A class of block smoothers for multigrid solution of saddle point problems with application to fluid flow*

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Abstract. We design and analyse an iterative method, which uses a specific block smoother for the multigrid cycle. Among many possibilities we choose a few multigrid iterations as the smoother's blocks. The result is a multilevel procedure that works for regular saddle point problems and features all good properties of the classical multigrid for elliptic problems, such as the optimal complexity and convergence rate independent of the number of levels.

1 Introduction

In many applications, one needs to solve an ill-conditioned, large discrete saddle point problem with a block matrix

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}, \tag{1}$$

which is non-symmetric, indefinite and ill-conditioned. For example, after a linearization of the Navier–Stokes system one ends up with a huge linear system with such a nonsymmetric block matrix, which ill-conditioned with respect to the mesh size h. We propose and analyse a new multilevel method for solving the linear system, based on inner and outer multigrid iteration.

Multigrid schemes for saddle point problems have been considered by many authors before, see for example [2], [14], [13], [10], [15]. These methods have usually been designed with a very specific equation in mind. Some of these works, e.g. [2], stressed the necessity of using sufficiently strong smoothers in order to achieve satisfactory performance of the multigrid. On the other hand, the block nature of (1) promotes the development of preconditioners exploiting this structure of the problem. Block preconditioning has also attained a lot of attention from many authors, see e.g. [6], [3], [11], [7], [12].

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We propose a method that combines these two approaches: the multigrid and the block preconditioning. Similar combination has been used in e.g. [2], however our approach looks a bit more flexible and makes the choice of concrete preconditioner broader. We design and analyse an iterative method, which may choose from a variety of blocked approximate solvers. An interesting option is to use an inner multigrid cycle as a smoother inside the outer multigrid cycle, which results in a multilevel procedure that resembles to some extent the W-cycle method but has better properties. We treat each variable separately, following the block approach used for preconditioning. Our method works for saddle point problems such as the Stokes problem, and it features all good properties of the classical multigrid for elliptic problems, such as the optimal complexity and convergence in "natural" norms.

In this paper, we describe the blocked smoother and derive from it a specific inner-outer multigrid iteration which uses two inner multigrid iterations to apply the smoother. We discuss the performance of these methods, including convergence theorems and numerical results. The details of the theoretical analysis, which is based on a combination of the multigrid and block preconditioners theory, will be given elsewhere.

2 Blocked multigrid framework for saddle point problems

Let $\bar{V}, \bar{W}$ be real Hilbert spaces with scalar products denoted by $((\cdot,\cdot))$ and $(\cdot,\cdot)$, respectively. The corresponding induced norms are $\|\cdot\|$ and $|\cdot|$. Let us consider two continuous bilinear forms, $a(\cdot,\,\cdot) : \bar{V}\times\bar{V} \to R$ and $b(\cdot,\,\cdot) : \bar{V}\times\bar{W} \to R$ and assume that $a(\cdot,\,\cdot)$ satisfies

$$\exists\alpha > 0 \quad a(u,\,u) \geq \alpha\|u\|^2 \quad \forall u \in V^0 = \{v \in \bar{V} : b(v,\,q) = 0 \quad \forall q \in \bar{W}\}, \tag{2}$$

and there holds the *inf-sup* condition:

$$\exists\beta > 0 \quad \sup_{v\in\bar{V}, v\neq 0} \frac{b(v,\,q)}{\|v\|} \geq \beta|q| \quad \forall q \in \bar{W}. \tag{3}$$

We consider a family of nested finite element spaces $V_0 \times W_0 \subset V_1 \times W_1 \subset \ldots \subset V_K \times W_K \subset \bar{V} \times \bar{W}$, where every $V_{k+1} \times W_{k+1}$ is obtained from $V_k \times W_k$ through mesh refinement procedure, $h_{k+1} = \frac{1}{2}h_k$. These spaces inherit their norms from $\bar{V} \times \bar{W}$, but in practice, one additionally uses another (mesh-dependent) inner products and norms, denoted by $((\cdot,\cdot))_k$ and $\|\cdot\|_k$ in V_k and analogously in W_k. Later on, we shall use these auxiliary inner products to define certain linear operators in V_k and W_k. We shall also denote for short $X_k = V_k \times W_k$; for $x = (u,p)^T \in X_k$ and $y = (v,q)^T \in X_k$, we define the natural inner product in X_k by $\langle x, y\rangle = ((u,v)) + (p,q)$, and the discrete one $\langle (u,p)^T, (v,q)^T\rangle_k = ((u,v))_k + (p,q)_k$, with corresponding norms denoted by $|||\cdot|||$ and $|||\cdot|||_k$.

