setminus (G_1 \cup \cdots \cup G_r) \). Candidate \( c_{r+1} \) is chosen as a candidate that maximizes \( \{|i \in N_{r+1} : c \in A(i)\}| \) (using a tiebreaking order on candidates if necessary). Now, if there are at most \( n_{r+1} \) not yet assigned voters that approve \( c_{r+1} \), then \( G_{r+1} = \{i \in N_{r+1} : c_{r+1} \in A(i)\} \); if there are more than \( n_{r+1} \) such voters, a tiebreaking order on voters is used to assign exactly \( n_{r+1} \) voters to \( G_{r+1} \). Greedy Monroe outputs the committee \( \{c_1, \ldots, c_k\} \).

**Example 7.** In our running example (Example 1) given by

\[
\begin{align*}
3 \times \{a, b\} & \quad 3 \times \{a, c\} & \quad 2 \times \{a, d\} & \quad 1 \times \{b, c, f\} & \quad 1 \times \{e\} & \quad 1 \times \{f\} & \quad 1 \times \{g\},
\end{align*}
\]

Greedy Monroe first picks candidate \( a \) as it is approved by most voters. We assume that ties among voters are broken in increasing order, so \( G_1 = \{1, 2, 3\} \). Now \( c \) is chosen since it is the only candidate with 4 supporters among the remaining voters \( |N_2| = \{4, \ldots, 12\} \). The corresponding group of voters is \( G_2 = \{4, 5, 6\} \) (again choosing voters with smaller index first). Now there are two candidates left that are approved by two voters in the remaining set \( |N_3| = \{7, \ldots, 12\} \): candidates \( d \) and \( f \). By alphabetic tiebreaking we choose \( d \), and so we set \( G_3 = \{7, 8\} \). Finally, there is one candidate that has two supporting voters in \( N_4 = \{9, \ldots, 12\} \): \( f \) is approved by voters 9 and 11; thus \( G_4 = \{9, 11\} \). A Monroe assignment corresponding to this committee \( \{a, c, d, f\} \) is, e.g., given by \( \phi^{-1}(a) = \{1, 2, 3\} \), \( \phi^{-1}(c) = \{4, 5, 6\} \), \( \phi^{-1}(d) = \{7, 8, 10\} \), and \( \phi^{-1}(f) = \{9, 11, 12\} \). In this instance, Greedy Monroe was able to find a committee with a maximum Monroe score, but this does not hold in general.

### 3.4 Phragmén’s Rules

Phragmén\(^7\) introduced a number of voting rules, most of which are based on a form of cost-sharing (or load balancing). The core idea is that placing a candidate in the winning committee incurs a cost, or load, that has to be shouldered by voters that approve this candidate. The goal is to choose a committee that allows for an as equal as possible distribution of its cost. In this way, the preferences of as many voters as possible are taken into account.

The most interesting rule invented by Phragmén is Phragmén’s Sequential Rule [166–168]. Although a variant of this rule exists that is based on global optimization (comparable to Thiele methods), the sequential rule exhibits overall much better properties, in particular regarding proportionality axioms. This is in contrast to Thiele methods, for which their sequential counterparts are arguably inferior. We will return to these claims and the corresponding axiomatic analysis in Sections 5.2 and 5.3.

---

\(^7\)Lars Edvard Phragmén (1863–1937) [58, 112, 157, 198] was a Swedish mathematician and an actuary. He was a professor of mathematics at Stockholm University and life-long editor of Acta Mathematica. His best known mathematical work is the Phragmén-Lindelöf principle in complex analysis [170], but he also published several works on election methods [165–169] and was involved in Swedish electoral reforms; see Janson’s survey [112] for a comprehensive summary of his work on election methods.
Even though Phragmén’s Sequential Rule can be considered one of the more appealing ABC rules, this rule remained unknown to many social choice researchers until recently. Few publications before 2017 mention Phragmén’s methods; notable exceptions are a survey by Janson [111] (in Swedish) and a paper by Mora and Oliver [153] (in Catalan). Since 2017 several papers have proven Phragmén’s method to be a particularly strong ABC rule, in particular being a proportional ABC rule that is both polynomial-time computable and committee monotone (see Section 4.3 for details).

We present two (equivalent) formulations of seq-Phragmén. The first is conceptually simpler, while the second gives a clearer picture how the rule is computed in practice.

**Rule 9. [Phragmén’s Sequential Rule, seq-Phragmén]** This ABC rule is based on the assumption that placing a candidate in the winning committee incurs a load (or cost) of 1, which is distributed to the set of voters that approve this candidate.

**Continuous formulation:** We assume that each voter has a budget which constitutes his or her voting power. Voters start with a budget of 0 and this budget continuously increases as time advances. At time $t$, the budget of each voter is $t$. As soon as a group of voters that jointly approve a candidate has a total budget of 1, the joint candidate is added to the winning committee. Then the budget of all involved voters is reset to 0; only voters that do not approve the selected candidate keep their current budget. This process continues until the committee is filled. If at some point two candidates could be moved in the committee at the same time, a tie-breaking order is used to decide which candidate is selected.

**Discrete formulation:** seq-Phragmén works in rounds; each round one candidate is added to the committee. Let $y_r(v)$ denote the load assigned to (or cost contributed by) voter $v$ after round $r \leq k$. We naturally start with $y_0(v) = 0$. Let $\{c_1, \ldots, c_{r-1}\}$ be the candidates added to the committee in rounds 1 to $r - 1$. To determine the next candidate $c_r$ to add, we compute for each candidate $c \in C \setminus \{c_1, \ldots, c_{r-1}\}$ the maximum load that would arise from adding $c_r$:

$$\ell_r(c) = \frac{1 + \sum_{v \in N(c)} y_{r-1}(v)}{|N(c)|};$$

the load of voters in $N(c)$ would increase to this amount if $c$ was added to the committee. Note that the load is distributed so that all voters approving $c$ end up with the same load; this is so to minimize the maximum load. Now, to keep the maximum load as small as possible, seq-Phragmén chooses the candidate $c$ with a minimum $\ell_r(c)$, i.e.,

$$c_r = \arg\min_{c \in C \setminus \{c_1, \ldots, c_{r-1}\}} \ell_r(c).$$

If two or more candidates yield the same maximum load, a tie-breaking method is required (typically some fixed order on $C$). After choosing $c_r$, the voter loads are
adapted accordingly:

\[
y_r(v) = \begin{cases} 
\ell_r(c_r) & \text{if } v \in N(c_r), \\
y_{r-1}(v) & \text{if } v \notin N(c_r). 
\end{cases}
\]

The rule returns the winning committee \(\{c_1, \ldots, c_k\}\).

To see that these two formulations are equivalent, note that for a winning committee \(W = \{c_1, \ldots, c_k\}\) (selected in this order) the maximum loads in each round \(\ell_r(c_r)\) directly corresponds to the points at which sufficient budget was available to pay for \(c_r\). From this point of view, the discrete formulation is only the explicit calculation of the time points at which sufficient budget is available to place a new candidate in the committee.

**Example 8.** Let us again consider our running example (Example 1):

\[
3 \times \{a, b\} \quad 3 \times \{a, c\} \quad 2 \times \{a, d\} \quad 1 \times \{b, c, f\} \quad 1 \times \{e\} \quad 1 \times \{f\} \quad 1 \times \{g\}.
\]

We use the continuous formulation to describe the method, but it is easy to repeat the calculations using the discrete formulation. Figure 4 shows a visualization of the procedure, which we will now explain step by step. The first time sufficient budget is available to move a candidate into the committee is at time \(t_1 = \frac{1}{8}\). At this point, voters \(\{1, \ldots, 8\}\) can jointly pay for candidate \(a\). Now the budget of voters 1 to 8 is reset to 0; the remaining voters have a budget of \(\frac{1}{8}\).

A second candidate can be added to the committee at time \(t_2 = \frac{11}{32}\). Voters 4, 5, 6, 9 approve candidate \(c\); their respective budgets are \((\frac{7}{32}, \frac{7}{32}, \frac{7}{32}, \frac{11}{32})\) (note that voters 4, 5, and 6 have a budget that is \(\frac{1}{8}\) less than that of voter 9).

The third candidate, \(b\), is added at time \(t_3 = \frac{55}{128}\). At this point, voters 1, 2, and 3 have a budget of \(\frac{39}{128}\), and voter 9 has a budget of \(\frac{11}{128}\); that’s in total 1. Note that these numbers follow from the fact that voters 1–3 already paid \(\frac{1}{8}\) for selecting candidate \(a\) and voter 9 paid \(\frac{11}{32}\) for selecting candidate \(c\).

Finally, at time \(t_4 = \frac{5}{8}\) the last candidate, \(d\), is added to the committee. At this point, the two voters approving \(d\) (voters 7 and 8) have budget of \(\frac{5}{8} - \frac{1}{8} = \frac{1}{2}\), in total 1. Thus, seq-Phragmén returns the committee \(\{a, b, c, d\}\). When repeating this calculation using the discrete formulation, one obtains the final loads \(y_4 = (t_3, t_3, t_3, t_2, t_2, t_4, t_4, t_3, 0, 0, 0)\). ⊢

As mentioned earlier, there are also optimization-based analogues of seq-Phragmén. We will discuss the most notable optimization-based method: lexmin-Phragmén\(^8\) [49, 112, 168].

---

\(^8\)Phragmén discusses optimization variants of his rule in [168] and proposes to minimize the maximum load (see [112]); this rule could be called min-Phragmén. Brill et al. [49] show that it is more sensible to use a lexicographic comparison of loads instead of only considering the maximum load. We thus only discuss lexmin-Phragmén (referred to as opt-Phragmén in [49]). Further optimization variants exist, such as minimizing the variance of loads [49, 112, 168].
Rule 10 (Phragmén’s Lexmin Rule, lexmin-Phragmén). Each candidate in the committee incurs a load (or cost) of 1 which has to be distributed to voters approving this candidate. Given a committee \( W = \{c_1, \ldots, c_k\} \), a valid load distribution for \( W \) is a function \( \ell : W \times N \to [0, 1] \) which satisfies (1) if \( \ell(c, i) > 0 \) then voter \( i \) approves \( c \), and (2) \( \sum_{i \in N} \ell(c, i) = 1 \) for all \( c \in W \). Let \( \ell(c) = \sum_{i \in N} \ell(c, i) \).

To compare two (valid) load distributions, we use a lexicographic order. Let \( W = \{c_1, \ldots, c_k\} \) and \( W' = \{c'_1, \ldots, c'_k\} \) be two committees and let \( \ell \) and \( \ell' \) be valid load distributions for \( W \) and \( W' \), respectively. Let \( x \) be the sorted \( k \)-tuple (starting with the largest entry) of \( (\ell(c_1), \ldots, \ell(c_k)) \), and let \( x' \) be the sorted \( k \)-tuple (starting with the largest entry) of \( (\ell'(c'_1), \ldots, \ell'(c'_k)) \). We say that \( x \) is lexicographically smaller than \( x' \) if there exists an index \( j \leq k \) such that \( y_1 = x'_1, y_2 = x'_2, \ldots, y_j = x'_j, \) and \( y_{j+1} < x'_{j+1} \).

Let \( \ell(W) \) denote the lexicographically smallest valid load distribution for committee \( W \). Then, lexmin-Phragmén returns all committees \( W \) for which \( \ell(W) \) is lexicographically minimal in the set \( \{\ell(W') : W' \subseteq C \text{ and } |W'| = k\} \).

Example 9. When considering our running example, we will see that lexmin-Phragmén behaves differently than seq-Phragmén. When looking for a committee that has the lexicographically smallest load distribution, we find committee \( W = \{a, b, c, f\} \) with \( \ell(W) = (3/8, 3/8, 3/8, 3/8, 3/8, 3/8, 3/8, 1/2, 0, 1/2, 0, 0) \). This load distribution is depicted in Figure 5. Committee \( W \) is the only winning committee; for example, committee \( W' = \{a, b, c, d\} \) (the winning committee of seq-Phragmén) has \( \ell(W') = (3/7, 3/7, 3/7, 3/7, 3/7, 1/2, 1/2, 3/7, 0, 0, 0, 0) \), which is lexicographically larger.

Figure 4: A visualization of seq-Phragmén (upper part) applied to the election instance of Example 1 (lower part). In the upper part all regions of the same color (corresponding to the same candidate) have an area of 1, which is the budget spent on this candidate.
Figure 5: A visualization of lexmin-Phragmén (upper part) applied to the election instance of Example 1 (lower part). In the upper part all regions of the same color (corresponding to the same candidate) have an area of 1, which is the budget spent on this candidate.

3.5 Phragmén-like Rules

We now discuss a very recent addition to the zoo of ABC rules, dubbed Rule X [164]. This rule can be viewed as a variant of seq-Phragmén, where the voters are given some budget upfront, rather than receiving it continuously. We include this rule in our survey as it is polynomial-time computable and even surpasses the proportionality guarantees of seq-Phragmén.

**Rule 11 (Rule X).** The rule proceeds in two phases. The first phase consists of at most \( k \) rounds; in each round one candidate is added to the committee. In the second phase the committee is completed in one of several possible ways.

For the first phase, we assume each voter is initially given a budget of \( \frac{k}{n} \). Let \( x_r(i) \) denote the budget of voter \( i \) after round \( r \); thus \( x_0(i) = \frac{k}{n} \). As with seq-Phragmén, putting a candidate in the committee incurs a cost of 1. In each round \( r + 1 \), we consider the set of candidates that have not yet been placed in the committee and whose supporters can afford to pay for them, i.e., the candidates \( c \) such that \( \sum_{i \in N(c)} x_r(i) \geq 1 \). Let this set be \( C_r \subseteq C \). If \( C_r \) is empty, then we conclude the first phase and move to phase two. Otherwise, for each candidate \( c \in C_r \) we ask what is the minimal budget \( \rho(c) \) such that each voter approving \( c \) pays at most \( \rho(c) \) and all voters who approve \( c \) pay 1 in total, i.e., what is the minimal value \( \rho(c) \) that satisfies:

\[
\sum_{i \in N(c)} \min(\rho(c), x_r(i)) = 1.
\]

(Such a \( \rho(c) \) always exists, since otherwise \( c \) would not be contained in \( C_r \).) We select the candidate \( c \) that minimizes \( \rho(c) \) (using some fixed tiebreaking if necessary), and reduce the
budget of voters who approve $c$ accordingly—for each $i \in N$ we set

$$x_{r+1}(i) = \begin{cases} x_i(r) - \rho(c) & \text{if } c \in A(i) \text{ and } x_i(r) \geq \rho(c), \\ 0 & \text{if } c \in A(i) \text{ and } x_i(r) < \rho(c), \\ x_i(r) & \text{if } c \notin A(i), \end{cases}$$

i.e., voters who approves $c$ either pay $\rho(c)$ or their remaining budget.

The second phase is only relevant if fewer than $k$ have been put in the committee $W$ so far. If $|W| < k$, we add have to add $k - |W|$ additional candidates in $W$. Many properties of Rule X do not depend on the specific way in which these $k - |W|$ candidates are selected.\(^9\) A concrete and recommendable way to fill the committee is to use seq-Phragmén but with initial budgets defined in the following fashion: We set the starting budget of each voter to their budget after the first phase of the Rule X. To be more concrete, assume phase one ends with round $r'$. In the discrete formulation of seq-Phragmén the starting loads are $y_0(i) = k/n - x_{r'}(i)$. Then seq-Phragmén proceeds as usual until to desired committee size is reached.

### Example 10

Consider once again our running example. Each voter is initially given a budget of $\frac4{12} = \frac13$. In the first round candidate $a$ is selected and each of the first 8 voters pays $\frac18$ for it. In the second round, $C_2 = \emptyset$ since no candidate has sufficiently endowed voters. For example, the amount of money that the voters who approve $b$ have in total is $3 \cdot (\frac13 - \frac18) + \frac13 < 1$ and thus insufficient to pay for $b$. This ends the first phase of the rule.

In the second phase, the voters start receiving additional budget. Voters 1 to 8 start with a budget of $\frac13 - \frac18$; voters 9 to 12 start with a budget of $\frac13$. At time $t_2 = \frac1{96}$ the voters who approve $b$ have enough money to pay for including $b$. Indeed:

$$3 \cdot (\frac13 - \frac18) + \frac13 + 4t_2 = 1.$$  

The same is true for the voters who approve $c$. Let us assume that we resolve the tie in favor of $b$: $b$ is selected and the voters 1, 2, 3 and 9 are left without budget. Next, at time $t_3 = \frac3{32}$ candidate $c$ is selected. Finally, at time $t_4 = \frac7{24}$ we select $d$. Committee $W = \{a, b, c, d\}$ is the only winning committee. In this example, Rule X returns the same committee as seq-Phragmén.

Since in Example 10 only one candidate is selected in the first phase of Rule X, we provide one additional example which better illustrates this phase of the rule.

### Example 11

Consider the following approval profile given by

- $11 \times \{a, b, c\}$
- $9 \times \{a, b, c, d, e\}$
- $4 \times \{a, b, d, e\}$
- $6 \times \{d, e\}$

\(^9\)An exception is the priceability axiom, see Section 5.3; this axiom is dependent on how to extend the committee to its full size. The proposed completion via seq-Phragmén fulfills priceability.
which is illustrated in the lower part of Figure 6. The goal is to select a committee of size \( k = 4 \). Thus, voters start with a budget of \( \frac{4}{30} \).

In this example, candidates \( a \) and \( b \) are selected in the first two rounds, and each of the first 24 voters pays two times \( \frac{1}{24} \) for these two candidates. Candidate \( c \) is selected in the third round; each of the first 20 voters pays \( \frac{1}{20} \) for \( c \), and after the third round these voters are left with no budget (indeed, \( \frac{1}{24} + \frac{1}{24} + \frac{1}{20} = \frac{4}{30} \)). In the fourth round \( d \) or \( e \) is selected (we assume it is \( d \) by tiebreaking). Voters 21–24 pay \( \frac{1}{20} \) for \( d \); the voters 25–30 pay \( \frac{4}{30} \). The payments of the voters for the particular candidates are depicted in the upper part of Figure 6. Since four candidates are selected in the first phase of Rule X, there is no second phase.

In this example, either committee \( \{a, b, c, d\} \) or \( \{a, b, c, e\} \) are winning according to Rule X (depending on tiebreaking). In contrast, seq-Phragmén would pick \( \{a, b, d, e\} \).

We mention three further rules that are related to Phragmén’s rules. The first is the Expanding Approvals Rule [10]. This rule is defined for weak order preferences and has favorable axiomatic properties in this setting. It is less convincing for purely dichotomous preferences [10] and thus we do not consider it further. The second rule is the maximin support method [181], which is similar to seq-Phragmén. It is an iterative rule based on a form of load balancing, but in contrast to seq-Phragmén all loads can be redistributed each round. A first analysis showed that the maximin support method and seq-Phragmén share many axiomatic properties [181]; it remains an open question which of the two rules is preferable. We focus in this survey on seq-Phragmén as it is better studied and
conceptually simpler.

Finally, Phragmén also introduced a method now referred to as either Phragmén’s first method, Eneström’s method, or method of Eneström–Phragmén\textsuperscript{11} [56, 87, 112]. We do not discuss this rule in this survey, even though it is a sensible rule. Indeed, it can be viewed as an analogue of STV with approval ballots. However, its main downside compared to seq-Phragmén is that it is not committee monotone (cf. Section 4.3) and satisfies only weaker proportionality requirements than Rule X (e.g., it fails EJR as defined in Section 5.2).

3.6 Non-Standard ABC Rules

As mentioned at the beginning of this section, most ABC rules coincide with (single-winner) Approval Voting for $k = 1$. If we understand an approval ballot as indicating those alternatives that a voter likes, then for $k = 1$ it is indeed very natural to select the most-approved alternative. Thus, we refer to rules that differ from Approval Voting for $k = 1$ as non-standard ABC rules. We present two such rules. The first one, Minimax Approval Voting (MAV) introduced by Brams, Kilgour, and Sanver [38], interprets approval ballots as the voter’s exact description of the desired outcome. If a voter approves a set $X$, then she indicates that all these alternatives should be chosen; any sub- or superset is suboptimal. In addition, MAV is an egalitarian rule in the sense that it only pays attention to the least-satisfied voter.

To measure the distance between an approval set and a committee, we rely on the Hamming distance:

**Definition 2.** Given two sets $X, Y$, we define the Hamming distance between $X$ and $Y$ as the size of their symmetric difference: $d_{\text{ham}}(X, Y) = |X \setminus Y| + |Y \setminus X|$.

**Rule 12** (Minimax Approval Voting, MAV). MAV selects committees $W$ that minimize the largest Hamming distance among all voters, i.e., MAV minimizes $\max_{i \in N} d_{\text{ham}}(A(i), W)$.

**Example 12.** To see that MAV does not correspond to Approval Voting for $k = 1$, consider the following approval profile:

$$99 \times \{a\} \quad 1 \times \{b, c\}.$$  

The Hamming distance $d_{\text{ham}}$ between the committee $W_1 = \{a\}$ and the approval set $\{b, c\}$ is 3. In contrast, for the committee $W_2 = \{b\}$ (or $\{c\}$) we have $d_{\text{ham}}(\{b, c\}, W_2) = 1$ and $d_{\text{ham}}(\{a\}, W_2) = 2$. Thus, MAV selects either $b$ or $c$, even though these alternatives are approved by only a single voter.\textsuperscript{10}

\textsuperscript{10}For example, in profiles where no candidate reaches a specified quota and every voter approves only one candidate, the Expanding Approvals Rule selects an arbitrary committee and thus ignores the voters’ preferences.

\textsuperscript{11}It is not completely clear whether Phragmén or Gustaf Eneström (1852–1923) should be credited with this method. However, it appears to be justifiable to simply credit both of them; see the historical summary provided by Janson [112, Footnote 38].
Remark 1. It is interesting to note that if we replace the max operator in the definition of MAV by a sum, we obtain the Multi-Winner Approval Voting rule (Rule 1).

Remark 2. MAV, as defined, has a major shortcoming. Consider the following slight modification of Example 12:

\[ 99 \times \{a\} \quad 1 \times \{a, b, c\}, \]

For all size-1 committees, the Hamming distance to \(\{a, b, c\}\) is 2. Hence, all three committees are equally preferable according to MAV—even though candidate a is approved by every voter (and b and c by only one voter). We see that MAV might disregard a unanimous choice. This problem can be remedied by also considering the second-least satisfied voter in case of ties, and the third-least in case there is still a tie, and so on until a difference between the committees is found. More formally, for each committee \(W\), we compute \(d_{\text{ham}}(A(1), W), d_{\text{ham}}(A(2), W), \ldots\) and sort this tuple of length \(|N|\) in decreasing order; we denote this tuple of distances \(D_W\). Instead of considering only the first entry in these tuples, we could lexicographically sort them. That is, a committee \(W_1\) is preferred to a \(W_2\) if there exists an index \(i \leq n\) such that \(D_{W_1}(i) < D_{W_2}(i)\) and \(D_{W_1}(j) = D_{W_2}(j)\) for all \(1 \leq j < i\). In our example, we have \(D_{\{a\}} = (2, 0, 0, \ldots)\) and \(D_{\{b\}} = D_{\{c\}} = (2, 2, 2, \ldots)\); with this modification \(\{a\}\) is the only winning committee. To the best of our knowledge this modification of MAV has not been studied in the context of voting. However, it is equivalent to the \(G_{\text{max}}\) belief merging operator for Hamming distance [118].

The second non-standard rule is Satisfaction Approval Voting\(^{12}\) (SAV). SAV is a variation of AV where each voter has one point and distributes it evenly among all approved candidates.

**Rule 13** (Satisfaction Approval Voting, SAV). The SAV-score of a committee \(W\) is defined as

\[ s_{\text{SAV}}(A, W) = \sum_{v \in V} \frac{|W \cap A(v)|}{|A(v)|}. \]

SAV returns all committees with a maximum SAV-score.

Note that SAV is not a Thiele method since the number of approved candidates influences the SAV-score.

**Example 13.** To see that SAV does not correspond to Approval Voting for \(k = 1\), consider

\[ 1 \times \{a\} \quad 3 \times \{b, c, d, e\}. \]

The SAV-score of \(a\) is 1 and of \(b, c, d, e\) it is \(3/4\). Thus, SAV selects \(\{a\}\) even though it is approved by only one voter.

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\(^{12}\)Satisfaction Approval Voting was introduced under this name by Brams and Kilgour [37], but the method has been discussed already in the 19th century (see Janson’s survey [112], Section E.1.5.). It is also known as Equal and Even Cumulative Voting.
Finally, let us note that rev-seq-PAV, introduced in Section 3.2, is also a non-standard method.

**Example 14.** To see that rev-seq-PAV is a non-standard method, consider, e.g.,

\[
1 \times \{a, b\} \quad 1 \times \{a, b, c\} \quad 1 \times \{a, b, d\} \quad 2 \times \{a, c, d\} \quad 1 \times \{b\} \quad 1 \times \{c\} \quad 1 \times \{d\}.
\]

In the first round, the marginal contribution of \(a\) is \(1/2 + 4 \cdot 1/3\); the marginal contribution from the other candidates is at least 2. Thus, candidate \(a\) is removed in the first round, even though it has the highest approval score.

## 4 Basic Properties of ABC Rules

In the previous section we have seen a wide array of ABC rules. Considering how much they differ in their definitions, it can be expected that they differ also in the properties they exhibit. In this section we consider basic properties of ABC rules. These properties describe the behavior of such rules and offer insights into the nature of specific ABC rules. **Table 1** offers an overview of most properties discussed in this section.
4.1 Anonymity, Neutrality, and Resoluteness

Anonymity and Neutrality are two of the most basic properties in the social choice literate [7, 146, 154]. Anonymity states that the identity of voters should not influence the outcome: it should be irrelevant whether voter \( i \) approves \( A(i) \) and voter \( j \) approves \( A(j) \) or vice versa. Formally, an ABC rule \( \mathcal{R} \) satisfies anonymity if for all election instances \((A, k)\) with voter set \( N \) and bijections \( \pi : N \to N \) it holds that \( \mathcal{R}(A, k) = \mathcal{R}(A \circ \pi, k) \). It is easy to see that all rules introduced in Section 3 satisfy anonymity. A typical example of a voting rule that fails anonymity is any dictatorial rule (a rule considering only the preferences of a single distinguished voter, e.g., of voter 1).

Neutrality is the counterpart to anonymity but applies to candidates: it states that all candidates should be treated equally. Formally, an ABC rule \( \mathcal{R} \) satisfies neutrality if for all election instances \((A, k)\) with candidate set \( C \) and bijections \( \pi : C \to C \) it holds that \( \mathcal{R}(A, k) = \mathcal{R}(\pi^\star \circ A, k) \), where \( \pi^\star \) is the natural extension of \( \pi \) to a bijection from \( \mathcal{P}(C) \) to \( \mathcal{P}(C) \) defined by \( \pi^\star(X) = \{\pi(c) : c \in X\} \) for each \( X \subseteq C \). The rules that fail neutrality are usually those that require some form of tiebreaking.

Anonymity and neutrality together are often called symmetry.

The third, equally fundamental property we want to discuss here is resoluteness. An ABC rule is resolute if it always returns exactly one winning committee. An ABC rule can either be resolute or neutral, but not both. To see this, consider an approval profile where all voters approve candidates \( \{a, b\} \) and \( k = 1 \): either a rule returns two winning committees or decides in favor of one of the two candidates. Clearly, any rule can be made resolute by imposing a tiebreaking between winning committees. Conversely, if a resolute rule is defined by a tiebreaking order over candidates (this includes all rules in Section 3 that fail neutrality), it can be made neutral by returning all committees that win according to some tiebreaking order. In this way, one can trade neutrality against resoluteness.

4.2 Pareto Efficiency and Condorcet Committees

Pareto efficiency\(^{13}\) is a very general concept to compare two outcomes given the preferences of individuals: outcome \( Y \) dominates outcome \( X \) if (1) every individual weakly prefers outcome \( Y \) to \( X \) (i.e., everyone likes \( Y \) at least as much as \( X \)), and (2) there is at least one individual that strictly prefers \( Y \) to \( X \). Pareto efficiency, broadly speaking, means that dominated outcomes are avoided. This concept can be directly translated to our setting to compare committees [128]:

**Definition 3.** A committee \( W_1 \) dominates a committee \( W_2 \) if

1. every voter has at least as many representatives in \( W_1 \) as in \( W_2 \) (for \( i \in N \) it holds that \( |A(i) \cap W_1| \geq |A(i) \cap W_2| \)), and
2. there is one voter with strictly more representatives (there exists \( j \in N \) with \( |A(j) \cap W_1| > |A(j) \cap W_2| \)).
A committee that is not dominated by any other committee (of the same size) is called Pareto optimal.

An ABC rule $R$ satisfies strong Pareto efficiency if $R$ never outputs dominated committees. An ABC rule $R$ satisfies weak Pareto efficiency if for all election instances $(A, k)$ it holds that if $W_2 \in R(A, k)$ and $W_1$ dominates $W_2$, then $W_1 \in R(A, k)$.

Table 1 shows an overview which rules satisfy Pareto efficiency (for details, in particular counterexamples, see [128]). It may be surprising that rather few ABC rules satisfy Pareto efficiency even though this is considered a basic property in other settings (or even a minimal requirement). Indeed, among the rules introduced in Section 3 only Thiele rules and MAV satisfy weak Pareto efficiency [128], and among those only AV, PAV, and SAV satisfy strong Pareto efficiency (Proposition A.1).

To see an example how a rule may fail Pareto efficiency, it is instructive to consider Monroe’s rule:

Example 15 (from [128]). Consider the approval profile

$$
\begin{align*}
2 \times \{a\} & \quad 1 \times \{a, c\} & \quad 1 \times \{a, d\} & \quad 10 \times \{b, c\} & \quad 10 \times \{b, d\}
\end{align*}
$$

For $k = 2$, Monroe selects $\{c, d\}$ as the (only) winning committee with a Monroe-score of 22. Committee $\{c, d\}$ is however dominated by $\{a, b\}$: everyone of the 24 voters has one representative in $\{a, b\}$ but only 22 voters have a representative in $\{c, d\}$. Thus, every voter is either equally satisfied or better off with committee $\{a, b\}$. This example shows that Pareto efficiency clashes with Monroe’s goal to assign representatives to groups of similar size.

One may wonder whether it is sensible to improve an ABC rule $R$ that is not Pareto efficient in the following way: given an election instance $E$, if $W \in R(E)$ is dominated by another committee, then instead output all Pareto optimal committees that dominate $W$. There are two main objections against this idea: First, this modification may destroy other axiomatic properties (e.g., perfect representation, as discussed in Section 5.3, and Pareto efficiency are incompatible). Second, finding Pareto improvements is a computationally hard task:

Theorem 1 ([11, Theorem 2]). Given an election instance $(A, k)$ and committee $W$, it is coNP-complete to determine whether $W$ is Pareto optimal.

As a consequence of Theorem 1, we cannot expect to obtain polynomial-time computable, Pareto efficient ABC rules by modifying existing rules as described above. Note, however, that polynomial-time computable, Pareto efficient ABC rules exist, e.g., AV and SAV. Thus, finding a Pareto optimal committee is possible in polynomial-time.

A related property to Pareto efficiency has been proposed by Darmann [71]: a committee $W$ is a Condorcet committee if for every other committee $W'$, for a majority of voters

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13Named after Vilfredo Pareto (1848–1923), an Italian economist [4, 80].
\( V \subseteq N \) it holds that \( |A(i) \cap W| > |A(i) \cap W'| \) for \( i \in V \). Similarly to Theorem 1, deciding whether a given committee \( W \) is a Condorcet committee is coNP-complete. However, in contrast to Pareto optimality, it is also coNP-complete to decide whether a Condorcet committee exists [71]. To the best of our knowledge, it has not been analyzed which ABC rules output a Condorcet committees if it exists.

### 4.3 Committee Monotonicity

Committee monotonicity (also referred to as house monotonicity) is property that is highly desirable in some settings: if the committee size \( k \) is increased to \( k + 1 \), then a winning committee of size \( k \) should be a subset of a winning committee of size \( k + 1 \). Since this property is particularly useful for resolute rules, we define it exclusively for resolute rules. Appropriate definitions for irresolute rules (called upward- and downward-accretive) can be found in [115].

**Definition 4.** A resolute ABC rule \( R \) is **committee monotone** if for all election instances \((A, k)\) it holds that \( W \subseteq W' \), where \( W \) is the single winning committee in \( R(A, k) \), and \( W' \) is the single winning committee in \( R(A, k + 1) \).

To see why committee monotonicity can be an essential requirement in applications, consider the following situation: a group can jointly acquire \( k \) items and uses an ABC rule to fairly select those. Once these \( k \) items are ordered, it turns out that one additional item can be afforded. If the used ABC rule is committee monotone, it is clear which item to acquire next. If the rule, however, is not committee monotone, then the selection for \( k + 1 \) items might include none of the already ordered items, a useless recommendation.

Another example is a hiring process where it is not determined up-front how many candidates are to be hired. Here it is useful that a committee monotone rule actually produces a *ranking* of candidates: which one should be hired if only one position is available, which one if a second position is to be filled, etc. This connection between committee monotone ABC rules and rankings has been explored in-depth by Skowron et al. [194].

Table 1 shows which of the considered rules are committee monotone, assuming that these rules are made resolute by fixing a tiebreaking order among candidates. AV, seq-PAV, seq-CC, rev-seq-PAV, seq-Phragmén, and SAV are committee monotone; this follows immediately from their corresponding definitions. Counterexamples for the remaining rules can be found in Appendix A, Proposition A.2.

### 4.4 Candidate and Support Monotonicity

Candidate monotonicity deals with a seemingly obvious requirement: if the support of a candidate increases (i.e., more voters approve this candidate), then this cannot harm the candidate’s inclusion in a winning committee. However, this property is not satisfied by some ABC rules, in particular, if we demand such a monotonicity to hold also for groups
of candidates. In addition, there is a difference whether an existing voter changes her ballot, or if a new voter enters the election.

Candidate monotonicity axioms for ABC rules have been considered in a number of papers [10, 112, 127], but the paper by Sánchez-Fernández and Fisteus [180] should be highlighted for the most in-depth analysis.

We use the following notation: For a profile $A$ and candidate set $X \subseteq C$, we write $A_{i+X}$ to denote the profile $A$ with voter $i$ additionally approving the candidates in $X$, i.e., $A_{i+X} = (A(1), \ldots, A(i-1), A(i) \cup X, A(i+1), \ldots, A(n))$. Further, we write $A_+X$ to denote the profile $A$ with one additional voter approving $X$, i.e., $A_+X = (A(1), \ldots, A(n), X)$.

**Definition 5 ([180]).** An ABC rule $\mathcal{R}$ satisfies support monotonicity without additional voters if for any election instance $(A, k)$, $i \in N$, and candidate set $X \subseteq C$ it holds that

1. if $X \subseteq W$ for all $W \in \mathcal{R}(A, k)$, then $X \subseteq W'$ for all $W' \in \mathcal{R}(A_{i+X}, k)$, and
2. if $X \subseteq W$ for some $W \in \mathcal{R}(A, k)$, then $X \subseteq W'$ for some $W' \in \mathcal{R}(A_{i+X}, k)$.

An ABC rule $\mathcal{R}$ satisfies support monotonicity with additional voters if for any election instance $(A, k)$ and candidate set $X \subseteq C$ the properties above hold for $A_+X$ instead of $A_{i+X}$.

If an ABC rule satisfies these axioms only for singleton sets ($X = \{c\}$), we speak of candidate monotonicity with/without additional voters.\(^\dagger\)

The analysis of ABC rules with respect to these axioms is mostly due to Janson [112], Sánchez-Fernández and Fisteus [180], Mora and Oliver [153]. We summarize the results in Table 1 with one column for support monotonicity with additional voters and one with additional voters. There, the symbol ✓ means that support monotonicity is satisfied, cand means that candidate monotonicity is satisfied but not support monotonicity, \(\geq\) cand means that candidate monotonicity is satisfied and it is unclear whether support monotonicity is satisfied, \(\times\) means that the rule fails even candidate monotonicity, and a question mark means that we do not know. Detailed counterexamples related to support monotonicity can be found in Proposition A.3 in the appendix.

### 4.5 Consistency

Consistency is an axiom describing whether a rule behaves consistently with respect to disjoint groups: if the outcome of an election is the same for two disjoint groups, then a voting rule should arrive at this outcome also if these two groups are joined into a single electorate. This axiom is a straightforward adaption of consistency as defined for single-winner rules by Smith [196] and Young [209] and was first discussed in the context of ABC rules by Lackner and Skowron [126].

\(^\dagger\)These properties are implied by weak support monotonicity with/without population increase as defined by Sánchez-Fernández and Fisteus [180].
Definition 6. An ABC rule \( \mathcal{R} \) satisfies consistency if for every \( k \geq 1 \) and two profiles \( A : N \to \mathcal{P}(C) \) and \( A' : N' \to \mathcal{P}(C) \) with \( N \cap N' = \emptyset \), if \( \mathcal{R}(A, k) \cap \mathcal{R}(A', k) \neq \emptyset \) then \( \mathcal{R}(A + A', k) = \mathcal{R}(A, k) \cap \mathcal{R}(A', k) \).

Monroe’s rule, for example, does not satisfy consistency:

Example 16. Let profile \( A \) be

\[
A(1) : \{a, y\} \quad A(2) : \{a, y\} \quad A(3) : \{b, y\} \quad A(4) : \{b, y\}
\]

and profile \( A' \) be

\[
A(5) : \{y\} \quad A(6) : \{a\} \quad A(7) = A(8) = A(9) = A(10) : \{a, x\} \\
A(11) : \{y\} \quad A(12) : \{b, y\} \quad A(13) = A(14) = A(15) = A(16) : \{b, x\}
\]

For \( k = 2 \), Monroe returns for profile \( A \) the winning committees \( \{a, b\} \), \( \{a, y\} \), and \( \{b, y\} \), all of which with a Monroe-score of 4. For profile \( A' \), Monroe returns the winning committee \( \{a, b\} \), with a Monroe-score of 10; the corresponding Monroe assignment groups voters 5–10 and 11–16. Now, let us consider the profile \( A + A' \). Consistency would demand that \( \{a, b\} \) is the unique winning committee, as it is the only committee winning in both \( A \) and \( A' \). Committee \( \{a, b\} \) has a Monroe-score of 14 in \( A + A' \). This score, however, is not optimal: \( \{x, y\} \) has a Monroe-score of 15; the corresponding Monroe assignment groups voters \( \{1, \ldots, 6, 11, 12\} \) and \( \{7, \ldots, 10, 13, \ldots, 16\} \). Thus, \( \{a, b\} \) is not winning and consistency is violated.

Broadly speaking, the only rules satisfying consistency are so-called ABC scoring rules [126]. These are defined similarly to Thiele methods but are more general, as the satisfaction of voter may depend on the number of candidates approved by this voter:

Definition 7. A scoring function is a function \( f : \mathbb{N} \times \mathbb{N} \to \mathbb{R} \) satisfying \( f(x, y) \geq f(x', y) \) for \( x \geq x' \). Given such a scoring function, we define the score of \( W \) in \( A \) as

\[
\text{sc}_f(A, W) = \sum_{v \in V} f(|A(v) \cap W|, |A(v)|).
\]

The ABC scoring rule defined by a scoring function \( f \) returns all committees with maximum score.

By definition, each Thiele method is an ABC scoring rule, whereas SAV is an example of an ABC scoring rule that is not a Thiele method. Further, it follows immediately from the definition of welfarist rules (Definition 1) that an ABC scoring rule is welfarist if and only if it is a Thiele method.

Lackner and Skowron [126] axiomatically characterized the class of ABC scoring rules. This characterization is in a slightly different model than the one we use in this survey: the characterization applies to ABC ranking rules instead of ABC rules (as defined in this survey).
survey). ABC ranking rules output a weak order over committees (a ranking with ties over committees) instead of just distinguishing between winning and losing committees (as we assume here). Note, however, every ABC ranking rule defines an ABC rule (top-ranked committees are winning).

The following characterization uses two axioms we have not mentioned so far: weak efficiency and continuity. Both are rather weak, technical axioms. Intuitively, weak efficiency requires that approved candidates are preferable to non-approved candidates, and continuity states that a sufficiently large majority can force a committee to win.

**Theorem 2 ([126]).** An ABC ranking rule is an ABC scoring rule if and only if it satisfies anonymity, neutrality, consistency, weak efficiency, and continuity.

As both weak efficiency and continuity are generally satisfied by sensible voting rules, one can conclude that ABC scoring rules essentially capture the class of consistent ABC ranking rules\(^{15}\) (see [126] for a more detailed discussion and exact definitions of the axioms).

To the best of the authors’ knowledge, this is the first axiomatic characterization of ABC rules. In Section 5.1, we will discuss how this result can be used to obtain further axiomatic characterizations, e.g., of PAV.

### 4.6 Strategic Voting

Strategic voting is a phenomenon central to social choice theory. Sometimes, it is preferable for voters to misrepresent their preferences to change the outcome of an election; this is often referred to as “manipulation”. The famous impossibility theorem by Gibbard [104] and Satterthwaite [184], showing that all “reasonable” single-winner voting rules are susceptible to manipulation, is considered one of the main results in the field. The Gibbard–Satterthwaite theorem applies to elections where voters provide linear rankings over alternatives. As our approval-based setting uses a much more restricted form of preferences, strategyproofness is not completely out of the picture.