Additionally, we assume that a uniform discrete *inf-sup* condition holds for all levels k:

$$\exists \beta > 0 \quad \forall k = 1, \ldots, K \qquad \sup_{v \in V_k, v \neq 0} \frac{b(v,\, q)}{\|v\|} \geq \beta |q| \quad \forall q \in W_k. \tag{4}$$

In what follows, for nonnegative scalars x, y, we shall write $x \lesssim y$ if there exits a positive constant C, independent of x, y and the level k, such that $x \leq Cy$. Similarly, $y \gtrsim x$ is equivalent to $x \lesssim y$. Finally, $x \simeq y$ means $x \lesssim y$ and $y \lesssim x$ simultaneously.

On the kth level, we consider the following saddle point problem:

Problem 1. Find $(u_k, p_k) \in V_k \times W_k$ such that

$$\mathcal{M}_k \begin{pmatrix} u_k \\ p_k \end{pmatrix} \equiv \begin{pmatrix} A_k & B_k^* \\ B_k & 0 \end{pmatrix} \begin{pmatrix} u_k \\ p_k \end{pmatrix} = \begin{pmatrix} F_k \\ G_k \end{pmatrix}. \tag{5}$$

The finite dimensional space operators in (5) are discretizations on the kth level mesh of the corresponding differential operators, that is,

$$\begin{aligned} A_k &: V_k \to V_k, \quad ((A_k u, v))_k = a(u,\, v) \quad \forall u, v \in V_k, \\ B_k &: V_k \to W_k, \quad (B_k u, p)_k = b(u,\, p) \quad \forall u \in V_k, p \in W_k, \end{aligned}$$

B_k^* denotes the formal adjoint operator to B_k, i.e. $(B_k u, p)_k = ((u, B_k^* p))_k$ for all $u \in V_k$, $p \in W_k$.

We introduce four more operators. $L_k : V_k \to V_k$ and $M_k : W_k \to W_k$ define the correspondence between the original and auxiliary inner products in V_k and W_k,

$$\begin{aligned} ((L_k u, v))_k &= ((u, v)) \qquad \forall u, v \in V_k, \\ (M_k p, q)_k &= (p, q) \qquad \forall p, q \in W_k. \end{aligned} \tag{6}$$

Usually, systems with L_k and M_k are not easy to solve. Therefore, we will need two more operators, spectrally equivalent to L_k and M_k: $L_{0k} : V_k \to V_k$ and $M_{0k} : W_k \to W_k$. We assume that they are self-adjoint, their inverses are easier to apply than those of L_k and M_k, and that

$$((L_{0k} u, u))_k \simeq ((L_k u, u))_k \qquad \forall u \in V_k, \tag{7}$$

$$(M_{0k} p, p)_k \simeq (M_k p, p)_k \qquad \forall p \in W_k. \tag{8}$$

In other words, we shall always assume that L_{0k} and M_{0k} define good preconditioners for L_k and M_k. Later, it will be important to choose these preconditioners as multigrid cycles.

3 Examples

We consider two problems in the CFD that lead to saddle point problem formulation as in Problem 1. Let Ω be a bounded, open polygon in R^2.

Example 1. Linearized Navier-Stokes equation

A reasonable model for a linearization of the Navier-Stokes equations is the Oseen equation,

$$\begin{cases} -\nu\Delta u + (\omega\cdot\nabla)u + \nabla p = f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega. \end{cases}$$

This problem may be expressed as a saddle point problem for $(u,p) \in H^1_0(\Omega)\times L^2_0(\Omega)$, [8]. Choosing inf-sup stable finite element functions, it follows that L_k is the discrete Laplacian matrix, M_k is the discrete mass matrix. A_k corresponds to a finite element approximation of a convection-diffusion operator, while B_k approximates the divergence operator. Note that A_k is nonsymmetric and the condition number of the saddle point problem grows proportionally to h^{-2}, making the finite element Oseen equations ill-conditioned.