We are going to consider two forms of strategyproofness here: Cardinality-strategyproofness and inclusion-strategyproofness (taken from [161], see [102, 201] for more general discussions of strategyproofness in social choice). Cardinality-strategyproofness assumes that voters are concerned only about the number of approved candidates in the committee (and do not distinguish them), whereas inclusion-strategyproofness assumes that voters may have more complex preferences so a successful manipulation must produce a committee including all approved candidates that were already included in the original committee.

To clarify what it means that a voter misrepresents their true preferences, we use the concept of \(i\)-variants: Given profiles \(A\) and \(A'\), both with the same set of voters \(N\), we

\(^{15}\)In the setting of single-winner rules a similar result holds: a social welfare function is a scoring rule if and only if it satisfies anonymity, neutrality, consistency, and continuity, as shown by Smith [196] and Young [209].
say that $A'$ is an $i$-variant of $A$ if $A(j) = A'(j)$ for all $j \in N \setminus \{i\}$ with $j \neq i$. Let us first define both notions for resolute ABC rules.

**Definition 8.** A resolute ABC rule $\mathcal{R}$ satisfies cardinality-strategyproofness if for all profiles $A$ and $A'$ where $A'$ is an $i$-variant of $A$ and $k \geq 1$ it holds that $|\mathcal{R}(A, k) \cap A(i)| \geq |\mathcal{R}(A', k) \cap A(i)|$.

**Definition 9.** A resolute ABC rule $\mathcal{R}$ satisfies inclusion-strategyproofness if for all profiles $A$ and $A'$ where $A'$ is an $i$-variant of $A$ and $k \geq 1$ it holds that $\mathcal{R}(A, k) \cap A(i) \not\subset \mathcal{R}(A', k) \cap A(i)$.

Inclusion-strategyproofness is a weaker notion than cardinality-strategyproofness, since fewer committees count as a successful manipulation, i.e., successful manipulations are harder to achieve.

Both axioms can be generalized to irresolute ABC rules via a set extension, i.e., by defining how voters compare sets of committees. A natural extension is the Kelly (or cautious) extension: a voter prefers $\mathcal{R}(A', k)$ to $\mathcal{R}(A, k)$ if every committee in $\mathcal{R}(A', k)$ is preferable to every committee in $\mathcal{R}(A, k)$. A more substantial discussion of strategyproofness of irresolute rules can be found in the paper of Kluiving et al. [117]. Moreover, Lackner and Skowron [127] propose a rather strong strategyproofness axiom (SD-strategyproofness) that applies to irresolute rules and implies cardinality-strategyproofness.

Among the rules considered in this paper, only AV satisfies any of the mentioned strategyproofness axioms, namely both inclusion-strategyproofness and cardinality-strategyproofness (see A.4 for counterexamples). Furthermore, AV also satisfies SD-strategyproofness and can even be characterized in the class of ABC scoring rules (as defined in Section 4.5) as the only rule satisfying SD-strategyproofness [127]. We note, however, that under more holistic models, e.g., models where voters have underlying non-dichotomous preferences, AV is no longer strategyproof (see, e.g., [72, 136, 148, 187]). Recently, Schuerman et al. [185] have conducted a behavioral experiment in which they analyzed how the voters vote under non-dichotomous preferences, when they are uncertain about other voters’ preferences, and when AV is used to select the winning candidates. These results suggest that the voters may use different (sometimes suboptimal) heuristics when making decisions which candidates they should approve.

We further discuss strategyproofness in Section 5.6 in the context of proportionality. We will see that even weakest forms of proportionality are incompatible with strategyproofness.

Finally, we mention that Amanatidis et al. [2] have considered a different variant of strategy-proofness, where each voter cares not only about the number of representatives in the elected committee, but also about the number of non-elected candidates that she does not approve.
4.7 Computational Complexity

As a last basic property of ABC rules, we want to discuss their computational complexity. That is, in short: how computationally expensive is it to find a winning committee according to a given ABC rule? Here, we distinguish only two types of complexity: ABC rules that are computationally easy, i.e., computable in polynomial time, and ABC rules that are computationally expensive, i.e., those that are NP-hard. This is only a coarse dichotomy; we will discuss computational issues in more detail in Section 6.

Let us first consider the class of Thiele methods. Out of the three most prominent Thiele methods, two are NP-hard (CC and PAV) and one is computable in polynomial time (AV). A polynomial-time algorithm for AV is straightforward: for each alternative $c$ we compute its approval score $s_{c_{AV}}(A, c) = |\{v \in N : c \in A(v)\}|$ and select the $k$ alternatives with the largest scores. To be able to claim NP-hardness of an ABC rule $\mathcal{R}$, we have to fix a decision problem; we choose the following for rules based on scores: given an approval profile, is there a committee with $\mathcal{R}$-score at least $s$? The NP-hardness of CC has been shown by Procaccia et al. [173]; the NP-hardness of PAV by Skowron et al. [192] and Aziz et al. [13] (for a different decision problem). A more general result [192, Theorem 5] shows which Thiele methods are NP-hard:

**Theorem 3.** Let $w : \mathbb{N} \rightarrow \mathbb{R}$ be a non-increasing function for which $w(i) > w(i + 1)$ for some $i \in \mathbb{N}$. Given an approval profile profile $A$, a committee size $k$, and a bound $s$, it is NP-hard to decide whether there exists a committee of size $k$ with a $w$-score of at least $s$, i.e., $s_{c_{w}}(A, W) \geq s$.

This theorem includes all Thiele methods except for AV, and thus AV is the only polynomial-time computable Thiele method.

Sequential and reverse sequential Thiele methods can be computed in polynomial time; this follows immediately from their definitions. The same holds for Greedy Monroe, seq-Phragmén, Rule X, and SAV. In contrast, appropriate decisions problems for Monroe’s rule [173], minimax-Phragmén [49], and MAV [138] are NP-hard. Further, these problems belong to NP. Indeed, computing the score of a given committee $W$ according to minimax-Phragmén and MAV is straightforward. For Monroe’s rule this is less obvious: a polynomial-time algorithm based on a reduction to the min-cost max-flow problem has been suggested by Procaccia et al. [173]. It works as follows: We build a graph with a single source $s$, and a single sink $t$. For each member of the given committee $c \in W$, we add a vertex and connect it to $s$ via an edge with cost 0, capacity $\lceil n/k \rceil$ and flow requirement of at least $\lfloor n/k \rfloor$. For each voter $i \in N$, we add a vertex and connect it to $t$ with an edge with cost 0 and capacity 1. Finally, we connect each candidate $c$ and each voter $i$ with an edge of capacity 1; if $c \in A(i)$ we set the cost of this edge to 1, otherwise we set it to 0. In such constructed graph we seek a min-cost flow of $n$ units from $s$ to $t$; the units of the optimal flow from the candidates to the voters give an optimal assignments.

To conclude, the complexity classification discussed here should not be misunderstood in implying that NP-hard ABC rules are impractical and should be avoided. There is a
wide range of algorithmic techniques available to solve NP-hard problems, and many disciplines in computer science encounter (and routinely solve) computationally hard problems. Instead the message here is the following: When using a polynomial-time computable rule, even very large instances can be expected to be solved quickly. For NP-hard rules, a more thorough analysis is necessary to determine how large instances can be solved. In Section 6.1, we discuss algorithmic techniques for computing NP-hard ABC rules.

4.8 Open Problems

We conclude this section by stating several interesting open problems.

1. We mentioned in Section 4.1 that ABC rules that require tiebreaking do not satisfy neutrality (e.g., sequential and reverse sequential Thiele methods, Greedy Monroe, seq-Phragmén, and Rule X). These rules can be made neutral with parallel universes tiebreaking \[48, 66, 100\]: a committee is winning under the neutral variant if and only if it is winning for some tiebreaking order under the original rule. This modification will have an algorithmic impact (trying all permutations of candidates would require exponential time), but the exact computational complexity of these neutral rules is not settled. Further, under which conditions can these rules be computed in polynomial time?

2. From Table 1, we can see that for some axioms and some ABC rules we still do not know whether a given rule satisfies the axiom or not. Filling these gaps would help us obtain a more complete picture of the differences between particular ABC rules.

3. In Section 4.7, we presented a coarse analysis of the computational complexity of ABC rules. This analysis could be refined by considering the Candidate Winner problem: given an election instance \((A,k)\) and a candidate \(c\), does there exist a winning committee \(W\) that contains \(c\). This problem has recently been shown to be \(\Theta^p_2\)-complete for Monroe and CC. A similar analysis for other computationally hard voting rules is missing.

5 Proportionality

A key difference among ABC rules is how they treat minorities of voters, i.e., small groups with preferences different from larger groups. Let us illustrate this issue through the following simple example.

Example 17. Consider the approval-based preference profile with 60 voters approving \(A = \{a_1, \ldots, a_{10}\}\), 20 voters approving \(B = \{b_1, \ldots, b_6\}\), 10 voters approving \(C = \{c_1, c_2\}\), 8 voter approving \(D = \{d_1, d_2, d_3, d_4\}\), and 2 voters who approve \(E = \{e_1, e_2, e_3\}\); assume our goal is to pick a committee of ten candidates. Given this instance AV returns committee \(A\), and in some cases this is a reasonable choice (e.g., when the goal of the election
Table 2: Proportionality of ABC rule. There are three rules which perform particularly well in terms of proportionality: PAV, Phragmén’s sequential rule, and Rule X. The mark † means that the result holds only when the number of voters \( n \) is divisible by the committee size \( k \). References of the form (A.x) refer to propositions in Appendix A.

is to select finalists of a contest). Yet, when the goal is to select a representative body that should reflect voters’ preferences in a proportional fashion, this committee violates very basic principles of fairness. Indeed, the voters who approve committee \( A \) constitute 60\% of the population, yet effectively they decide about the whole committee; at the same time the group of 20\% who approve \( B \) is ignored. A committee that consists of six candidates from \( A \), two candidates from \( B \), one candidate from \( C \), and one candidate from \( D \) is, for example, a much more proportional choice.

In Example 17, picking an outcome that is intuitively proportional is easy due to a very specific structure of voters’ approval sets—each two approval sets are either the same or disjoint. Finding a proportional committee in the general case, when any two approval sets can arbitrarily overlap, is by far less straightforward, and to some extent ambiguous. Several approaches that allow one to formally reason about proportionality have been proposed in the literature.

The goal of this section is to review these approaches and to identify ABC rules which can be considered proportional. Table 2 provides an overview of this analysis; the corresponding concepts are explained in this section.

<table>
<thead>
<tr>
<th>Rule</th>
<th>Proportionality degree</th>
<th>EJR</th>
<th>PJR</th>
<th>JR</th>
<th>Laminar prop.</th>
<th>Price-ability</th>
<th>Apportionment</th>
</tr>
</thead>
<tbody>
<tr>
<td>AV</td>
<td>0 [189]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>none</td>
</tr>
<tr>
<td>PAV</td>
<td>( \ell - 1 ) [15]</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>[14]</td>
<td></td>
<td>D’Hondt [50]</td>
</tr>
<tr>
<td>seq-PAV</td>
<td>( \approx 0.7\ell - 1 ) (for ( k \leq 200 )) [189]</td>
<td></td>
<td></td>
<td></td>
<td>[14]</td>
<td></td>
<td>D’Hondt [50]</td>
</tr>
<tr>
<td>rev-seq-PAV</td>
<td>?</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>D’Hondt [50]</td>
</tr>
<tr>
<td>CC</td>
<td>( \leq 1 ) (Ex. 22)</td>
<td>✓</td>
<td></td>
<td>✓</td>
<td>[14]</td>
<td></td>
<td>none</td>
</tr>
<tr>
<td>seq-CC</td>
<td>( \leq 1 ) (Ex. 22)</td>
<td></td>
<td></td>
<td>✓</td>
<td>[14]</td>
<td></td>
<td>none</td>
</tr>
<tr>
<td>seq-Phragmén</td>
<td>((\ell-1)/2) [189]</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>[49]</td>
<td>✓  [49]</td>
<td>D’Hondt [50]</td>
</tr>
<tr>
<td>Rule X</td>
<td>((\ell+1)/2) (A.9)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>[164]</td>
<td>✓  [164]</td>
<td>D’Hondt [164]</td>
</tr>
<tr>
<td>lexmin-Phragmén</td>
<td>1 [189]</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>[49]</td>
<td>✓  [49]</td>
<td>D’Hondt [50]</td>
</tr>
<tr>
<td>Monroe</td>
<td>( \leq 1 ) (Ex. 22)</td>
<td>✓</td>
<td></td>
<td>✓</td>
<td>[164]</td>
<td>✓  [164]</td>
<td>LRM† [50]</td>
</tr>
<tr>
<td>Greedy Monroe</td>
<td>( \leq 1 ) (Ex. 22)</td>
<td></td>
<td></td>
<td>✓</td>
<td>(A.7)</td>
<td></td>
<td>LRM† (A.5)</td>
</tr>
<tr>
<td>MAV</td>
<td>0 (A.9)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>none</td>
</tr>
<tr>
<td>SAV</td>
<td>0 (A.9)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>none</td>
</tr>
</tbody>
</table>
5.1 Apportionment

One approach to reasoning about proportionality of voting rules is to first identify a class of well-structured preference profiles where the concept of proportionality can be intuitively captured, and then to examine the behavior of voting rules on such well-structured profiles. We focus here on so-called party-list profiles, which are election instances of the form as we have seen in Example 17.

Definition 10 (Party-list profiles). We say that an approval profile \( A = (A(1), \ldots, A(n)) \) is a party-list profile if for each two voters \( i, j \in N \) we have that either \( A(i) = A(j) \) or that \( A(i) \cap A(j) = \emptyset \). We say that an election instance \((A, k)\) is a party-list instance if (i) \( A \) is a party-list profile, and (ii) for each voter \( i \in N \) we have that \( |A(i)| \geq k \).

Party-list profiles closely resemble political elections with political parties, hence the name of the domain. Indeed, if \( A \) is a party-list profile, then the sets of voters and candidates can be divided into \( p \) disjoint groups each, \( N = N_1 \cup \ldots \cup N_p \) and \( C \supseteq C_1 \cup \ldots \cup C_p \), so that all voters from group \( N_i, i \in [p] \), approve exactly the candidates from \( C_i \) (and no others). The candidates from \( C_i \) can be thought of as members of some (virtual) party, and the voters from \( N_i \) are those who cast their single vote on party \( C_i \) (i.e., assuming plurality ballots).

In such elections, where the voters do not vote for individual candidates but rather each individual casts a single vote for one political party, the problem of distributing seats to political parties is called the apportionment problem. The concept of proportionality in the apportionment setting has been extensively studied in the literature and is well understood—for a detailed overview we refer the reader to the comprehensive books by Balinski and Young [19] and by Pukelsheim [175].

We see from Definition 10 that the apportionment problem can be viewed as a strict subdomain of approval-based committee elections, and consequently ABC rules can be viewed as functions that extend apportionment methods to the more general setting of approval profiles. This connection was already known and referred to by Thiele [203] and Phragmén [167]. In a more systematic fashion, Brill et al. [50] showed such relations between various ABC rules and methods of apportionment. To properly explain this relation, let us first define three prominent apportionment methods, used in parliamentary elections all over the world.

In the following, we assume that there are \( p \) political parties, consisting of the candidate sets \( C_1, \ldots, C_p \). By \( n_i \) we denote the number of votes cast on party \( C_i \). Further, recall that \( k \) denotes the number of committee seats that we want to distribute among the parties.

**Apportionment Rule 1** (D’Hondt method\(^\text{16}\)). The D’Hondt method proceeds in \( k \) rounds, in each round allocating one seat to some party. Consider round \( r \), and let \( s_i(r) \)

---

\(^{16}\) Victor D’Hondt (1841–1901) was a Belgian professor of law and active proponent of proportional representation [73, 74]. The D’Hondt method is also known as Jefferson method. Thomas Jefferson (1743–1826) was president of the United States, and proposed this method to allocate seats in the House of Representatives to states. D’Hondt developed this method independently of Jefferson’s earlier and largely similar proposal, and D’Hondt’s proposal was specifically meant for proportional representation.
be the number of seats that are currently assigned to party \( C_i \); thus, \( \sum_{i \in [p]} s_i(r) = r - 1 \). The D’Hondt method assigns the \( r \)-th seat to the party \( C_i \) with the maximal ratio \( \frac{n_i}{s_i(r)+1} \) (using a tie-breaking order between parties if necessary).

**Apportionment Rule 2** (Sainte-Laguë\(^{17}\) method). The Sainte-Laguë method is defined analogously, but in the \( r \)-th round it allocates the \( r \)-th seat to the party \( C_i \) which maximizes the ratio \( \frac{n_i}{2s_i(r)+1} \).

**Apportionment Rule 3** (Largest remainder method, LRM\(^{18}\)). The largest remainder method first assigns to each party \( \left\lfloor \frac{k \cdot n_i}{n} \right\rfloor \) seats—this way at least \( k - p + 1 \) seats are assigned. Second, it assigns the remaining \( r < p \) seats to the \( r \) parties with the largest remainders \( k \cdot \frac{n_i}{n} - \left\lfloor \frac{k \cdot n_i}{n} \right\rfloor \), assigning each party at most one seat.

**Example 18.** Consider a party-list representation of the profile from Example 17. We have five parties, \( A, B, C, D, \) and \( E \), each getting, respectively, 60, 20, 10, 8, and 2 votes; the committee size is \( k = 10 \). The computation of the D’Hondt method can be represented through the following left table:

<table>
<thead>
<tr>
<th>( n_i )</th>
<th>( A )</th>
<th>( B )</th>
<th>( C )</th>
<th>( D )</th>
<th>( E )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n_i/2 )</td>
<td>30</td>
<td>10</td>
<td>5</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>( n_i/3 )</td>
<td>20</td>
<td>6( \frac{2}{3} )</td>
<td>3( \frac{1}{3} )</td>
<td>2( \frac{2}{3} )</td>
<td>2/3</td>
</tr>
<tr>
<td>( n_i/4 )</td>
<td>15</td>
<td>5</td>
<td>2( \frac{1}{2} )</td>
<td>2</td>
<td>1( \frac{1}{2} )</td>
</tr>
<tr>
<td>( n_i/5 )</td>
<td>12</td>
<td>4</td>
<td>2</td>
<td>3( \frac{1}{5} )</td>
<td>2( \frac{1}{5} )</td>
</tr>
<tr>
<td>( n_i/6 )</td>
<td>10</td>
<td>3( \frac{1}{3} )</td>
<td>1( \frac{2}{3} )</td>
<td>1( \frac{1}{3} )</td>
<td>1/3</td>
</tr>
<tr>
<td>( n_i/7 )</td>
<td>8( \frac{4}{7} )</td>
<td>2( \frac{6}{7} )</td>
<td>1( \frac{3}{7} )</td>
<td>1( \frac{1}{7} )</td>
<td>2/7</td>
</tr>
<tr>
<td>( n_i/8 )</td>
<td>7( \frac{1}{2} )</td>
<td>2( \frac{1}{2} )</td>
<td>1( \frac{1}{4} )</td>
<td>1</td>
<td>1/4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( n_i )</th>
<th>( A )</th>
<th>( B )</th>
<th>( C )</th>
<th>( D )</th>
<th>( E )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n_i/3 )</td>
<td>20</td>
<td>6( \frac{2}{3} )</td>
<td>3( \frac{1}{3} )</td>
<td>2( \frac{2}{3} )</td>
<td>2/3</td>
</tr>
<tr>
<td>( n_i/5 )</td>
<td>12</td>
<td>4</td>
<td>2</td>
<td>3( \frac{1}{5} )</td>
<td>2( \frac{1}{5} )</td>
</tr>
<tr>
<td>( n_i/7 )</td>
<td>8( \frac{4}{7} )</td>
<td>2( \frac{6}{7} )</td>
<td>1( \frac{3}{7} )</td>
<td>1( \frac{1}{7} )</td>
<td>2/7</td>
</tr>
<tr>
<td>( n_i/9 )</td>
<td>6( \frac{2}{3} )</td>
<td>2( \frac{2}{9} )</td>
<td>1( \frac{1}{9} )</td>
<td>1( \frac{8}{9} )</td>
<td>2/9</td>
</tr>
<tr>
<td>( n_i/11 )</td>
<td>5( \frac{5}{11} )</td>
<td>1( \frac{9}{11} )</td>
<td>1( \frac{10}{11} )</td>
<td>8( \frac{8}{11} )</td>
<td>2( \frac{2}{11} )</td>
</tr>
<tr>
<td>( n_i/13 )</td>
<td>4( \frac{8}{13} )</td>
<td>1( \frac{7}{13} )</td>
<td>1( \frac{10}{13} )</td>
<td>8( \frac{13}{13} )</td>
<td>2( \frac{13}{13} )</td>
</tr>
<tr>
<td>( n_i/15 )</td>
<td>4</td>
<td>1( \frac{1}{3} )</td>
<td>3/2</td>
<td>8( \frac{15}{15} )</td>
<td>2( \frac{15}{15} )</td>
</tr>
</tbody>
</table>

In the subsequent rounds the D’Hondt method allocates seats to parties \( A, A \) (by tie-breaking), \( B, A, A \) (by tie-breaking), \( B \) (by tie-breaking), and \( C \). For example, in the fourth round, when \( A \) is already allocated 3 seats and \( B \) is allocated none, the rule will give the next seat to \( B \) rather than to \( A \), because \( \frac{20}{0+1} > \frac{60}{3+1} \). Summarizing, 7 seats will be allocated to party \( A \), two seats to party \( B \), and one seat to party \( C \); the remaining parties in parliaments. The name “Jefferson method” is typically used in the U.S., while “D’Hondt method” is prevalent in Europe.

\(^{17}\)As it is the case with the D’Hondt/Jefferson method, this rule has been developed independently in Europe and in the U.S. and goes by different names: Sainte-Laguë is used in Europe (in particular in the context of proportional representation in parliaments) and Webster is the name used in the U.S. literature. Sainte-Laguë (1882–1950) was a French mathematician and proposed this method in 1910 [179]. Daniel Webster (1782–1852) was a U.S. statesman and proposed this method in 1832 [19].

\(^{18}\)The largest remainder method is also known as Hamilton method, as it was proposed in the U.S. by Alexander Hamilton (1755–1804). His proposal was abandoned in favor of Jefferson’s method [19].
will get no seats. In the diction of ABC rules, winning committees are exactly those that consist of 7 candidates from A, 2 candidates from B and one candidate from C.

The computation of the Sainte-Laguë method is represented through the above right table. It will allocate 6 seats to A, 2 seats to B, 1 seat to C, and 1 seat to D.

The largest remainder method first assigns to parties A, B, C, D, and E—respectively—6, 2, 1, 0, and 0 seats. Then, the remainders are considered:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_i$</td>
<td>60</td>
<td>20</td>
<td>10</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>$\left\lfloor k \cdot \frac{n_i}{n} \right\rfloor$</td>
<td>6</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>remainder</td>
<td>0</td>
<td>0</td>
<td>0.8</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>seats</td>
<td>6</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

There is one unassigned seat which will be given to the party with the largest remainder, namely to D. Thus, LRM will allocate 6 seats to A, 2 seats to B, 1 seat to C, and 1 seat to D.

The D’Hondt method, the Sainte-Laguë method, and the LRM exhibit particularly appealing properties. For example, the D’Hondt method satisfies lower quota, which means that a party $i$ which receives $n_i$ out of $n$ votes must be allocated at least $\left\lfloor k \cdot \frac{n_i}{n} \right\rfloor$ committee seats. The largest remainder method satisfies not only lower quota but also upper quota: a party $i$ with $n_i$ out of $n$ votes must not receive more than $\left\lceil k \cdot \frac{n_i}{n} \right\rceil$ seats. However, the largest remainder method fails an important axiom called population monotonicity, which states that an increase in support must not harm a party. For further details, we refer the interested reader to the aforementioned books on apportionment methods [19, 175].

We are now ready to formulate the main results of Brill et al. [50]:

**Theorem 4 ([50]).** PAV, sequential PAV, seq-Phragmén, and lexmin-Phragmén extend the D’Hondt method of apportionment. Phragmén’s variance-minimizing rule\(^{19}\) extends the Sainte-Laguë method of apportionment. If $n$ is divisible by $k$, then Monroe’s rule extends the largest remainders method.

The three methods of apportionment that are mentioned in Theorem 4 are all well-known, commonly used, and considered very proportional; they differ only in the way how the fractional seats are rounded. Thus, the corresponding ABC rules listed in Theorem 4 behave all well on party-list profiles; these rules can be considered good contenders for being proportional. In addition, we show in the appendix that also Greedy Monroe extends the largest remainder method when $n$ is divisible by $k$ (Proposition A.5), but both Monroe’s rule and Greedy Monroe do not if $n$ is not divisible by $k$ (Proposition A.6).

\(^{19}\)This rule is similar to lexmin-Phragmén but minimizes the variance of loads instead of the maximum load, see [49, 112] for a precise definition.
Lackner and Skowron [126], then extended the results of Brill et al. [50], providing strong arguments in favor of PAV:

**Theorem 5 ([126])**. PAV is the unique extension of the D’Hondt method of apportionment that satisfies neutrality, anonymity, consistency, and continuity.

### 5.2 Cohesive Groups

In party-list profiles (Definition 10), voters can be arranged in groups with identical preferences. Then, proportionality requires that a large-enough group of voters with identical preferences deserves a certain number of representatives in the elected committee (proportional to the size of the group). This approach can be generalized to groups with non-identical but similar preferences. In this section, we discuss axioms that relax the requirements for groups of voters to be eligible for representatives. These axioms are based on the concept of \( \ell \)-cohesiveness:

**Definition 11.** A group \( V \subseteq N \) is \( \ell \)-cohesive if: 
(i) \( |V| \geq \ell \cdot \frac{n}{k} \), and 
(ii) \( |\bigcap_{i \in V} A(i)| \geq \ell \).

An \( \ell \)-cohesive group consists of a \( \ell/k \)-th fraction of voters, thus, intuitively, such a group should be able to control at least \( \ell/k \cdot k = \ell \) committee seats. Further, an \( \ell \)-cohesive group agrees on \( \ell \) candidates, so one can ensure each member of the group gets \( \ell \) representatives by selecting only \( \ell \) candidates. It is, hence, tempting to require that for each \( \ell \)-cohesive group \( V \), each voter from \( V \) should be given at least \( \ell \) representatives in the elected committee. Unfortunately, this would be too strong—there exists no rule that would satisfy this property.

**Example 19.** Consider a profile \( A \) with four candidates \( (a, b, c, d) \) and 12 voters, with the following approval sets:

\[
\begin{align*}
A(1) & : \{a, d\} & A(4) & : \{a, b\} & A(7) & : \{b, c\} & A(10) & : \{c, d\} \\
A(2) & : \{a\} & A(5) & : \{b\} & A(8) & : \{c\} & A(11) & : \{d\} \\
A(3) & : \{a\} & A(6) & : \{b\} & A(9) & : \{c\} & A(12) & : \{d\}
\end{align*}
\]

Let \( k = 3 \). The group \( \{1, 2, 3, 4\} \) is \( 1 \)-cohesive, as it has a commonly approved candidate \( a \) and is of size \( \frac{12}{3} = 4 \). If we want to give each voter in this group a representative, candidate \( a \) has to be in the winning committee (voters 2 and 3 only approve \( a \)). Now observe that also the groups \( \{4, 5, 6, 7\}, \{7, 8, 9, 10\}, \) and \( \{10, 11, 12, 1\} \) are also \( 1 \)-cohesive. Thus, also candidates \( b, c, \) and \( d \) have to be in any winning committee. This is impossible as we are interested in committees of size 3. We see that it is impossible to satisfy every voter in \( 1 \)-cohesive groups.

This formulation can be weakened a bit without losing much of its intuitive appeal. The two weaker concepts that we start our discussion with are extended justified represen-
tation (EJR) \cite{14} and Proportionality Degree \cite{15, 182, 189, 194}.\footnote{Proportionality degree was initially referred to as \textit{average satisfaction of $\ell$-cohesive groups} \cite{15, 182}. Skowron et al. \cite{194} called an almost equivalent property $\kappa$-group representation.} The former concept is formulated as an axiom, the latter as a proportionality guarantee specified by a function.

**Definition 12** (Extended justified representation, EJR). An ABC rule $R$ satisfies extended justified representation (EJR) if for each election instance $E = (A, k)$, each winning committee $W \in R(E)$, and each $\ell$-cohesive group of voters $V$ there exists a voter $v \in V$ with at least $\ell$ representatives in $W$, i.e., $|A(i) \cap W| \geq \ell$.

**Example 20.** Let us revisit \textit{Example 19}. The committee $\{a, b, c\}$ satisfies the condition of EJR: every 1-cohesive group contains at least one voter with one representative in $\{a, b, c\}$. For example, for the 1-cohesive group $\{10, 11, 12, 1\}$, the voters 10 and 1 have a representative in the committee. Note that in this example actually all size-3 committees satisfy the EJR condition; also there are no $\ell$-cohesive groups for $\ell \geq 2$.  \hfill $\Box$

**Definition 13** (Proportionality Degree). Fix a function $f : \mathbb{N} \to \mathbb{R}$. An ABC rule $R$ has a proportionality degree of $f$ if for each election instance $E = (A, k)$, each winning committee $W \in R(E)$, and each $\ell$-cohesive group of voters $V$, the average number of representatives that voters from $V$ get in $W$ is at least $f(\ell)$, i.e.,

$$\frac{1}{|V|} \cdot \sum_{i \in V} |A(i) \cap W| \geq f(\ell).$$

At first, it might appear that even for large cohesive groups, EJR gives a guarantee only to a single voter within this group. However, the EJR property applies to any group of agents: Let $V$ be an $\ell$-cohesive group. If we remove the voter with $\ell$ representatives (which, by EJR, is guaranteed to exist), the resulting group will be at least $(\ell - 1)$-cohesive. Consequently, in such a group there must exist a voter with at least $\ell - 1$ representatives, etc. As a consequence of this argument, EJR implies a proportionality degree of at least $f_R(\ell) = \frac{\ell - 1}{2}$ \cite{182}.

**Example 19** also shows that there exists no rule with a proportionality degree of $f(\ell) = \ell$:

**Example 21.** Consider again the profile of \textit{Example 19}. Assume, there exists a rule $R$ with a proportionality degree of $f_R(\ell) = \ell$ and let $k = 3$. The group $\{1, 2, 3, 4\}$ is 1-cohesive, so in order to ensure that these voters get on average one representative, candidate a must be selected. By applying the same reasoning to $\{4, 5, 6, 7\}$ we infer that $b$ must be selected. Analogously, we conclude that $c$ and $d$ must be selected. However, there are only three seats in the committee, a contradiction.  \hfill $\Box$
Theorem 6 ([14, 15]). \( PAV \) has a proportionality degree of \( \ell - 1 \). It also satisfies EJR.

In contrast, the two sequential variants of PAV do not satisfy EJR: both seq-PAV and rev-seq-PAV fail even the much weaker justified representation axiom mentioned below [8, 14]. However, the proportionality guarantees of Theorem 6 also hold for a local-search variant of PAV [15], which—in contrast to PAV itself—runs in polynomial time. Thus, EJR and a proportionality degree of \( \ell - 1 \) are achievable in polynomial time. Aziz et al. [15] also construct a second polynomial-time computable (but rather involved) rule that satisfies EJR. More recently, Peters and Skowron [164] proved that Rule X satisfies EJR, which is also computable in polynomial time. Among the rules introduced in Section 3, PAV and Rule X are the only ones that satisfy EJR. An overview of the proportionality degree of rules can be found in Table 2.

Let us consider two other, strictly weaker properties than EJR, which have been considered in the literature.

**Definition 14** (Proportional justified representation, PJR [182]). An ABC rule \( \mathcal{R} \) satisfies proportional justified representation (PJR) if for each election \( E = (A, k) \), each winning committee \( W \in \mathcal{R}(E) \), and each \( \ell \)-cohesive group of voters \( V \) it holds that \( \bigcup_{i \in V} |A(i) \cap W| \geq \ell \).

**Definition 15** (Justified representation, JR [14]). An ABC rule \( \mathcal{R} \) satisfies justified representation (JR) if for each election \( E = (A, k) \), each \( W \in \mathcal{R}(E) \), and each 1-cohesive group of voters \( V \) there exists a voter \( v \in V \) who is represented by at least one member of \( W \).

PJR and JR are much weaker properties than EJR; in particular EJR implies PJR, which in turn implies JR. Example 22, below, illustrates that the stronger of the two axioms, PJR, can be satisfied even by rules that could be considered very bad from the perspective of proportionality degree (and, thus, also from the perspective of approximating EJR). On the other hand, there exist rules with good proportionality degree that do not satisfy even JR—this happens, e.g., when a rule does not provide sufficient guarantees for 1-cohesive groups (although it might satisfy EJR for \( \ell \geq 2 \)). Generally, justified representation should not be viewed as a proportionality axiom but rather a property capturing the idea of diversity, which we will discuss in Section 5.5. In contrast, PJR can be viewed as a moderate proportionality requirement, significantly weaker than EJR but stronger than, e.g., lower quota on party-list profile. We refer to Table 2 for an overview which rules satisfy JR and PJR.

**Example 22.** Fix \( k \) and consider the following instance:

<table>
<thead>
<tr>
<th></th>
<th>( c_1 )</th>
<th>( c_2 )</th>
<th>( c_3 )</th>
<th>( \cdots )</th>
<th>( c_k )</th>
<th>( c_{k+1} )</th>
<th>( \cdots )</th>
<th>( c_{2k} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_1 )</td>
<td>( V_2 )</td>
<td>( V_3 )</td>
<td>( \cdots )</td>
<td>( V_k )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

38
There are $2k$ candidates. The voters can be divided into $k$ equal-size groups so that the voters from the $i$-th group, in the diagram denoted as $V_i$, approve $c_i$ and $\{c_{k+1}, \ldots, c_{2k}\}$. Committee $\{c_1, \ldots, c_k\}$ (marked blue) satisfies PJR, but clearly, $\{c_{k+1}, \ldots, c_{2k}\}$ (marked green) is a much better choice from the perspective of proportionality degree. Also, $\{c_{k+1}, \ldots, c_{2k}\}$ satisfies the EJR condition while $\{c_1, \ldots, c_k\}$ does not. This example shows that PJR implies no better proportionality degree than $f(\ell) = 1$.

Given that there are rather few rules satisfying EJR, Bredereck et al. [46] performed computer simulations for several distributions of voters' preferences and checked how hard it is on average to find a committee that satisfies the condition imposed by EJR. They concluded that $\ell$-cohesive groups for $\ell \geq 2$ are quite rare, and that a random committee among those that satisfy the much weaker condition of JR is quite likely to satisfy the one of EJR as well. Their second conclusion was that JR, PJR, and EJR, while highly desired, do not guarantee on their own a sensible selection of committees, and one needs to put forward additional criteria. Otherwise, there are often many committees satisfying these conditions, and these committees may vary significantly.

To sum up, when considering proportionality axioms based on cohesive groups, PAV stands out as the most proportional rule. Rule X comes at a close second (its proportionality degree is lower) but is computable in polynomial time. If we desire a committee monotone rule, then seq-Phragmén is the best known choice: it has a proportionality degree of $f_{\text{Phrag}}(\ell) = \frac{\ell - 1}{2}$ [189], i.e., the proportionality degree that is implied by EJR, and satisfies PJR [49].

### 5.3 Laminar Proportionality and Priceability

The properties that we discussed in Section 5.2 (extended justified representation and the proportionality degree) and the axiomatic characterization given in Theorem 5 all indicate that PAV provides particularly strong proportionality guarantees. Specifically, one could interpret these results as suggesting that PAV is a better rule—in terms of proportionality—than Phragmén’s sequential rule and Rule X. However, drawing such a conclusion based on the so-far presented results would be too early—in this section we explain that proportionality can be understood in at least two different ways and that the axioms we discussed so far capture and formalize only one specific form of proportionality. We explain that Phragmén’s sequential rule and Rule X provide very strong proportionality guarantees, but with respect to an interpretation of proportionality that is not captured by the properties based on cohesive groups, and which is—to some extent—incomparable with the type of proportionality guaranteed by PAV.

Let us start by illustrating the difference in how PAV and Phragmén’s sequential rule (and Rule X) operate through the following example.

**Example 23 ([164]).** There are 15 candidates and 6 voters—the voters’ approval sets are depicted in the diagram below. The committee shaded in blue in the left-hand side picture is the one that is selected by the Phragmén’s sequential rule and by Rule X. The committee shaded in the right-hand side picture is chosen by PAV.
(a) Phragmén’s sequential rule and Rule X

(b) PAV

The approval sets of voters 1, 2, and 3 are disjoint from those of voters 4, 5, and 6. It seems intuitive that the first three voters, who together form half of the society, should be able to decide about half of the elected candidates. Phragmén’s sequential rule and Rule X select committees, where the first three voters approve in total half members, thus the behavior of these rules is consistent with the aforementioned understanding of proportionality. PAV follows a different principle: In the committee depicted in (a), each of the first three voters approves 4 candidates; each of the remaining three voters approves only 2 committee members. PAV notices that this is the case, and tries to reduce the societal inequality of voters’ satisfaction by removing one representative of voter 1 and adding one to 4; similarly, PAV considers that it is more fair to remove the representatives of 2 and 3, and add the candidates liked by 5 and 6. On the one hand, PAV prefers to pick a committee that minimizes the societal inequality in the voters’ satisfactions (measured as the number of approved committee members). On the other hand, it punishes voters 1, 2, and 3 for being agreeable and “easy to satisfy” with fewer committee members—PAV allows them to decide only about one quarter of the committee.

Example 23 illustrates that PAV and Phragmén’s sequential rule (and Rule X) follow two different types of proportionality. PAV implements a welfarist type of proportionality which is primarily concerned with the welfare (satisfaction) of the voters. This type of proportionality is captured, e.g., by the properties discussed in Section 5.2. PAV also satisfies the Pigou–Dalton principle of transfers, which says that given an election \((A, k)\) and two committees, \(W\) and \(W'\), which in total get the same numbers of approvals \((\text{sc}_{AV}(A,W) = \text{sc}_{AV}(A,W'))\), the one which minimizes the societal inequality should be preferred. Phragmén’s sequential rule and Rule X, on the other hand, implement proportionality with respect to power, which—informally speaking—says that a group consisting of an \(\alpha\) fraction of voters should be given a voting power that enables to decide about an \(\alpha\) fraction of the committee. In other words, the type of proportionality of Phragmén-like rules is not mainly concerned with the welfare of groups but also with the justification of welfare, i.e., voting power.

Peters and Skowron [164] discuss two properties—laminar proportionality and priceability—which aim at formally capturing the high-level idea of proportionality with respect to power. The first of the two properties—laminar proportionality—is very similar in spirit to proportionality on party-list profiles. The corresponding axiom identifies a class of well-structured election instances—called laminar elections—and specifies how
a laminar proportional rule should behave on these profiles. Laminar profiles are more general than party-list profiles and are defined by a recursive structure, similar to the election from Example 23.

The second property, which we will discuss in more detail, is priceability. Intuitively, we say that a committee $W$ is priceable if we can endow each voter a fixed budget and if for each voter there exists a payment function such that: (1) each voter pays only for the elected candidates she approves of, (2) each candidate gets a total payment of 1, and (3) there is no group of voters who approve a non-elected candidate, and who in total have more than one unit of unspent budget.

**Definition 16 (Priceability).** Given an election instance $(A, k)$, a committee $W$ is priceable if there exists a voter’s budget $p \in \mathbb{R}^+$ and $p_i : C \rightarrow [0, 1]$ for each voter $i \in N$ such that:

1. $\sum_{c \in C} p_i(c) \leq p$ for each $i \in N$,
2. $p_i(c) = 0$ for each $i \in N$ and $c \not\in A(i)$,
3. $\sum_{i \in N} p_i(c) = 1$ for each $c \in W$,
4. $\sum_{i \in N} p_i(c) = 0$ for each $c \not\in W$, and
5. $\sum_{i \in N(c)} (p - \sum_{c' \in W} p_i(c')) \leq 1$ for each $c \not\in W$.

An ABC rule is priceable if it returns only priceable committees.

**Example 24.** Consider the election instance from Example 23. The committees returned by Phragmén’s sequential rule and by Rule X are priceable. For example, consider $W_1 = \{c_1, \ldots, c_6, c_7, c_8, c_{10}, c_{11}, c_{13}, c_{14}\}$ (the committee shaded blue in the left figure in Example 23). This committee is priceable as witnessed by the following price system: the voters’ budget is $p = 2$, and the payment functions are as follows (we only specify the non-zero payments): $p_1(c_i) = p_2(c_i) = p_3(c_i) = 1/3$ for $i \in \{1, 2, 3\}$ and $p_1(c_4) = p_2(c_5) = p_3(c_6) = p_4(c_7) = p_4(c_8) = p_5(c_{10}) = p_5(c_{11}) = p_6(c_{13}) = p_6(c_{14}) = 1$. Each voters fully spends their budget of 2.

On the other hand, the committee $W_2 = \{c_1, c_2, c_3, c_7, \ldots, c_{15}\}$ returned by PAV (the one shaded blue in the right figure in Example 23) is not priceable. Indeed, if the voters’ budget $p$ were $\leq 2$, then the voters 4, 5, 6 could not afford to pay for 9 candidates $c_7, \ldots, c_{15}$. If $p > 2$, then some of the voters 1, 2, 3, say voter 1, would have a remaining budget of more than 1. Hence, this voter would have more budget than needed to buy a candidate outside of $W_2$ (e.g., $c_4$), which contradicts condition (5) in Definition 16.