Example 2. Biharmonic equation

The Ciarlet-Raviart method for a first Dirichlet biharmonic problem [5] reads:

$$\begin{aligned} (\sigma,v)_{L^2(\Omega)} - (\nabla v,\nabla u)_{L^2(\Omega)} &= 0 \qquad \forall v\in H^1(\Omega), \\ -(\nabla\sigma,\nabla w)_{L^2(\Omega)} &= -(f,v)_{L^2(\Omega)} \qquad \forall w\in H^1_0(\Omega). \end{aligned} \tag{9}$$

Then, in our notation, A_k corresponds to the usual mass matrix, while L_k is a matrix corresponding to Helmholtz operator $-\Delta + I$ discretization, and M_k is the Laplacian $-\Delta$ (with boundary constraints) representation. Note that A_k is uniformly elliptic only on $\ker B_k$, while its global ellipticity constant decays proportionally to h. The condition of the saddle point problem matrix is proportional to h^{-4}.

4 Block smoothed multigrid method for Problem 1

In order to solve the k-th level problem,

$$\mathcal{M}_k x_k = g_k$$

where $x_k, g_k \in X_k = V_k \times W_k$, we use classical W-cycle multigrid scheme $\mathcal{MG}_k(x_k^0, g_k)$ with m pre- and post-smoothing iterations using smoother S_k, see e.g. [9] or [1] for details. Here, x_k^0 denotes the initial approximation to x_k. We recall the MG scheme briefly mainly for the notational purposes.

On the zeroth level, $k = 0$, we define $\mathcal{MG}_0(x_0^0, g_0) = \mathcal{M}_0^{-1} g_0$ (direct solve). For $k > 0$ we define $\mathcal{MG}_k$ recursively. First we apply m smoother iterations in the pre-smoothing step

for $i = 1, \ldots, m$
$x_k^j = x_k^{j-1} - S_k(\mathcal{M}_k x_k^{j-1} - g_k)$,

and then follow with the coarse grid correction: for $g_{k-1} \in X_{k-1}$ defined by the identity

$$\langle g_{k-1}, y \rangle k - 1 = \langle g_k - \mathcal{M}_k x_k^{j-1}, y \rangle_k \quad \forall y \in X_{k-1},$$

and we compute $\tilde{x}_{k-1}$ by applying two iterations of $(k-1)$ level method (with zero initial guess) to problem

$$\mathcal{M}_{k-1} \tilde{x}_{k-1} = g_{k-1},$$

so that $\tilde{x}_{k-1} = \mathcal{MG}_{k-1}(\mathcal{MG}_{k-1}(0, g_{k-1}), g_{k-1})$. Finally, we set

$$\mathcal{MG}_k(x_k^0, g_k) = x_k^m + \tilde{x}_{k-1}.$$

The key ingredient of the above procedure is of course the smoother. It should be easy to apply to a vector, and it should remove effectively high frequency components of the error. The simplest choice used in practice is the Richardson iteration; however, it turns out that sometimes more efficient smoothers are necessary, [2]. Therefore, in what follows we shall consider smoothers based on block preconditioned Richardson iteration.

We focus here on the W-cycle iteration, note however, that it is also possible to use other variants of the multigrid (see e.g. [1]), making use of such concepts as the V-cycle, the post-smoothing, or using smoothers other than the Richardson method.