Peters and Skowron [164] generalized Example 24 and showed that no welfarist rule (see Definition 1) is priceable, hence that indeed priceability is inherently not a welfarist concept. The same is true for laminar proportionality.
Theorem 7 ([164]). Phragmén’s sequential rule and Rule X are laminar proportional and priceable. No welfarist rule is laminar proportional nor priceable. No rule satisfying the Pigou–Dalton principle of transfers is laminar proportional nor priceable.

While priceability is not a welfarist concept, it implies proportional justified representation. Further, all priceable rules must be equivalent to the D’Hondt method of apportionment on party-list profiles (cf. Theorem 4).

We mention one more property—perfect representation [182]—which is loosely related to priceability. It also requires an explanation how voters can distribute their support/power in a way that justifies electing a committee; however, the axiom applies only in very specific situations.

Definition 17 (Perfect representation [182]). We say that a committee $W$ satisfies perfect representation if the set of voters can be divided into $k$ equal-sized disjoint groups $N = N_1 \cup \ldots \cup N_k$ ($|N_i| = n/k$ for each $i \in k$) and if we can assign a distinct candidate from $W$ to each of these groups in a way that for each $i \in k$ the voters from $N_i$ all approve their assigned candidate. An ABC rule $R$ satisfies perfect representation if $R$ returns only committees satisfying perfect representation whenever such committees exist.

Perfect representation is incompatible with EJR [182] and with weak (and strong) Pareto efficiency (Proposition A.8), and it is not implied by (nor implies) priceability. Among the rules considered in this paper, only Monroe [182] and lexmin-Phragmén [49] satisfies perfect representation, as does the variance-based rule by Phragmén mentioned in Theorem 4 [49].

To sum up, if we are mainly interested in the welfarist interpretation of proportionality, as captured by axioms that specify how cohesive groups of voters should be treated, then PAV is the best among the considered rules. Yet, sequential PAV, seq-Phragmén, and Rule X perform also reasonably well with respect to these criteria, and they are computable in polynomial time. Sequential PAV, however, can be considered slightly worse than these other rules, since it does not satisfy JR, and so it might discriminate small cohesive groups of voters. If we are interested in proportionality with respect to power, then Rule X and Phragmén’s sequential rule are the two superior rules. It is not entirely clear which one of the two rules is better. On the one hand, Rule X satisfies the appealing axiom of EJR; on the other hand, Phragmén’s sequential rule is committee monotone (see Section 4.3). In Table 2, we highlighted the three rules that—with the current state of knowledge—we consider the best ABC rules in terms of proportionality; we use two colors to distinguish the two types of proportionality.

5.4 The Core

An important concept of group fairness that has been extensively studied in the context of ABC rules is the core. This notion is adopted from cooperative game theory.\footnote{The definition used in the literature on multiwinner voting is based on the definition of the core for games with non-transferable payoffs [158].}
Definition 18 (Core [14]). Given an instance \((A, k)\) we say that a committee \(W\) is in the core if for each \(V \subseteq N\) and each \(T \subseteq C\) with
\[
\frac{|T|}{k} \leq \frac{|V|}{n}
\]there exists a voter \(i \in V\) such that \(|A(i) \cap T| \leq |A(i) \cap W|\). We say that an ABC rule \(\mathcal{R}\) satisfies the core property if for each instance \((A, k)\) each winning committee \(W \in \mathcal{R}(A, k)\) is in the core.

Informally speaking, the core property requires that a group \(V\) constituting an \(\alpha\) fraction of voters should be able to control an \(\alpha\) fraction of the committee. If such a group can propose a set \(T\) of \(\lfloor \alpha k \rfloor\) candidates such that each voter in \(V\) is more satisfied with the proposed set \(T\) than with the winning committee \(W\), then the group \(V\) would have an incentive to deviate, hence would witness that committee \(W\) is not stable (and, in some sense, also not fair). If a winning committee is in the core, then no such a deviation is possible.

The core property implies extended justified representation (Definition 12): Assume an ABC rule \(\mathcal{R}\) satisfies the core property and consider an instance \((A, k)\), a winning committee \(W\), and an \(\ell\)-cohesive group of voters \(V\). Let \(T\) be the set of \(\ell\) candidates that are approved by all the voters in \(V\) (such candidates exist because \(V\) is \(\ell\)-cohesive). Since \(W\) is in the core, there must exists a voter \(i \in V\) such that \(|A(i) \cap W| \geq |A(i) \cap T| = \ell\), hence the condition of EJR must be satisfied. While the notion of core strictly generalizes EJR and thus implies strong satisfaction guarantees for cohesive groups, it can be also viewed as a concept formalizing the idea of proportionality with respect to power (cf. Section 5.3).

It is an important open question whether there exists an ABC rule that satisfies the core property, or—equivalently—whether the core is always non-empty. For the time being only partial answers to this intriguing question are known:

1. None of the rules mentioned in Section 3 satisfies the property. Since a rule satisfying the core must satisfy EJR, only PAV and Rule X come into consideration. However, counterexamples for both are known [14, 164]. For PAV, the instance from Example 23 shows a violation of the core.

2. No welfarist rule (Definition 1) can satisfy the core property [164].

3. If one restricts the attention to a special subclass of approval profiles, so-called approval-based party-list profiles [51], the situation changes. Approval-based party-list profiles are approval profiles where each candidate appears with at least \(k\) copies, i.e., for every candidate \(c\) it holds that \(|\{c' \in C: N(c) = N(c')\}| \geq k\). Brill et al. [51] proved that PAV satisfies the core property on approval-based party-list profiles; as mentioned before, PAV does not satisfy the core property in the general case.

As it remains unclear whether an ABC rule satisfying the core property is an achievable goal, several works in the most recent literature analyzed relaxed notions of the core. We review these notions in the subsequent sections.
5.4.1 Relaxation by Randomization

The first type of relaxation that we consider is a probabilistic variant of the notion, i.e., the question becomes: “can core-like properties be guaranteed in expectation (ex-ante)?”. Cheng et al. [63] proved that there always exists a lottery over committees that satisfies the core property in expectation. Let $\mathbb{E}_{X \sim \Delta}(X)$ denote the expected value of a random variable $X$ distributed according to a lottery (probability distribution) $\Delta$.

**Theorem 8 ([63]).** For each election instance $(A, k)$ there exists a lottery over committees $\Delta$ such that for each group of voters $V \subseteq N$ and each $T \subseteq C$ it holds that

$$\frac{|T|}{k} > \mathbb{E}_{W \sim \Delta}(N(T, W)) \frac{n}{|V|},$$

where $N(T, W)$ is the set of voters who prefer $T$ over $W$:

$$N(T, W) = \{i \in N: |A(i) \cap T| > |A(i) \cap W|\}.$$

Note that Equation (2) is indeed a negated, probabilistic version of Equation (1), showing that in expectation there are too few voters to propose a different committee. While it is not known whether such a lottery $\Delta$ can be found in a polynomial time, Cheng et al. [63] prove that if we restrict our attention only to sets $T$ of size bounded by a constant, then for each $\epsilon > 0$ there is a polynomial-time algorithm that computes $\Delta$ such that $(1 + \epsilon) \cdot \frac{|T|}{k} > \mathbb{E}_{W \sim \Delta}(N(T, W)) \frac{n}{|V|}$.

5.4.2 Relaxation by Deterministic Approximation

Another approach is to ask whether the core property can be well approximated. A few notions of approximation have been proposed; **Definition 19** below unifies the definitions considered in the literature.

**Definition 19.** We say that an ABC rule $R$ provides a $\gamma$-multiplicative-$\eta$-additive-satisfaction $\beta$-group-size approximation to the core if for each instance $(A, k)$, each winning committee $W \in R(A, k)$, each subset of voters $V \subseteq N$, and each subset of candidates $T \subseteq C$ with

$$\beta \cdot \frac{|T|}{k} \leq \frac{|V|}{n},$$

there exists a voter $i \in V$ such that $|A(i) \cap T| \leq \gamma \cdot |A(i) \cap W| + \eta$.

There are two components in **Definition 19**: The satisfaction-approximation component says that a voter $i \in V$ has an incentive to deviate towards $T$ only if her gain in satisfaction is sufficiently large, that is, if $i$‘s satisfaction in $T$ is greater at least by a multiplicative factor of $\gamma$ and an additive factor of $\eta$ than her satisfaction in $W$. The group-size-approximation component prohibits deviations towards sets $T$ which are (by a multiplicative factor of $\beta$) smaller than $k \cdot \frac{|V|}{n}$, as imposed by the core. If $\gamma = 1$, then
we omit the term “\(\gamma\)-multiplicative” from the name of the property. Similarly, if \(\eta = 0\) we omit the term “\(\eta\)-additive”, and if \(\beta = 1\), then we omit the term “\(\beta\)-group-size”. The satisfaction-approximation and the group-size approximation are incomparable.

When considering the problem of approximating the core, we can distinguish two classes of algorithms that have been considered in the literature. The first class contains dedicated algorithms, which are mostly based on dependent rounding of fractional committees. The second class consists of well-known rules, such as PAV or Rule X.

Jiang et al. [113] present an algorithm that provides 32-group-size approximation to the core. Their approach is based on dependent rounding of lotteries that are in expectation in the core (the existence of such lotteries is guaranteed by Theorem 8). Notably, the approach of Jiang et al. [113] extends much beyond the approval-based preferences.

Fain et al. [90] present a family of algorithms based on dependent rounding of fractional committees (returned by a linear program that closely resembles the formulation of PAV as an integer linear program). For each \(\lambda \in (1, 2]\) they provide an algorithm that guarantees \(\lambda\)-multiplicative-\(\eta\)-additive-satisfaction \(\frac{1}{2\eps}\)-group-size approximation to the core, where \(\eta = O\left(\frac{1}{\sqrt{\lambda}} \log\left(\frac{k}{\lambda}\right)\right)\). Their algorithm naturally extends to a more general model related to participatory budgeting.

For commonly known rules the following results are known: Cheng et al. [63] prove that PAV does not guarantee \(\beta\)-group-size approximation to the core even for \(\beta = \Theta(\sqrt{k})\). On the other hand, Peters and Skowron [164] prove that PAV gives 2-multiplicative-satisfaction approximation to the core. Further, for each \(\epsilon > 0\) no rule that satisfies the Pigou–Dalton principle can provide a \((2 - \epsilon)\)-multiplicative-satisfaction approximation to the core. Thus, PAV can be viewed as giving the strongest multiplicative-satisfaction approximation to the core subject to satisfying the Pigou–Dalton principle of transfers. Finally, they show that Rule X provides \(O(\log(k))\)-multiplicative-1-additive-satisfaction approximation to the core.

### 5.4.3 Relaxation by Constraining the Space of Deviations

Yet another approach to relaxing the core property is to prohibit only certain types of deviations. As we have already explained at the beginning of this section, EJR can be viewed as a restricted variant of the core property: It prohibits the deviations of groups of voters towards outcomes \(T\) on which the deviating voters unanimously agree. Intuitively, if a group \(V\) agrees on all candidates from \(T\), then it is easier for such a group to synchronize and to deviate, thus EJR can be viewed as the minimal restricted variant of the core. Motivated by the same arguments Peters and Skowron [164] considered other restricted variants of the core property.

A **committee property** is a set of triples \((A, k, W)\), where \((A, k)\) is an election instance and \(W\) is a size-\(k\) committee. We write \(A|_{V}\) for profile \(A\) restricted to voters in \(V \subseteq N\).

**Definition 20** ([164]). *Let \(\mathcal{P}\) be a committee property. Given an instance \((A, k)\) we say that a pair \((V, T)\), with \(V \subseteq N, T \subseteq C\), is an allowed deviation from a committee \(W\) if:

1. \(\frac{|V|}{k} \leq \frac{|V|}{n}\),
2. \(|A(i) \cap T| > |A(i) \cap W|\) for each \(i \in V\), and
3. \(T\) has property \(\mathcal{P}\), i.e.,

\(\)
An ABC rule $\mathcal{R}$ satisfies the core subject to $\mathcal{P}$ if for each instance $(A, k)$ and each winning committee $W \in \mathcal{R}(A, k)$ there exists no allowed deviation.

For example, let $\mathcal{P}_{\text{coh}}$ be a committee property such that $(A, k, W) \in \mathcal{P}_{\text{coh}}$ if and only if $A(i) = W$ for all voters $i$ in the domain of $A$; we call $\mathcal{P}_{\text{coh}}$ cohesiveness (cf. Definition 11). Then, EJR can be equivalently defined as the core subject to cohesiveness.

Rule X satisfies core subject to priceability with equal payments, which is a stronger variant of the priceability (Definition 16), yet weaker than cohesiveness [164]. It is an open question whether the core subject to weaker (yet still natural) types of constraints is always non-empty.

5.5 Degressive and Regressive Proportionality

The notions of proportionality that we discussed in Sections 5.1 to 5.4 aimed at capturing the following intuitive idea: An $\alpha$ fraction of voters should be able to decide about an $\alpha$ fraction of the committee—in this approach the relation between the size of the group and its eligibility is linear. In this section we discuss two alternative concepts: degressive and regressive proportionality. These two concepts should be viewed more as high-level ideas than formal properties. We first explain them intuitively, providing an illustrative example, and next we will discuss a few formal approaches to reasoning about degressive and regressive proportionality.

According to degressive proportionality smaller groups of voters should be favored, i.e., be eligible to more representatives in the elected committee than in the case of linear proportionality. An extreme form of degressive proportionality is diversity [93]—there, if possible, each voter should be represented by at least one candidate in the elected committee. At the other end is the idea of regressive proportionality, where the emphasis is put on well-representing large groups. An extreme form of regressive proportionality is individual excellence [93], where it is assumed that only the candidates with the highest total support from the voters should be elected. In fact, these two notions—the diversity and the individual excellence—are extreme to the extent that they can no longer be considered notions of proportionality. Example 25, below, illustrates the ideas of degressive and regressive proportionality, and the two extreme variants of them—diversity and individual excellence.

Example 25. Consider the approval-based preference profile from Example 17:

| 60 voters: | $\{a_1, \ldots, a_{10}\}$ | 20 voters: | $\{b_1, \ldots, b_{6}\}$ | 10 voters: | $\{c_1, c_2\}$ |
| 8 voters: | $\{d_1, \ldots, a_{4}\}$ | 2 voters: | $\{e_1, e_2, e_3\}$. |

Degressive proportional apportionments are often used for distributing parliamentary seats among geographical regions, e.g., in the division of the European Parliament seats among EU countries (see [178] for the discussion on arguments and negotiations that resulted in the degressive apportionment rule used for assembling the European Parliament).
<table>
<thead>
<tr>
<th># votes</th>
<th>60</th>
<th>20</th>
<th>10</th>
<th>8</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>linear proportionality (Sainte-Lagué)</td>
<td>6</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>linear proportionality (D'Hondt)</td>
<td>7</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>degressive proportionality (example 1)</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>degressive proportionality (example 2)</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>diversity (example 1)</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>diversity (example 2)</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>regressive proportionality (example)</td>
<td>8</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>individual excellence</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3: Flavors of (dis)proportionality

A linearly-proportional committee $W_1$ could consist of six candidates from $A$, two candidates from $B$, one candidate from $C$, and one candidate from $D$ (this is the committee selected by the Sainte-Lagué apportionment method). Another linearly-proportional committee could consist of seven candidates candidates from $A$ but none from $D$ (this is the committee selected by the D’Hondt apportionment method).

In contrast, a degressive-proportional committee $W_2$ could, for example, consist of four candidates from $A$, three candidates from $B$, two candidates from $C$, and one candidate from $D$. Another example of a degressive-proportional committee would be $W_3$ with three candidates from each of the sets $A$, $B$, and $C$, and one from $D$. Committees $W_2$ and $W_3$, however, are not diverse, since two voters who support $E = \{e_1, e_2, e_3\}$ are not represented at all. A diverse committee could consist, e.g., four candidates from $A$, three candidates from $B$, one candidate from $C$, one candidate from $D$, and one candidate from $E$. A regressive-proportional committee would include more candidates from the set $A = \{a_1, \ldots, a_{10}\}$ at the cost of groups supported by less voters. For example, a committee that consists of eight candidates from $A$ and two candidates from $B$ would be regressive-proportional. Table 3 shows the example relations between a size of a group and its number of representatives for different forms of proportionality:

The arguments in favor of degressive proportionality usually come from the analysis of probabilistic models describing how the decisions made by the elected committee map to the satisfaction of individual voters participating in the process of electing the committee (for party-list preferences an excellent exposition is given by Koriyama et al. [122]; see also [133, 145]). An interesting concrete example of degressive proportionality is square-root proportionality devised by Penrose [159] (see also [195]), where the idea is that the groups of voters should be represented proportionally to the square-roots of their sizes.\(^{23}\) Further, degressive proportionally in general, and diversity in particular, are

\(^{23}\)This method has been proposed for a United Nations Parliamentary Assembly [52] and for allocating
particularly appealing ideas in the context of deliberative democracy—there, the goal is to select a committee that should discuss and deliberate on various issues rather than make majoritarian decisions. It is argued that for deliberative democracy it is particularly important to represent as many various opinions in the committees as possible [61, 152], which can be achieved by maximizing the number of voters who are represented in the elected committee.

On the other hand, the idea of regressive proportionality is particularly appealing when the goal is to select a committee of candidates based on their individual merits, for example when the goal of an election is to select finalists in a contest or to choose a set of grants that should be funded (then, the voters act as judges/experts).

In the remaining part of this section we discuss two approaches to formalizing the ideas of degressive and regressive proportionality: axiomatic approaches and a quantitative approach.

5.5.1 Axiomatic Approaches to Diversity and Individual Excellence

The axiomatic approach generally applies only to the extreme forms of the degressive and regressive proportionality, i.e., to diversity and individual excellence, respectively. This approach is similar to the one we discussed in Section 5.1: by formalizing the concepts of diversity and individual excellence on party-list profiles (Definition 10), we obtain axiomatic characterizations for the more general domain of ABC rules.

Intuitively, disjoint diversity requires that in party-list profiles as many voters as possible have at least one representative in the elected committee. Disjoint equality says that each approval carries the same strength, and so all candidates that are approved once have the same right for being elected.

**Definition 21** (Disjoint diversity). An ABC rule $\mathcal{R}$ satisfies disjoint diversity if for each party-list instance $(A, k)$ with voter sets $(N_1, \ldots, N_p)$ and $|N_1| \geq |N_2| \geq \ldots \geq |N_p|$, there exists a winning committee $W \in \mathcal{R}(A, k)$ that contains one candidate for each of the $k$ largest parties, i.e., for each $r \leq \min(p, k)$ and each $i \in N_r$ we have that $A(i) \cap W \neq \emptyset$.

**Definition 22** (Disjoint equality). An ABC rule $\mathcal{R}$ satisfies disjoint equality if for each election instance $(A, k)$ where each candidate is approved at most once and the number of approved candidates is at least $k$ (i.e., $|\bigcup_{i \in N} A(i)| \geq k$), a committee $W$ is winning if and only if it contains only approved candidates, $W \subseteq \bigcup_{i \in N} A(i)$.

The following theorems show that, similarly as in case of linear proportionality, the concepts of disjoint diversity and disjoint equality uniquely extend to the full domain of approval-based preferences if one assumes the natural axioms of symmetry and consistency (and a few more technical axioms).

voting weights in the Council of the European Union [27].
The Approval Chamberlin–Courant rule is the only non-trivial ABC rule that satisfies symmetry, consistency, weak efficiency, continuity, and disjoint diversity. Multi-Winner Approval Voting is the only ABC ranking rule that satisfies symmetry, consistency, weak efficiency, continuity, and disjoint equality.

Lackner and Skowron [126] provided a similar analysis for intermediate notions of degressive and regressive proportionality. They conclude that \( w \)-Thiele methods based on \( w \)-scoring functions that have a larger slope than the \( w \)-function of PAV, \( w_{PAV}(x) = \sum_{i=1}^{x} \frac{1}{i} \), are more oriented towards regressive proportionality, whereas \( w \)-functions that have a smaller slope, are closer in spirit to the idea of degressive proportionality. This relation is symbolically visualized in Figure 7.

Finally, Subiza and Peris [199] propose an axiom called \( \alpha \)-unanimity (parameterized with \( \alpha \in [0,1] \)), which can be seen as a strong diversity axiom. The authors propose a voting rule (Lexiunanimous Approval Voting) that satisfies this axiom; this rule is a refined version of CC. Thiele methods (including CC itself) do not satisfy this axiom for arbitrary \( \alpha \).

5.5.2 Quantifying Degressive and Regressive Proportionality

The second approach to formally reason about degressive and regressive proportionality is quantitative in nature. Lackner and Skowron [128] define two measures—the utilitarian guarantee and the representation guarantee—that can be used to quantify how well a given rule performs in terms of individual excellence and diversity.

Recall that \( sc_{AV}(A,W) \) denotes the total number of approvals a given committee receives in profile \( A \); \( sc_{CC}(A,W) \) denotes the number of voters who approve at least one
member of $W$.

**Definition 23** (Utilitarian and Representation Guarantee [128]). The utilitarian guarantee of an ABC rule $R$ is a function $\kappa_{AV}: \mathbb{N} \rightarrow [0, 1]$ that takes as input an integer $k$, representing the committee size, and is defined as:

$$\kappa_{AV}(k) = \inf_{A \in A(C)} \min_{W \in R(A,k)} \frac{\text{sc}_{AV}(A,W)}{\max_{W: |W| = k} \text{sc}_{AV}(A,W)}.$$ 

The representation guarantee of an ABC rule $R$ is a function $\kappa_{CC}: \mathbb{N} \rightarrow [0, 1]$ defined as:

$$\kappa_{CC}(k) = \inf_{A \in A(C)} \min_{W \in R(A,k)} \frac{\text{sc}_{CC}(A,W)}{\max_{W: |W| = k} \text{sc}_{CC}(A,W)}.$$ 

Note that the utilitarian and the representation guarantee of an ABC rule $R$ measure how well rule $R$ approximates Multi-Winner Approval Voting and the Approval Chamberlin–Courant rule, respectively. These two rules embody the principles of diversity and individual excellence (cf. Theorem 9).

Lackner and Skowron [128] show that the utilitarian guarantee of PAV, sequential PAV, and seq-Phragmén is $\Theta(1/\sqrt{k})$; their representation guarantee $1/2 + \Theta(1/k)$. CC and seq-CC achieve a better representation guarantee (of 1 and $1 - 1/e$, respectively), but their utilitarian guarantee is only $\Theta(1/k)$. In that sense, these three proportional rules (PAV, sequential PAV, and seq-Phragmén) can be viewed as a desirable compromise between the two guarantees. On the other, the authors also show that proportional rules are never an optimal compromise. Finally, $p$-geometric rules—the Thiele rules defined by $w_{p,\text{geom}}(x) = \sum_{i=1}^{x} (1/p)^i$—for different values of the parameter $p$ span the whole spectrum from AV to CC. By adjusting the parameter $p$, one can obtain any desired compromise between the utilitarian and representation goals.

### 5.6 Proportionality and Strategic Voting

The ABC rules that we have considered in the context of proportionality are all prone to manipulations (cf. Section 4.6). In this section we explain that this is not a coincidence—achieving proportionality and strategy-proofness at the same time is inherently impossible. This impossibility was first proved by Peters [161, 162] for resolute rules (rules that always return a single winning committee), even for very weak formulations of the desired axioms. (Earlier work by Aziz et al. [13] and Janson [112] already showed that certain proportional rules—such as PAV, seq-PAV, and seq-Phragmén—are not strategy-proof.)

**Theorem 10** ([161, 162]). Suppose $k \geq 3$, the number $n$ of voters is divisible by $k$, and $m \geq k + 1$. Then there exists no resolute ABC rule $R$ which satisfies the following three axioms:

1. weak proportionality: for each party-list election $(A,k)$ where some singleton ballot $\{c\}$ appears at least $n/k$ times (i.e., $|\{i: A(i) = \{c\}\}| \geq n/k$), candidate $c$ must belong to the winning committee: $c \in R(A,k)$
2. weak efficiency: a candidate who is approved by no voter may not be part of the winning committee, unless fewer than \(k\) candidates receive at least one approval.

3. inclusion-strategyproofness (as defined in Section 4.6) \(^{24}\)

Kluiving et al. [117] prove a similar result for irresolute rules (i.e., when rules are allowed to output multiple tied winning committees), using cardinality-strategyproofness and Pareto efficiency. Further, Duddy [77] proves a related impossibility result for irresolute rules using slightly different axioms; this result also requires a form of Pareto efficiency.

Lackner and Skowron [127] showed that AV is the only ABC scoring rule (Section 4.5) that satisfies SD-strategyproofness; this result can also be seen as an impossibility result concerning proportionality and strategyproofness within the class of ABC scoring rules. Further, they quantified the tradeoff between strategyproofness and proportionality. For various ABC rules they empirically measured their level of strategyproofness by assessing the fraction of profiles, for which there exists a voter who has an incentive to misreport her approval set. They concluded that rules which are more similar to AV (i.e., rules that follow the principle of progressive proportionality) are less manipulable than proportional rules. The rules that follow the principle of degressive proportionality are the most manipulable. A similar conclusion was obtained by Barrot et al. [21], but there the authors analyzed a different class of rules—namely those based on the Hamming distance, and spanning the spectrum from AV to Minimax Approval Voting.

### 5.7 Proportionality with Respect to External Attributes

In Sections 5.1 to 5.6, we have considered formal concepts that capture, in various ways, what it means that the structure of the elected committee proportionally reflects the (approval-based) preferences of the voters. In other words, we have considered proportionality with respect to the preferences given by the voters. In this section, we briefly consider a framework that approaches the concept of proportionality quite differently: to analyze proportionality with respect to external attributes of the candidates.

Let us start by recalling the apportionment setting that we discussed in Section 5.1. In the apportionment model we are given a set of candidates, each candidate belonging to a single political party; for each political party we are given a desired fraction of seats the party should ideally get in the elected committee (typically, this is the fraction of votes cast on the party). The goal is to pick the committee that matches the desired fractions as closely as possible. Thus, one can say that in the apportionment setting there is one external attribute, which is the party affiliation, each candidate has a certain value of this attribute, and the goal is to pick the committee where the different values of the attribute are represented proportionally to the given desired fractions.

\(^{24}\)This axiom can be further weakened to allow voters only to manipulate by reporting subsets of their true approval sets.
Now, assume that there are two attributes—each candidate has a political affiliation and a geographic region that she represents. For each value of each attribute we are given a desired fraction of seats that the candidates with this attribute value should get. This setting is called bi-apportionment, and it is discussed in detail in a book chapter by Pukelsheim [174] (several more recent articles study the bi-apportionment setting from a computational perspective [132, 176, 186]). The model of bi-apportionment has been further extended to an arbitrary number of attributes by Lang and Skowron [130]. There, the authors analyzed axiomatically and algorithmically two rules that extend the D’Hondt method and the largest remainder method to the multi-attribute apportionment.

The desired fractions in the (multi-attribute) apportionment model can be based on the voters preferences, or they might be given exogenously (e.g., by imposing certain quotas). In the remaining part of this section we will consider a model which takes into account both the voters’ preferences and external constraints based on attributes of the candidates. Instead of defining this model formally, we provide an illustrative example.

Example 26. Assume we want to select a representative committee. Such a committee should be gender-balanced, containing 50% of male (M) and 50% of female (F) committee members. Additionally, the committee should represent people from different educational backgrounds: at least 25% and at most 50% of its members should have the bachelor degree (H), between 40% and 60% should have an upper-secondary education (S), and between 10% and 25%—a primary or lower-secondary education (P). Finally, the selected committee should contain at least 25% young people (Y) and at least 50% senior people (S). The pool of candidates from which we can select members of such a committee is given in the table below. Additionally, seven voters express their preferences via the following approval ballots.

<table>
<thead>
<tr>
<th>Name</th>
<th>Gender</th>
<th>Education</th>
<th>Age</th>
<th>A(1) = {c_1, c_2, c_3}</th>
<th>A(2) = {c_3, c_5}</th>
<th>A(3) = {c_7, c_8}</th>
<th>A(4) = {c_3, c_4, c_5, c_7}</th>
<th>A(5) = {c_1, c_8}</th>
<th>A(6) = {c_6}</th>
<th>A(7) = {c_1, c_2, c_6}</th>
</tr>
</thead>
<tbody>
<tr>
<td>c_1</td>
<td>F</td>
<td>H</td>
<td>Y</td>
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<td></td>
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</tr>
<tr>
<td>c_2</td>
<td>M</td>
<td>S</td>
<td>Y</td>
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</tr>
<tr>
<td>c_3</td>
<td>M</td>
<td>S</td>
<td>S</td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c_4</td>
<td>F</td>
<td>P</td>
<td>S</td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c_5</td>
<td>M</td>
<td>S</td>
<td>Y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c_6</td>
<td>M</td>
<td>S</td>
<td>Y</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c_7</td>
<td>M</td>
<td>S</td>
<td>Y</td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c_8</td>
<td>F</td>
<td>H</td>
<td>S</td>
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</tr>
</tbody>
</table>

Assume we want to select $k = 4$ committee members. The winning committee according to AV would be $W_1 = \{c_1, c_3, c_7, c_8\}$ (for simplicity, we assume the ties are broken lexicographically $c_8 \succ c_7 \succ \ldots \succ c_1$), and according to PAV, the winning committee would be $W_2 = \{c_1, c_3, c_6, c_3\}$. However, each of these two committees violates the attribute-level constraints. The committee maximizing the AV-score and the PAV-score subject to these constraints would be, respectively, $W_3 = \{c_1, c_3, c_4, c_7\}$ and $W_4 = \{c_3, c_4, c_6, c_8\}$. ⊣
As can be seen in this example, score-based ABC rules (in particular Thiele methods) are suitable for this approach: the winning committee is the one with the highest score that satisfies all external constraints. Note that this approach is not compatible with rules that do not naturally provide a ranking of committees by scores (e.g., seq-Phragmén or Rule X). We will consider algorithmic aspects of this and related approaches in Section 6.2.

5.8 Open Problems

We conclude this section with a few interesting open problems:

1. Does there exist an ABC rule that satisfies EJR and committee monotonicity? Only partial answers are known to this question. For example, it is known that such a rule can be defined for approval-based party-list elections (see [51]; mentioned in Section 5.4), but there is no clear generalization of this rule to the setting of ABC rules.

2. What is the proportionality degree of rev-seq-PAV?

3. Does there exist an ABC rule that satisfies the core property? Equivalently, is the core always non-empty? In case the core can be empty, what is a (sensible) ABC rule that outputs a committee in the core whenever it exists?

4. Does there exist an ABC rule that satisfies priceability and Pareto efficiency?

5. We have seen in Section 5.6 that proportionality and strategyproofness is typically incompatible. The corresponding impossibility result for arbitrary, i.e., irresolute, ABC rules [117] relies on Pareto efficiency. Since this is a property that many sensible ABC rules do not satisfy (see Section 4.2) it would be desirable to strengthen this result by relaxing this condition, e.g., by replacing Pareto efficiency with weak efficiency. Is this possible or are there ABC rules that are irresolute, strategyproof, proportional, but not Pareto efficient?

6 Algorithms

In this section, we discuss computational problems related to ABC rules and algorithms that solve these problems. As we have seen in Section 4.7, many ABC rules are difficult to compute. Hence, a thorough algorithmic analysis is paramount to a practical application of these rules.

Many algorithms that we discuss in this section are available in the Python abcvoting library [129].
6.1 How to Compute Winning Committees?

The arguably most central algorithmic question is: how to compute winning committees for an ABC voting rule? Clearly, the answer significantly differs from rule to rule. Rules that can be computed in polynomial time generally do not require sophisticated algorithms; In particular, algorithms for AV, SAV, as well as for sequential and reverse sequential Thiele methods follow immediately from their corresponding definitions. Algorithms for Phragmén’s sequential rule and Rule X are slightly more involved but also do not require more than a careful adaption of the corresponding mathematical definitions. (Note that for seq-Phragmén it is more convenient to implement its discrete formulation.)

For rules that are NP-hard to compute, we discuss four algorithmic methods in the following: integer linear programs, approximation algorithms, fixed-parameter algorithms, and algorithms for structured domains.

6.1.1 Integer Linear Programs (ILPs)

The most common approach to compute NP-hard ABC rules is to employ integer linear program (ILP) solvers, such as Gurobi or CPLEX. These are fast, general purpose solvers used for hard optimization problems. To use such a solver, one has to encode an ABC rule as a integer linear program, i.e., a system of linear inequalities constraining a linear expression that is maximized or minimized. We will see examples of ILPs in the following; all these ILPs are available in the Python `abcvoting` library [129].

The ILP displayed in Figure 8 shows how Thiele methods, in particular PAV, can be expressed in such a form; this particular ILP formulation for PAV is taken from [163]. Two types of variables are used here: $x_{i,\ell}$ intuitively encodes that voter $i$ approves at least $\ell$ candidates in the committee, and $y_c$ encodes that candidate $c$ is contained in the winning committee. Given an election instance $(A, k)$, this ILP maximizes the PAV-score expressed in (3). Further it ensures that exactly $k$ candidates are selected with
minimize $D$

subject to: $d_{i,c} \in \{0, 1\}$ for $i \in [n]$, $c \in C$

$y_c \in \{0, 1\}$ for $c \in C$

$\sum_{c \in C} y_c = k$

$d_{i,c} = 1 - y_c$ for $c \in A(i)$ (8)

$d_{i,c} = y_c$ for $c \in C \setminus A(i)$ (9)

$\sum_{c \in C} d_{i,c} \leq D$ (10)

Figure 9: An ILP for computing MAV

Equation (6) and that $x_{i,\ell}$ indeed encodes that voter $i$ approves at least $\ell$ candidates in the committee with Equation (7). Note that it can occur $x_{i,\ell} = 0$ but $x_{i,\ell+1} = 1$, but this is never an optimal solution since $\frac{1}{\ell} > \frac{1}{\ell+1}$. A somewhat different ILP, for the more general when the voters assign cardinal utilities to candidates rather than point those candidates that they approve, can be found in [192].

As a second example of an ILP encoding, see Figure 9 for MAV. Here, $y_c$ encodes whether candidate $c$ is contained in the winning committee, $d_{i,c}$ encodes whether voter $i$ disagrees with the decision of whether $c$ is in the committee or not, and $D$ is the maximum Hamming distance between a voter and the chosen committee. Constraints (8) and (9) fix the value of $d_{i,c}$, i.e.,

$$d_{i,c} = \begin{cases} 0 & \text{if } (c \in A(i) \text{ and } y_c = 1) \text{ or } (c \notin A(i) \text{ and } y_c = 0), \\ 1 & \text{otherwise.} \end{cases}$$

Then, $\sum_{c \in C} d_{i,c}$ is the Hamming distance between the committee defined by $y_c$ and $A(i)$. Due to Constraint (10), these sums are $\leq D$ for all voters. Hence, by minimizing $D$, we minimize the maximum distance.

Lastly, for Monroe’s rule, Potthoff and Brams [172] discuss ILP formulations, and for lexmin-Phragmén an ILP can be found in [49].

6.1.2 Approximation Algorithms

The most natural approximation algorithm for Thiele methods are their sequential variants, as described in Section 3.2. Sequential $w$-Thiele provides a very good approximation of $w$-Thiele [143, 192]; this follows directly from a more general approximation result for submodular set functions by Nemhauser et al. [155].
Theorem 11 ([143, 155, 192]). Sequential $w$-Thiele is a factor-0.63 approximation algorithm for $w$-Thiele. More specifically, Sequential $w$-Thiele achieves a $w$-score of at least $1 - (1 - 1/k)^k \geq 1 - 1/e \geq 0.63$ times the optimal $w$-score.

One can also find approximation algorithms for the corresponding minimization problem: for $w$-Thiele, instead of maximizing the $w$-score, one can equivalently minimize the difference to the theoretical optimum of $n \cdot (1 + \frac{1}{2} + \cdots + \frac{1}{k})$, i.e., to minimize the $w$-loss defined as

$$\text{loss}_{\text{PAV}}(A, W) = n \cdot (1 + \frac{1}{2} + \cdots + \frac{1}{k}) - \text{sc}_w(A, W).$$

Byrka et al. [55] present a factor-2.35 approximation algorithm for PAV. This result requires a more complex algorithm based on dependent rounding of a linear program solution. It is notable that this result does not hold for weights other than the harmonic numbers; in particular, such an approximation algorithm does not exist for CC under the assumption that $P \neq \text{NP}$. While seq-PAV can be viewed as a voting rule in its own right, this is more debatable for this rounding scheme. In particular, it cannot be expected to satisfy nice axiomatic properties such as committee monotonicity, and thus constitutes first and foremost an approximation of PAV.

Skowron [188] describe two alternative algorithms that for certain Thiele methods (including PAV and CC) can provide arbitrarily good approximation guarantees and that work in FPT time for the parameter $(k, t)$, where $t$ is the upper-bound on the number of candidates each voter approves. Thus, these algorithms are practical only when the desired size of the committee $k$ and the approval sets of the voters are all small. On contrary, Skowron [188] show that if each voter approves sufficiently many candidates, then sequential $w$-Thiele provides an even better approximation guarantee than 0.63. Analogous results, but with the focus on CC, have been given by Skowron and Faliszewski [190].

For MAV, stronger approximation results hold: Byrka and Sornat [54] and Cygan et al. [70] present polynomial-time approximation schemes (PTAS) for MAV, i.e., polynomial-time approximation algorithms that achieves arbitrary (but fixed) precision; previous work established first factor-3 [138] and then factor-2 [57] approximation algorithms.

### 6.1.3 Fixed-Parameter Algorithms

Fixed-parameter algorithms have received some attention for ABC rules. The main idea is to identify a parameter of the problem (ideally one that is small in practice) and search for algorithms that require only polynomial time when this parameter is constant. A fixed-parameter tractable (FPT) algorithm for a parameter $p$ is one with a runtime of $O(f(p) \cdot \text{poly}(m, n))$, where $f$ is an arbitrary, typically exponential function. A very natural parameter for multiwinner voting is $k$, the committee size. Alas, results for this parameter are negative: First, Betzler et al. [29] show for Monroe and CC that it is W[2]-hard to verify whether a committee exists with at least a certain Monroe-/CC-score (some of these results have been strengthen by Skowron and Faliszewski [190], who show
that the hardness holds even if the voters’ approval sets are small). Second, Misra et al. [151] show an analogous result for MAV. Third, Aziz et al. [13] show for arbitrary Thiele methods that testing whether a committee is winning is coW[1]-hard. These results imply that one cannot hope for an FPT algorithm computing these ABC rules, i.e., it is unlikely that an algorithm exists with a runtime of, e.g., \( O(2^k \cdot \text{poly}(m, n)) \).

Moreover, Betzler et al. [29] and Skowron and Faliszewski [190] provide a thorough and detailed parameterized complexity analysis for CC and Monroe for further parameters (e.g., the number of unrepresented voters) but find mostly hardness results. Yang and Wang [208] give an overview of further parameterized results; however, the concrete algorithms announced in this paper are not published. Further fixed-parameter tractability results have also been obtained by Bredereck et al. [47]; these results apply to the more general setting of OWA voting rules.

To conclude, let us report on a positive result for MAV: MAV can be computed in time \( O(d^{2d}) \), where \( d \) is the optimal MAV-score, as shown by Misra et al. [151].\(^{25}\) This runtime is essentially optimal subject to a standard complexity theoretic assumption [70].

### 6.1.4 Algorithms for Structured Domains

The fourth and final algorithmic technique is to consider structured preference domains. Here, the assumption is that preferences are not arbitrary but satisfy structural conditions. We refer the interested reader to a survey by Elkind et al. [85] that discusses this topic more broadly. For our purpose here, we would like to discuss only two restrictions: candidate and voter interval (defined in [82], based on previous work [75, 92, 140]), but we note that many other restrictions exist and have been studied extensively [82, 83, 99, 160, 202, 207].

A profile \( A \) belongs to candidate interval (CI) domain if there exists a linear order of candidates such that for each voter \( i \in N \), the set \( A(i) \) appears contiguously on the linear order. Similarly, a profile \( A \) belongs to the voter interval (VI) domain if there exists a linear order of voters such that for each voter \( c \in C \), the set \( N(c) \) appears contiguously on the linear order. The CI domain is closely related to the single-peaked domain for arbitrary ordinal preferences and the VI domain closely related to the single-crossing domain; this is analyzed in more detail by Elkind and Lackner [82].

Under the assumption that preferences belong either to the CI or VI domain, the computational complexity can change dramatically: MAV is solvable in polynomial time if the given approval profile belongs either to the CI or VI domain [142]. Further, Thiele methods [163] and Monroe’s rule [29] can be solved in polynomial time if the given approval profile belongs to the CI domain. It remains an open problem whether the same holds for the VI domain.

\(^{25}\)Misra et al. [151] claimed that the runtime of their algorithm is \( d^d \); this was corrected later [70, 142].
6.2 The Algorithmic Perspective of Proportionality

In this section, we briefly review the literature that deals with the computational problem of finding a proportional committee.

6.2.1 Finding Proportional Committees for Cohesive Groups

We first look at the proportionality concepts that formalize the behavior of rules with respect to cohesive groups of voters; see Section 5.2.

Note that even the problem of deciding whether in a given instance of election there exists an \( \ell \)-cohesive group of voters is NP-complete [194]. Similarly, given a committee \( W \) deciding whether \( W \) satisfies the EJR condition is coNP-complete [14]; the same holds for the problem of deciding whether \( W \) satisfies the PJR condition [15]. Checking if a given committee \( W \) satisfies JR is computationally easy—for each candidate one needs to check whether the group of voters approving this candidate is 1-cohesive, and if so, to check if less than \( n/k \) voters from such a group are left without a representative in \( W \). Checking whether a given committee satisfies perfect representation (Definition 17) is also computationally easy—the problem reduces to finding a perfect constrained matching in a bipartite graph [182].