We shall consider a block preconditioned Richardson smoother, that is,

$$S_k = \frac{1}{\omega_k} \mathcal{M}_{0k}^{-1} \mathcal{M}_k^* \mathcal{M}_{0k}^{-1}, \tag{10}$$

where $\omega_k > 0$ is a prescribed parameter and

$$\mathcal{M}_{0k} = \begin{pmatrix} L_{0k} & \\ & M_{0k} \end{pmatrix}. \tag{11}$$

Theorem 1. *Let x_k be the accurate solution of Problem 1 and let $\tilde{x}_k = \mathcal{MG}(x_k^0, g_k)$ be its approximation after one iteration of the kth level W-cycle method with m inner smoother iterations defined by (10) and with initial guess x_k^0. Then, for any $0 < \delta < 1$, there exists m large enough such that the multigrid iteration is convergent linearly with rate δ. The convergence rate is independent of k.*

5 Smoother based on inner multigrid

In order to solve Problem 1 on the kth level in $\mathcal{O}(N_k)$ floating point operations, where $N_k = \dim X_k$, we use the multigrid procedure $\mathcal{MG}_k$ described above, but with specific choice of the smoothing preconditioners L_{0k}, M_{0k}.

For $F_k \in V_k$ we define

$$L_{0k}^{-1} F_k = U_k, \tag{12}$$

where U_k is a result of one classical V-cycle multigrid applied to solve $L_k U_k = F_k$. Similarly, for $G_k \in W_k$,

$$M_{0k}^{-1} G_k = P_k, \tag{13}$$

where P_k is derived from one V-cycle multigrid for $M_k P_k = G_k$. In other words, the smoother S_k defined by (10) amounts to applying two kth level multigrid cycles to each variable separately, interlaced with multiplication by the transpose of $\mathcal{M}$.

The resulting procedure uses an inner multigrid cycle in an outer multigrid iteration. It also applies the outer multigrid to a squared preconditioned system, so that in one outer iteration, two inner multigrid cycles are performed, see Figure 1.

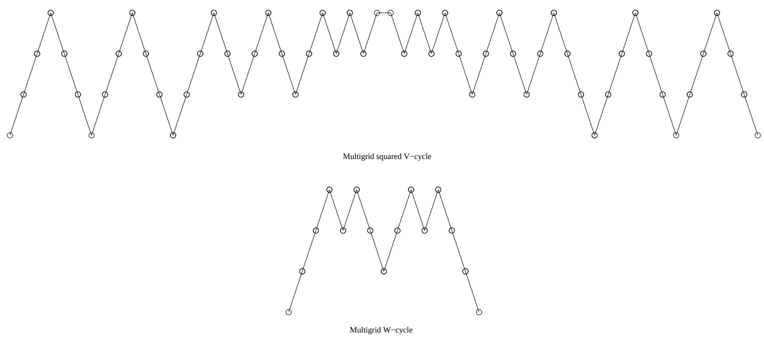

Fig. 1. The new scheme using a V-cycle inner and outer iteration (top), versus the usual W-cycle multigrid.

Theorem 2. *Under the above additional assumptions, and for sufficiently large number m of smoother iterations, the kth level MG iteration, consisting of the W-cycle multigrid for Example 2 with a smoother defined by* (10) *and with block solvers as above, is convergent. The convergence rate is independent of the level k and the arithmetic complexity of one iteration is $\mathcal{O}(N_k)$.*

6 Numerical experiments

Let us consider a saddle point problem which is an *ad hoc* modification of the Ciarlet-Raviart saddle point formulation of the first biharmonic equation. Since in its original form, the A matrix (the mass matrix in this case) is not uniformly H^1-elliptic with respect to the mesh size h [4], we replace this matrix with a matrix that corresponds to the discretization of the H^1 inner product. It is clear that after such a modification the uniform *inf-sup* condition remains to hold.

We report on the convergence factors of our block smoothed multigrid method in the following configuration: the outer iteration uses a 2-level multigrid V-cycle, with $m = 1, \ldots, 4$ pre- and postsmoothings. The inner (that is, the smoothing iteration) is either a direct solve or again a two-grid V-cycle, with k smoothings. The convergence factor is calculated as the mean value of $||r_{i+1}||/||r_i||$ in three consecutive iterations. The $|| \cdot ||$ norm is the usual Euclidean norm.