While the problem of checking if a given committee satisfies the EJR/PJR condition is computationally hard, for a given election instance one can find in polynomial time some committee that satisfies the two conditions (e.g., through Rule X [164], or through a local-search algorithm for PAV [15]). The situation is quite different for perfect representation (PR): it is NP-complete to check whether there exists a PR committee for a given election instance [182]. Consequently, unless P = NP, there exists no polynomial-time ABC rule that satisfies perfect representation.

6.2.2 Finding Committees with Attribute-Level Constraints

Next, we move to the model with external attribute-level constraints from Section 5.7.

We start by considering the model from Example 26, where we have a set of voters with approval-based preferences over the candidates, the candidates have attribute values (the attributes can be, e.g., gender, age group, education level, etc.) and for each attribute value we are given quotas specifying upper- and lower- limits on the number of committee members with this particular attribute value. Two recent works [45, 59] considered algorithmic aspects of the problem of finding committees maximizing the score according to a given Thiele method, subject to given attribute-level constraints. The authors considered the problem from the perspective of approximation algorithms and parameterized complexity theory, as well as considered variants of the problem, where the attribute-level constraints have certain special structures. We do not describe their results in detail, but it is worth mentioning that the problem is computationally hard even when there are no voters, and the question boils down to finding a committee satisfying the attribute-level constraints. The (approximation and fixed-parameter tractable)
algorithms for this simpler setting were studied by Lang and Skowron [130].

A very similar model to the one from Example 26 is the one of constrained approval voting (CAP) [33, 171]. The main difference is that, there, the constraints are formulated for combinations of attributes. For example, a constrain can have the following form: “the proportion of young (Y) males (M) with higher education (H) in the committee should not exceed 14%”. Specifically, Brams [33], Potthoff [171] suggest to pick the committee that maximizes the AV score subject to the aforementioned combinatorial constraints. A simple translation of CAP into an ILP problem was given by Straszak et al. [197]. However, the setting of constrained approval voting has not been thoroughly studied in its full generality, and the model is fairly unexplored from the computational perspective.

We conclude this section by discussing two related lines of research. The computational problem of finding a committee subject to attribute-level constraints is related to the multidimensional knapsack problem (the main difference is that in the multidimensional knapsack the candidates can contribute more than a unit weight to each attribute-level constraint) and to the generic problem of optimizing a submodular function subject to constraints (see, e.g., [123]). However, this literature usually deals with more general types of constraints, while the literature we discussed before often provides more dedicated solutions.

6.3 The Algorithmic Perspective of Strategic Voting

Other types of computational problems arise when one analyzes how the results of ABC elections change in response to changes in voters’ preferences. There are several reasons to study this type of computational problems, and we briefly summarize them below. Historically, the first motivation was to use the computational complexity as a shield protecting elections from strategic manipulations—the reasoning was the following: if we cannot construct a good rule that would be strategy-proof (e.g., due to known impossibility theorems; cf. Section 5.6), then we could at least aim at proposing a rule for which it is computationally hard for a voter to come up with a successful strategic manipulation. This motivation originated in the context of single-winner elections, and was first proposed by Bartholdi et al. [22]; it was later contested since the analysis of computational complexity is worst-case in spirit—even for rules for which the problem of finding a successful strategic manipulation is NP-hard, such manipulations can be found easily in average case, in particular for many real-life preference profiles (for a more detailed discussion of these arguments, but with a focus on single-winner elections, we refer the reader to a survey by Faliszewski and Procaccia [91] and a book by Meir [147]).

Nonetheless, problems related to finding strategic manipulations have various connections to other important questions. Examples are the question of whether one can stop eliciting preferences and safely determine the winners of an election, deciding whether the result of an election is robust to small changes in the given preference profile [103], and measuring the success of individual candidates in a committee election [94].

Before we move further, we note that for the case of selecting a single winner ($k = 1$)
under approval-based preferences, an excellent overview of computational issues related to strategic voting is given by Baumeister et al. [24]. Thus, in the remaining part of this section we focus on the literature that deals with the case when \( k > 1 \).

### 6.3.1 Computational Complexity of Manipulation

We first consider the computational problem of finding a successful manipulation. Recall that we write \( A + X \) to denote the profile \( A \) with one additional voter approving \( X \), i.e., \( A + X = (A(1), \ldots, A(n), X) \).

**Definition 24.** Consider an ABC rule \( \mathcal{R} \). In the Utility-Manipulation problem we are given an election instance \((A, k)\), a utility function \( u : C \to \mathbb{R} \), and a threshold value \( t \in \mathbb{R} \). We ask whether there exists an additional ballot \( X \subseteq C \) such that \( \sum_{c \in W} u(c) \geq t \), where \( W \in \mathcal{R}(A + X, k) \).

In Subset-Manipulation we are given an election \((A, k)\), a subset of candidates \( L \subseteq C \), and a positive integer \( r \). We ask whether there exists a profile \( A' \) that extends \( A \) by \( r \) additional voters such that \( L \subseteq W \) for some \( W \in \mathcal{R}(A', k) \).

Intuitively, in Utility-Manipulation we have a single manipulator with a utility function describing her level of appreciation for different candidates; the utility function is additive. The question is whether the manipulator can send an approval ballot such that she gets the utility of at least \( t \) from an elected committee. In Subset-Manipulation there are \( r \) manipulators and their goal is slightly different—they want to ensure that the candidates from a given set \( L \) are all selected. For \( r = 1 \) Subset-Manipulation can be represented as Utility-Manipulation: we assign the utility of one to the candidates from \( L \) and the utility of zero to the other candidates, and set \( t = |L| \). Observe that it makes sense to consider Utility-Manipulation also in the context of AV—this is because AV is strategy-proof only for dichotomous preferences, while the definition of Utility-Manipulation assumes the manipulator has more fine-grained preferences.

Meir et al. [148] have shown that Utility-Manipulation is solvable in polynomial time for Multi-Winner Approval Voting with adversarial tie-breaking\(^{26}\). Baumeister et al. [25] proved that also Subset-Manipulation is solvable in polynomial time for AV. Aziz et al. [13] have shown that Utility-Manipulation is computationally hard for SAV and PAV with a given tie-breaking order on candidates. They have also proved that Subset-Manipulation is NP-hard for SAV and coNP-hard for PAV. Further, for PAV the problem stays hard even if there is only a single manipulator (\( r = 1 \)), while for SAV with a single manipulator the problem becomes computable in polynomial time.

Bredereck et al. [44] studied a more general version of Utility-Manipulation, where the goal is to check whether there exists a coalition of voters that could jointly perform a successful manipulation. The authors focused on the \( \ell \)-Bloc rule, which is a variant of Multi-Winner Approval Voting, where each voter approves exactly \( \ell \) candidates. Then,\(^{26}\)

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\(^{26}\)Adversarial tie-breaking means that ties between candidates are broken in disfavor of the manipulators.
the coalition-manipulation problem is computationally hard in its all variants studied by
the authors. On the other hand, if we look at an egalitarian version of $\ell$-Bloc (maximizing
the AV-score of the worst-off voter), then the problem becomes computationally tractable.
Another problem related to UTILITY-MANIPULATION has been considered by Barrot et al.
[20]: given utility functions of all voters, is there an approval profile consistent with the
utility functions in which a given committee wins.

6.3.2 Computational Complexity of Robustness

The next computational problem that we look at is ROBUSTNESS. In the definition
below, we consider the following three operations: the operation Add consists in adding a
candidate to the approval set of some voter, Remove deletes a candidate from the approval
set of a voter, and Swap is a combination of Add and Remove applied simultaneously to
the approval set of a single voter.

Definition 25. Consider an ABC rule $\mathcal{R}$ and an operation $\text{Op} \in \{\text{Add}, \text{Remove}, \text{Swap}\}$. In the $\text{Op}$-ROBUSTNESS problem we are given an election instance $(A, k)$ and an integer $b$. We ask whether there exist a sequence $S$ of $b$ operations of type $\text{Op}$ such that $\mathcal{R}(A, k) \neq \mathcal{R}(A', k)$, where $A'$ is the preference profile obtained from $A$ by applying the operations from sequence $S$.

Gawron and Faliszewski [103] have shown that the Op-ROBUSTNESS problem is com-
putationally hard for PAV and CC, for each type of the three operations. On the other
hand, the problem can be solved in polynomial time for AV and SAV. The authors also
computed the robustness radius—a measure that says how much the result of an election
can change in response to a single change in the preference profile—for several ABC rules.
Gawron and Faliszewski [103] and Misra and Sonar [150] also considered the parameter-
ized complexity of the ROBUSTNESS problem, and have designed several parameterized
algorithms for natural parameters, such as the number of voters $n$ and the number of can-
didates $m$. Faliszewski et al. [94] considered a similar problem, but they asked whether,
through a sequence of operations of a given type, one can make a particular candidate a
member of a winning committee.

6.4 Open Problems

Also from a computational point of view, many questions concerning ABC rules remain
unanswered:

- Sequential PAV approximates the optimal PAV-score by a factor of at least $1 - \frac{1}{e}$. What is the factor for Reverse Sequential PAV? Is it better? The same question can be asked for CC.

- Several approximation algorithms and heuristics exist for PAV, such as seq-PAV, rev-
seq-PAV, the approximation algorithm based on dependent rounding ([55], discussed
in Section 6.1.2), and a local-search algorithm used for finding EJR committees in polynomial time [15]. The difference between these algorithms has not been investigated from a practical point of view. The main question is which of these algorithms such be chosen to approximate PAV given a very large election?

- Is it possible to compute Thiele methods and Monroe's rule in polynomial time if the given preference profile belongs to the voter interval (VI) domain [82]?

- Often, voting rules have to be computed given incomplete information (such as missing ballots or incomplete ballots). This problem has not been studied from a computational point of view for ABC rules, but received substantial attention in the single-winner literature [31]. One open question in particular is which of the algorithmic approaches discussed in Section 6.1 can be generalized to this setting.

- The computation of some polynomial-time ABC rules can clearly be parallelized. For example, for AV each candidate can be processed independently of others. The framework of P-completeness [106] can be used to determine which ABC rules are inherently sequential (by showing P-completeness) and which can be parallelized (by showing, e.g., NL-containment). Such work has been done for single-winner rules [42, 68, 69] but not for multiwinner rules.

7 Related Formalisms and Applications

In this section, we briefly discuss connections of approval-based committee voting with a number of other formalisms.

7.1 Multiwinner Voting for Other Types of Votes

In this survey we consider the variant of the committee election model where agents vote by specifying sets of approved candidates. Several recent works study (mostly from an algorithmic point of view) an extended variant of this model, where the ballots are trichotomous, that is where each voter can approve, disapprove or remain neutral with regard to a candidate. Zhou et al. [210] focuses on CC, PAV, and SAV, and generalize these rules to trichotomous votes; Baumeister and Dennisen [23] and Baumeister et al. [25] focus on AV and MAV. Baumeister et al. [26] further extend MAV to the case where each voter assigns each candidate to one of \( \ell \) predefined buckets, where \( \ell \) is a parameter.

Another classic multiwinner election model is when the voters vote by ranking the candidates from the most to the least preferred one. The recent survey by Faliszewski et al. [93] provides an up-to-date overview of the literature on multiwinner election with ranking-based preferences. The variant where the voters submit weak orders over candidates (i.e., rankings with ties) has been considered, e.g., by Aziz and Lee [10]. While multiwinner voting based on weak orders generalizes both the ABC model of this survey and ranking-based multiwinner elections, it should be noted that the concepts introduced in this survey...
(e.g., notions of proportionality) do not easily generalize to this more expressive setting, in particular not without a substantial increase in required notation.

7.2 Multiwinner Voting with a Variable Number of Winners

Throughout this paper, we assume that the committee size is fixed. In the literature on multiwinner voting with a variable number of winners \[116\], this assumption is dropped and a voting rule can return an arbitrary number of candidates—depending on the given election instance. This question has been first considered by Kilgour \[114, 116\], and later from a computational point of view by Faliszewski et al. \[95\]. The special case of shortlisting rules has been analyzed by Lackner and Maly \[125\]. Finally, we note that so-called \textit{social dichotomy functions} \[39, 79\] are largely equivalent to multiwinner rules with a variable number of winners.

7.3 Participatory Budgeting

In participatory budgeting (PB), we assume that candidates come with different costs, and that the sum of the costs of the selected candidates cannot exceed a given budget. Thus, committee elections can be viewed as a special case of participatory budgeting, where the costs of the candidates are all equal. Typically, candidates correspond to projects in this setting that require a certain amount of money to be funded. For an overview of different models and approaches to participatory budgeting, we refer the reader to a recent survey by Aziz and Shah \[12\].

Given that in PB the costs of different candidates can vary significantly, it seems natural to assume that the voters’ preferences are fine-grained. Nevertheless, some of works \[17, 105, 200\] study approval-based voting rules for PB, which include rules used in practice. Knapsack voting suggested by \[105\] closely resembles AV. Aziz et al. \[17\] generalize PAV, seq-Phragmén, and CC to the case where the candidates can have arbitrary costs, and provide a taxonomy of axioms aimed at formalizing proportionality; those axioms are adaptations of JR and PJR, which we discuss in Section 5.2. Talmon and Faliszewski \[200\] study other axioms, mostly pertaining to different forms of monotonicity (cf. Sections 4.3 and 4.4) and through experiments provide visualizations of the kind of committees returned by different participatory budgeting rules.

7.4 Budget Division and Probabilistic Social Choice

The goal of a probabilistic social choice function is to divide a single unit of a global resource between the candidates. Thus, committee elections can be viewed as instances of probabilistic social choice with the additional requirement that each candidate gets either $1/k$-th fraction of the global resource, or nothing. For an overview of recent results on probabilistic social choice functions, we refer to a book chapter by Brandt \[41\],

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Several works [18, 30, 40, 78, 89, 149] study probabilistic social choice functions for approval votes. The particular focus of some of these works is put on formalizing the concepts of fairness and proportionality. Some of these concepts closely resemble the ones that we discussed in the context of approval-based committee elections (Section 5). For example, Aziz et al. [18] and Fain et al. [89] study the concept of the core (Section 5.4), Aziz et al. [18] additionally consider the axioms of average fair share, group fair share, and individual fair share—the properties that closely resemble—respectively—proportionality degree, PJR, and JR (Section 5.2), Michorzewski et al. [149] show the relation between these fairness properties and the utilitarian welfares of outcomes (cf. Section 5.5.2). Bogomolnaia et al. [30] focuses on mechanisms which are strategyproof, and Duddy [78] proves that strategyproofness is incompatible with certain forms of proportionality—an impossibility result similar to the ones that we discuss in Section 5.6.

7.5 Voting in Combinatorial Domains

Multiwinner election rules output fixed-size subsets of available candidates. An alternative way of thinking of such rules is that (1) for each candidate \( c \) they make a decision whether \( c \) should be selected to the winning committee or not, and (2) there is a constraint which specifies that exactly \( k \) decisions must be positive. Thus, with \( m \) candidates there are \( m \) dependent binary decisions (each decision is of the form “include a candidate in the winning committee or not”) that are made by a multiwinner rule. These decisions are dependent (related) because of the constraint on the number of positive decisions.

The literature on voting in combinatorial domains studies a general setup, where a number of decisions (not necessarily binary) need to be made, and where there exist (possibly complex) relations between the decisions. Similarly, the preferences of the voters might have complex forms. For example, consider two issues—\( I_1 \) with two possible decisions \( Y_1 \) and \( N_1 \), and \( I_2 \) with possible decisions \( Y_2 \) and \( N_2 \). A voter might prefer decision \( Y_2 \) only if the decision with respect to issue \( I_1 \) is \( Y_1 \); otherwise, that is if decision \( N_1 \) on \( I_1 \) is made, a voter might prefer \( N_2 \) over \( Y_2 \) (see [32] for a detailed discussion of this example). Various languages have been studied that allow voters to express such complex combinatorial preferences. For example, in the context of approval-based committee elections, some of these languages would allow voters to express the view that a certain group of candidates works particularly well together, so they should either be all selected as members of the winning committee or none of them should be chosen, or the view that some candidates should never be chosen together. In the literature on committee elections, on the other hand, it is assumed that the preferences of the voters are separable, thus the voters can only make statements about their levels of appreciation for different candidates.

For an overview of the literature on voting in combinatorial domains, we refer to a book chapter by Lang and Xia [131]. Here, we mention three works from this literature that deal with models particularly related to the model of approval-based committee elections. In public decision making [67] the decisions are not related, the preferences of the voters
with respect to decisions on various issues are separable, thus the model closely resembles the one studied in this survey. The main difference is that in the model for public decisions there is no constraint specifying the number of decisions that can be positive. There, the authors focus on designing fair (i.e., proportional) rules. The model of sub-committee elections [9] generalizes the ones of multiwinner elections and public decisions. There, it is assumed that the set of candidates is partitioned and for each group of candidates there is a threshold bounding the number of candidates selected from this group.

Another formalism closely related to ABC voting is perpetual voting [124]. Here, instead of a committee we have time steps and in each step one candidate is selected. Consequently, after $k$ rounds $k$ candidates are selected, which can be viewed as a committee. The main difference is that the set of available candidates and voters' preferences can change each round. The goal is to provide proportionality over time, which requires that the decision in round $k$ is made under consideration of the voters' satisfaction in previous rounds. This formalism can be viewed as a special case of voting in combinatorial domains (with a very specific sequentiality constraint). Further, due to the sequential structure imposed by time, perpetual voting rules have close connections with committee monotonic ABC rules (such as seq-Phragmén and seq-PAV). Similar questions in a utility-based model have been studied in on- [101] and offline [67] settings.

### 7.6 Judgment Aggregation and Propositional Belief Merging

In judgment aggregation, we are given a set of logical propositions and a set of voters providing true/false valuations for these propositions; the goal is to find a collective consistent valuation. Sometimes it is also required that the collective valuation must be consistent with exogenous logical constraints. Committee elections can be represented as instances of judgment aggregation, where for each candidate we have a single Boolean variable representing whether the candidate is elected or not; the exogenous constraints can be used to enforce that exactly $k$ from these variables are set true, as discussed in Section 7.5. A chapter by Endriss [86] in the Handbook of Computational Social Choice discusses this framework in detail and reviews rules for judgment aggregation; see also the survey by List and Puppe [141].

Propositional belief merging [119–121] is a very general framework, which allows agents to aggregate their individual positions (beliefs, preferences, judgments, goals) on a set of issues. Also here this combined, collective outcome has to satisfy given exogenous logical constraints. Approval-based committee voting can be seen as a special case of propositional belief merging, although the focus of these two directions of research has little overlap: belief merging operators are analyzed with respect to a set of postulates that only partially reflect desiderata in a voting context. A few works have made an explicit effort to connect voting and belief merging: A particular focus has been the study of belief merging and strategyproofness [64, 88, 108]. Further, Haret et al. [109] consider classic axioms from social choice theory in the context of belief merging. Finally, Haret et al. [110] introduce and analyze proportional belief merging operators.
7.7 Proportional Rankings

The theory of committee elections can be applied in a seemingly unrelated setting, where the goal is to find a ranking of candidates based on voters’ preferences. One can observe that every committee monotonic (Definition 4), resolute ABC rule $\mathcal{R}$ can be used to obtain a ranking of candidates: we put in the first position in the ranking the candidate that $\mathcal{R}$ returns for $k = 1$; call this candidate $c$. Committee monotonicity guarantees that the set of two candidates returned by $\mathcal{R}$ for $k = 2$ contains $c$; the other candidate is put in the second position in the ranking, etc.

In particular, if we use a proportional committee-monotonic rule (for example, seq-Phragmén or seq-PAV) then the obtained ranking will proportionally reflect the views of the voters in the sense that each prefix of such a ranking, viewed as a committee, will be proportional; this idea has been studied in detail by Skowron et al. [193]. Proportional rankings are desirable, e.g., when one wants to provide a list of recommendations or search results that accommodate different types of users (cf. diversifying search results [76, 183]), or in the context of liquid democracy [28], where an ordered list of proposals is presented to voters for their consideration.

8 Outlook

This section does not exist yet.

Acknowledgements

We are thankful to Jan Maly for providing helpful feedback.

9 Bibliography


Additional Proofs

In this section, we provide some proofs and counterexamples that we were not able to find in the published literature.

Proposition A.1. AV, PAV, and SAV satisfy strong Pareto efficiency; CC and MAV fail strong Pareto efficiency.

Proof. Observe that if $W_1$ dominates $W_2$ then the AV-, PAV-, and SAV-score of $W_1$ is strictly larger than that of $W_2$. Thus, $W_2$ is not a winning committee for these ABC rules.

To see that CC fails strong Pareto efficiency, consider the approval profile

\[ 1 \times \{a, c, d\} \quad 1 \times \{b, c, d\} \]
For $k = 2$, $\{a, b\}$ is a winning committee even though it is dominated by $\{c, d\}$.

To see that MAV fails strong Pareto efficiency, consider the approval profile

$$1 \times \{a, c\} \quad 1 \times \{b, c\} \quad 1 \times \{d, e\}$$

For $k = 1$, there is always one voter with Hamming distance 3 to any size-1 committee. Consequently, all size-1 committees are winning even though $\{c\}$ dominates $\{a\}$ and $\{b\}$.

**Proposition A.2.** CC, PAV, Monroe, Greedy Monroe, lexmin-Phragmén, Rule X, and MAV do not satisfy committee monotonicity.

**Proof.** First, let us consider the approval profile

$$2 \times \{a\} \quad 3 \times \{a, c\} \quad 3 \times \{b, c\} \quad 2 \times \{c\},$$

CC, PAV, Monroe, lexmin-Phragmén, and MAV choose $\{c\}$ for $k = 1$ and $\{a, b\}$ for $k = 2$. For Greedy Monroe, consider the approval profile $A$ defined as

$A(1) = \cdots = A(6) = \{a\}, \quad A(7) = \cdots = A(10) = \{a, c\}, \quad A(11) = A(12) = \{a, b, c\}$

$A(13) = A(14) = \{a\}, \quad A(15) = \{a, d\}, \quad A(16) = \cdots = A(18) = \{b, d\}.$

We assume that Monroe breaks ties between candidates in alphabetic order and between voters in increasing order. For $k = 2$ groups have a size of 9, for $k = 3$ groups have a size of 6. Now, for $k = 2$, Greedy Monroe first chooses $a$ and assigns voters 1–9 and then candidate $b$ assigning voters $\{11, 12, 16, 17, 18\}$. For $k = 3$, Greedy Monroe first chooses $a$ and assigns voters 1–6, then candidate $c$ assigning voters 7–12, and finally candidate $d$ assigning voters 15–18. We see that $\{a, b\}$ is not a subset of $\{a, c, d\}$.

For Rule X, consider

$$4 \times \{a, c\} \quad 2 \times \{a, d\} \quad 3 \times \{b, d\} \quad 1 \times \{b\},$$

For $k = 2$, the budget of voters is 0.2. Candidate $a$ is selected in the first round, reducing the budget of voters 1–6 to $\frac{1}{30}$. The remaining budget is insufficient to pay for another candidate, so Phase 2 starts. Now, candidate $b$ is selected via seq-Phragmén.

For $k = 3$, the budget of voters is 0.3. Candidate $a$ is selected in the first round, reducing the budget of voters 1–6 to $0.3 - \frac{1}{6} = \frac{2}{15}$. Next, $b$ would be affordable for $q = 0.25$, but $d$ is cheaper for $q = \frac{11}{45}$. The third candidate is selected in Phase 2. Here candidate $c$ is selected: voters 1–4 just need an additional budget of $\frac{7}{60}$ (since $4 \cdot (\frac{2}{15} + \frac{7}{60}) = 1$).

We see that Rule X selects $\{a, b\}$ and $\{a, c, d\}$ and is thus not committee monotone.

**Proposition A.3.** Thiele methods and SAV satisfy support monotonicity with additional voters, seq-PAV, rev-seq-PAV, seq-Phragmén, and lexmin-Phragmén satisfy candidate monotonicity with additional voters. Further, seq-Phragmén and lexmin-Phragmén
do not satisfy support monotonicity with additional voters, and Monroe, Greedy Monroe,
and Rule X fail candidate monotonicity with additional voters.

AV and SAV satisfy support monotonicity without additional voters, PAV, CC, seq-
PAV, rev-seq-PAV, Monroe, seq-Phragmén, and lexmin-Phragmén satisfy candidate mono-
tonicity without additional voters; none of these satisfy support monotonicity without ad-
ditional voters.

Proof. Support monotonicity with additional voters: Sánchez-Fernández and Fis-
teus [180] show that Thiele methods and SAV satisfy support monotonicity with additional
voters (referred to support monotonicity with population increase in this paper). Janson
[112] proves that seq-PAV, rev-seq-PAV, and seq-Phragmén satisfy candidate monotonic-
ity with additional voters. Further, lexmin-Phragmén satisfy candidate monotonicity with
additional voters; this is a consequence that it satisfies weak support monotonicity with
population increase [180]. A counterexamples showing that seq-Phragmén does not sat-
isfy support monotonicity with additional voters can be found in [112, 153]. Further,
counterexamples for lexmin-Phragmén and Monroe can be found in [180].

To see that Greedy Monroe fails candidate monotonicity with additional voters, con-
sider the following election instance: profile with 9 votes and 8 candidates:

<table>
<thead>
<tr>
<th>Vote 1</th>
<th>Vote 2</th>
<th>Vote 3</th>
<th>Vote 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>b, c, d</td>
<td>a, c, f</td>
<td>a, d, e</td>
<td>c, e</td>
</tr>
<tr>
<td>a, b</td>
<td>d, f</td>
<td>b, e</td>
<td>b, f</td>
</tr>
</tbody>
</table>

For \( k = 3 \), the winning committee according to Greedy Monroe is \{b, e, f\}. If an additional
voter approves \{e\}, the winning committees changes to \{b, c, d\}. This committee does not
contain \( e \) and hence Greedy Monroe fails candidate monotonicity with additional voters.

To see that Rule X fails candidate monotonicity with additional voters. Consider

<table>
<thead>
<tr>
<th>Vote 1</th>
<th>Vote 2</th>
<th>Vote 3</th>
<th>Vote 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>b, d</td>
<td>a, b</td>
<td>b, d, e</td>
<td>a, e</td>
</tr>
<tr>
<td>c, d, e</td>
<td>c, e</td>
<td>a, c, e</td>
<td>b, c, d</td>
</tr>
</tbody>
</table>

For \( k = 3 \), the winning committee according to Rule X is \{a, d, e\}. If an additional voter
approves \{a\}, the winning committee changes to \{b, c, e\}. As this committee does not
contain \( a \), Rule X fails candidate monotonicity with additional voters.

Support monotonicity without additional voters: AV and SAV satisfy support
monotonicity without additional voters [180]. PAV, CC, seq-PAV, rev-seq-PAV, Monroe,
seq-Phragmén, and lexmin-Phragmén satisfy candidate monotonicity without additional
voters [112, 180]. Counterexamples that these rules do not satisfy the stronger axiom can be
found in [112] for seq-Phragmén, in [180] for PAV, CC, Monroe, and lexmin-Phragmén.

For seq-PAV, consider

<table>
<thead>
<tr>
<th>Vote 1</th>
<th>Vote 2</th>
<th>Vote 3</th>
<th>Vote 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>b, c</td>
<td>b, d</td>
<td>e, f</td>
<td>a, e</td>
</tr>
<tr>
<td>c, f</td>
<td>a, b</td>
<td>b, f</td>
<td></td>
</tr>
</tbody>
</table>
For \( k = 3 \), the winning committee according to seq-PAV is \( \{b, e, f\} \). If the first voters changes her ballot from \( \{b, c\} \) to \( \{b, c, e, f\} \), there are two possible winning committees (depending on the chosen tiebreaking): \( \{a, b, f\} \) and \( \{b, e, f\} \). As the first one does not contain \( e \), seq-PAV fails support monotonicity without additional voters.

For rev-seq-PAV, consider

\[
\begin{align*}
2 \times \{a, e\} & \quad 2 \times \{b, c, d\} & \quad 1 \times \{d, e\} & \quad 3 \times \{c, e\} & \quad 1 \times \{b, d, e\} \\
1 \times \{a, b, c\} & \quad 1 \times \{c, d, e\} & \quad 2 \times \{a, d, e\} & \quad 1 \times \{b, d\} & \quad 1 \times \{a, b\} \\
1 \times \{a, d\} & \quad 1 \times \{a, b, d\} & \quad 1 \times \{b, c\}
\end{align*}
\]

For \( k = 3 \), the winning committee according to rev-seq-PAV is \( \{c, d, e\} \). If the first voter changes her ballot from \( \{a, e\} \) to \( \{a, c, d, e\} \), the winning committee changes to \( \{b, d, e\} \). As this committee does not contain \( c \), rev-seq-PAV fails support monotonicity without additional voters.

**Proposition A.4.** AV with fixed tie-breaking order on candidates satisfies cardinality-strategyproofness and thus inclusion-strategyproofness. PAV, seq-PAV, seq-CC, rev-seq-PAV, Monroe, Greedy Monroe, seq-Phragmén. Rule X, MAV, and SAV do not satisfy inclusion-strategyproofness.

**Proof.** To see that AV satisfies cardinality-strategyproofness, consider a fixed voter \( i \). Observe that if \( i \) disapproves one of the (truly) approved candidates, say \( c \), then it may cause at most one additional candidate get to the winning committee; however, this will happen only if \( c \) is removed from the winning committee. In such case, the satisfaction of \( i \) cannot increase. If \( i \) approves a not-yet approved candidate, then this might only cause this candidate replace some other candidate in the committee. Again, such a change cannot increase the satisfaction of the voter. Finally, a voter changing his ballot can be decomposed into a sequence of changes which consists in either approving a disliked candidate or disapproving a candidate that is actually liked. Each such a change cannot increase the satisfaction of the voter.

For SAV consider the following profile with 2 votes and 5 candidates:

\[
\begin{align*}
1 \times \{a, b, c\} & \quad 1 \times \{d, e\}
\end{align*}
\]

For \( k = 1 \) the winning committees according to SAV are \( \{d\} \) and \( \{e\} \). If the first voter changes her ballot to \( \{a\} \), the winning committee will change to \( \{a\} \), an outcome which is preferred by the first voter.

For revseq-PAV consider the following profile with 5 votes and 6 candidates:

\[
\begin{align*}
1 \times \{a, b, c\} & \quad 1 \times \{b, d\} & \quad 1 \times \{b, c\} & \quad 1 \times \{a, d, e\} & \quad 1 \times \{b, e\}
\end{align*}
\]

For \( k = 2 \) the winning committees are \( \{b, d\} \) and \( \{b, e\} \). If the first voter changes her ballot from \( \{a, b, c\} \) to \( \{a\} \), then \( \{a, b\} \) will become the only new winning committee. The first voter prefers this committee to both \( \{b, d\} \) and \( \{b, e\} \), thus she has an incentive to misreport her preferences.
For seq-Phragmén consider the following profile with 6 votes and 6 candidates:

\[ 1 \times \{a, b, c\} \quad 1 \times \{a, b\} \quad 1 \times \{b, f\} \quad 1 \times \{c, e\} \quad 1 \times \{b, e, f\} \quad 1 \times \{b, d, f\} \]

For \( k = 2 \) the only winning committee is \( \{b, f\} \). If the first voter changes her ballot from \( \{a, b, c\} \) to \( \{c\} \), then the winning committee changes to \( \{b, c\} \), an outcome that the voter strictly prefers to the original winning committee.

For seq-PAV consider the following profile with 6 votes and 6 candidates:

\[ 1 \times \{a, b\} \quad 1 \times \{b, d\} \quad 1 \times \{c, f\} \quad 1 \times \{a, b, f\} \quad 1 \times \{b, f\} \quad 1 \times \{b, c\} \]

For \( k = 3 \) the unique winning committee is \( \{b, c, f\} \). The first voter can successfully manipulate by changing her ballot to \( \{a\} \)—then the winning committee changes to \( \{a, b, f\} \).

For MAV consider the following profile with 6 votes and 6 candidates:

\[ 1 \times \{a, b, c\} \quad 1 \times \{b, d\} \quad 2 \times \{a, b, e\} \quad 1 \times \{a, b, d\} \quad 1 \times \{a, b\} \]

For \( k = 3 \) the unique winning committee is \( \{a, b, d\} \). If the first voter changes her ballot to \( \{c\} \), then \( \{a, b, c\} \) becomes the only winning committee.

For PAV consider the following profile with 6 votes and 6 candidates:

\[ 1 \times \{c, d, e\} \quad 1 \times \{a, b\} \quad 1 \times \{b, f\} \quad 1 \times \{a, c, d\} \quad 1 \times \{b, c, f\} \quad 1 \times \{c, e, f\} \]

For \( k = 3 \) the only winning committee is \( \{b, c, f\} \). If the first voter submits \( \{e\} \) instead of \( \{c, d, e\} \), then \( \{b, c, e\} \) will become the only winning committee.

For Rule X consider the following profile with 6 votes and 6 candidates:

\[ 1 \times \{b, c, d\} \quad 1 \times \{a, b\} \quad 1 \times \{b, d\} \quad 1 \times \{c, d\} \quad 2 \times \{d, e\} \]

For \( k = 3 \) the only winning committee is \( \{b, d, e\} \). The first voter can successfully manipulate by changing her ballot to \( \{c\} \)—then the winning committee changes to \( \{b, c, d\} \).

For Greedy Monroe consider the following profile with 4 votes and 6 candidates:

\[ 1 \times \{a, b\} \quad 1 \times \{a, c, f\} \quad 1 \times \{a, c, d\} \quad 1 \times \{e, f\} \]

For \( k = 2 \) the only winning committee is \( \{a, c\} \). If the first voter changes her ballot to \( \{b\} \), then \( \{a, b\} \) becomes the only winning committee.

For Monroe consider the following profile with 12 votes and 6 candidates:

\[ 1 \times \{b, d\} \quad 1 \times \{a, b, c\} \quad 1 \times \{b, e\} \quad 1 \times \{d, e\} \quad 1 \times \{e, f\} \quad 1 \times \{b, c, e\} \]
\[ 1 \times \{c, d, e\} \quad 1 \times \{b, c\} \quad 2 \times \{a, f\} \quad 1 \times \{b, c, d\} \quad 1 \times \{a, d\} \]

For \( k = 3 \) the only winning committee is \( \{a, b, e\} \). If the first voter changes her ballot to \( \{f\} \), the winning committee changes to \( \{b, d, f\} \).

For seq-CC consider the following profile with 12 votes and 6 candidates:

\[ 1 \times \{b, e, f\} \quad 1 \times \{a, b\} \quad 1 \times \{d, e, f\} \quad 1 \times \{d, e\} \quad 1 \times \{b, f\} \quad 2 \times \{c, d\} \]
\[ 1 \times \{a, b, c\} \quad 1 \times \{a, c\} \quad 1 \times \{a, b, e\} \quad 1 \times \{a, e, f\} \quad 1 \times \{b, c, d\} \]

For \( k = 3 \) the only winning committee is \( \{a, b, d\} \). The first voter can successfully manipulate by changing her ballot to \( \{c\} \)—then the winning committee changes to \( \{b, c, e\} \). \qed
Proposition A.5. If $k$ divides $n$, then Greedy Monroe extends the largest remainders method.

Proof. Consider an apportionment instance with $p$ political parties, $C_1, \ldots, C_p$, and let $n_i$ denote the number of votes cast on party $C_i$. Since $n$ is divisible by $k$, Greedy Monroe always tries to assign a candidate to $\frac{n}{k}$ voters. Observe that:

$$n_i - \frac{n}{k} < \left[k \cdot \frac{n_i}{n}\right] \cdot \frac{n}{k} \leq n_i.$$

Let $k_1 = \sum_{i=1}^{p} \lfloor k \cdot \frac{n_i}{n} \rfloor$. In the first $k_1$ rounds Greedy Monroe assigns to each party $C_i$ exactly $\lfloor k \cdot \frac{n_i}{n} \rfloor$ seats. This is consistent with the first phase of the largest remainders method. During these rounds, whenever Greedy Monroe assigns a seat to a party, it removes $n/k$ of its supporters. Then, each party $C_i$ is left with less than $n/k$ supporters. Specifically, party $C_i$ is left with the following number of supporters:

$$n_i - \left[k \cdot \frac{n_i}{n}\right] \cdot \frac{n}{k} = \frac{n}{k} \left(k \cdot \frac{n_i}{n} - \left[k \cdot \frac{n_i}{n}\right]\right).$$

Next, Greedy Monroe will assign the remaining seats to the parties in the order of decreasing values $k \cdot \frac{n_i}{n} - \left[k \cdot \frac{n_i}{n}\right]$, that is, it will proceed exactly as the largest remainders method. $\square$

Proposition A.6. In the general case (when $k$ does not have to divide $n$), Greedy Monroe and Monroe do not extend the largest remainders method.

Proof. Consider an apportionment instance with 2 parties with, respectively, 50 votes and 31 votes. Assume the committee size is $k = 4$. For this instance LRM gives 2 seats to each party. Greedy Monroe can proceed as follows. It starts by giving the second party a representative and removing the group of 21 voters. Next it can give 3 representatives to the first party (depending on tie-breaking). The Monroe rule can also select 3 candidates from the first party and one candidate from the second party. $\square$

Proposition A.7. Greedy Monroe satisfies justified representation (JR).

Proof. Sánchez-Fernández et al. [182] show that Greedy Monroe satisfies PJR if $k$ divides $n$, hence it also satisfies JR under this condition. However, Greedy Monroe satisfies JR also without this additional constraint. Assume towards a contradiction that Greedy Monroe fails JR for the election instance $(A, k)$ and let $W$ be the winning committee according to Greedy Monroe. As $W$ does not satisfy JR, there exists a group of voters $V$ of size at least $n/k$ and a candidate $c \notin W$ approved by all of them. Adding candidate $c$ would have increased the Monroe score of the committee by at least $n/k$ in all rounds. Hence, the candidates contained in $W$ also increased the score by at least $n/k$ each. Thus, $W$ has a Monroe score of $n$, i.e., all voters have an approved candidate in $W$, which implies that JR is satisfied. $\square$
Proposition A.8. An ABC rule cannot satisfy both perfect representation and weak Pareto efficiency.

Proof. Consider the profile

\[2 \times \{a, c\} \quad 1 \times \{a, c, d\} \quad 1 \times \{a, d\} \quad 1 \times \{b, d\} \quad 3 \times \{b, c\}\]

For \(k = 2\), there is exactly one committee that satisfies perfect representation: \(W_1 = \{a, b\}\). This committee, however, is dominated by \(W_2 = \{c, d\}\). An ABC rule \(R\) satisfies PR if it exclusively returns committees satisfying PR; hence \(W_1\) is the only winning committee and thus \(R\) fails weak Pareto efficiency.

Proposition A.9. The proportionality degree of Rule X is between \(\frac{\ell - 1}{2}\) and \(\frac{\ell + 1}{2}\). The proportionality degree of SAV and MAV is 0.

Proof. For SAV fix \(\ell \in \mathbb{N}\), set the committee size to \(k = 2\ell + 1\), and consider the following profile with \(m = 2k\) candidates and \(n = k\) voters: the first \(\ell\) voters approve candidates \(a_1, \ldots, a_k\) and the next \(k - \ell\) voters approve \(b_1, \ldots, b_k\). SAV will select the committee \(\{b_1, \ldots, b_k\}\). The group of the first \(\ell\) voters is \(\ell\)-cohesive, but no voter gets any representative in the elected committee.

For MAV fix \(\ell \in \mathbb{N}\), set the committee size to \(k = \ell + 1\), and consider the following profile with \(m = 4k + 1\) candidates and \(n = k\) voters: the first \(\ell\) voters approve candidates \(a_1, \ldots, a_k\) and the next voter approves \(b_1, \ldots, b_{3k+1}\). MAV will select a \(k\)-element subset of \(\{b_1, \ldots, b_{3k+1}\}\). The group of the first \(\ell\) voters is \(\ell\)-cohesive, but no voter gets any representative in the elected committee.

Finally, we consider Rule X. Since Rule X satisfies EJR [164] and EJR implies a proportionality degree of at least \(f(\ell) = \frac{\ell - 1}{2} [182]\), we get the lower-bound. For the upper bound consider the following instance. Fix \(\ell \in \mathbb{N}\). We set \(n = k = \frac{\ell(\ell + 1)}{2}\) and \(m = k + \ell\). The voters are divided into \(\ell\) groups \(N = N_1 \cup N_2 \cup \ldots \cup N_\ell\) such that \(|N_i| = i\) for each \(i \in [\ell]\). The set of the first \(k\) candidates is also divided into \(\ell\) groups \(C = C_1 \cup C_2 \cup \ldots \cup C_\ell\) such that \(|C_i| = i\) for each \(i \in [\ell]\). The set of remaining \(\ell\) candidates is denoted by \(A\). The voters from \(N_i\) approve \(C_i\). Additionally the first voter from each group \(N_i\) approves \(A\). Rule X can select the candidates from \(C_\ell\) first. Then the voters from \(N_\ell\) have no money left. Next the candidates from \(C_{\ell-1}\) are selected, etc. Consequently, Rule X can return committee \(C_1 \cup C_2 \cup \ldots \cup C_\ell\). Consider the voters who approve \(A\). They form an \(\ell\)-cohesive group, but the average number of representatives that they get equals \(1 + 2 + \ldots + \ell = \frac{\ell(\ell + 1)}{2}\). This completes the proof.