Table 1. Left: A block smoothed multigrid with a directly solved preconditioner; Right: A block smoothed multigrid with inner multigrid which is a 2-grid V-cycle.

	m			
N	1	2	3	4
9	0.76	0.58	0.44	0.34
17	0.76	0.58	0.44	0.33
33	0.76	0.58	0.44	0.34

	m			
N	1	2	3	4
9	0.91	0.85	0.79	0.73
17	0.89	0.79	0.71	0.65
33	0.87	0.77	0.68	0.60

An interesting observation, see Table 2, is that the method still works very well for the *original* Ciarlet-Raviart method with a compatible right hand side, despite the global ellipticity constant is proportional to h. Here we report on a two grid outer iteration, F = [0,rand(f)], with exactly solved block preconditioner. This suggests that it is only the V^0–ellipticity which controls the behaviour of the method under consideration.

Table 2. A block smoothed inner-outer V-cycle multigrid for the original Ciarlet-Raviart problem with random right hand side f. Exactly solved block preconditioner. Instead of the average, we report on the convergence factor on the 4-th iteration (as we obtained extremely good convergence factors in 3 previous iterations).

	m			
N	1	2	3	4
9	0.47	0.29	0.36	0.34
17	0.46	0.32	0.35	0.29
33	0.45	0.36	0.34	0.26

7 Conclusions

The new flexible multilevel scheme for saddle point problems makes efficient use of a block smoother. The method has optimal complexity $\mathcal{O}(N_k)$, where N_k is the kth level problem size, and the smoother error reduction is proportional to $\frac{1}{m}$.

The method can be applied to several saddle point problems encountered in the CFD, including Stokes, Oseen equations or the Ciarlet-Raviart method for the biharmonic problem and reuses simple multigrid schemes for *elliptic* problems in the saddle point problem context.

A potential drawback of the proposed scheme is its sensitivity to the ellipticity constant in the nonsymmetric case.

References

1. D. Braess, *Finite elements*, Cambridge University Press, Cambridge, 1997. Theory, fast solvers, and applications in solid mechanics, Translated from the 1992 German original by Larry L. Schumaker.
2. D. Braess and R. Sarazin, *An efficient smoother for the Stokes problem*, Appl. Numer. Math., 23 (1997), pp. 3–19. Multilevel methods (Oberwolfach, 1995).
3. J. Bramble and J. Pasciak, *A preconditioning technique for indefinite problems resulting from mixed approximation of elliptic problems*, Math. Comp., 50 (1988), pp. 1–17.
4. F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, Hong Kong, Barcelona, 1991.
5. P. G. Ciarlet and P.-A. Raviart, *A mixed finite element method for the biharmonic equation*, in Mathematical aspects of finite elements in partial differential equations (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1974), Math. Res. Center, Univ. of Wisconsin-Madison, Academic Press, New York, 1974, pp. 125–145. Publication No. 33.
6. E. D'yakonov, *On iterative methods with saddle operators*, Soviet Math. Dokl., 35 (1987), pp. 166–170.
7. H. C. Elman, D. J. Silvester, and A. J. Wathen, *Iterative methods for problems in computational fluid dynamics*, in Iterative methods in scientific computing (Hong Kong, 1995), Springer, Singapore, 1997, pp. 271–327. (also available as a University of Maryland report).
8. V. Girault and P. Raviart, *Finite Element Method for Navier–Stokes Equations. Theory and Algorithms*, Springer-Verlag, Berlin, Heidelberg, New York, 1986.
9. W. Hackbusch, *Multigrid methods and applications*, Springer-Verlag, Berlin, 1985.
10. M. R. Hanisch, *Multigrid preconditioning for the biharmonic Dirichlet problem*, SIAM J. Numer. Anal., 30 (1993), pp. 184–214.
11. A. Klawonn, *Preconditioners for Indefinite Problems*, PhD thesis, Universität Münster, Germany, 1996.
12. P. Krzyżanowski, *On block preconditioners for nonsymmetric saddle point problems*, SIAM J. Sci. Comp., 23 (2001), pp. 157–169.
13. P. Peisker, *A multilevel algorithm for the biharmonic problem*, Numer. Math., 46 (1985), pp. 623–634.
14. R. Verfürth, *A multilevel algorithm for mixed problems*, SIAM J. Numer. Anal., 21 (1984), pp. 264–271.
15. G. Wittum, *Multi-grid methods for Stokes and Navier-Stokes equations. Transforming smoothers: algorithms and numerical results*, Numer. Math., 54 (1989), pp. 543–563.