Selected Topics in Combinatorics, tutorial 8

Matchings with low crossing number, cutting lemma, Haussler's packing theorem

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Theorem 1. Let $\mathcal{S}$ be a set system on an n-point set X, n even, with $\pi^*_{\mathcal{S}}(m) \le Cm^d$ for all m, where C and $d > 1$ are constants. Then there exists a perfect matching M on X (i.e. a set of $\frac{n}{2}$ vertex-disjoint edges) whose crossing number is at most $C_1 n^{1-1/d}$, where $C_1 = C_1(C, d)$ is another constant.

Problem 1. Prove that any set system $(X, \mathcal{S})$ as in Theorem 1 admits a spanning path with crossing number $\mathcal{O}(n^{1-1/d})$, i.e. there is a path connecting all the points of X in some order such that any set $S \in \mathcal{S}$ crosses at most $\mathcal{O}(n^{1-1/d})$ edges of the path.

Problem 2. Prove that there exist 2n points in the plane such that for any perfect matching on them there is a line crossing at least $c\sqrt{n}$ edges. This means that Theorem 1 is asymptotically optimal for $d = 2$.

Lemma 1 (Cutting lemma)**.** Let L be a set of n lines in the plane, and let r be a parameter, $1 < r < n$. Then the plane can be subdivided into t generalized triangles (this means intersections of three half-planes) $\Delta_1, \Delta_2, \dots, \Delta_t$ in such a way that the interior of each Δ_i is intersected by at most $\frac{n}{r}$ lines of L, and we have $t \le Cr^2$ for a certain constant C independent of n and r.

Problem 3. Prove that the cutting lemma is asymptotically optimal for $r \to \infty$.

Problem 4. Prove a weaker version of the cutting lemma, with $t \le Cr^2(\log n)^2$.
Hint: Sample randomly $6r \log n$ lines from L and triangulate them.

Problem 5. Let L be a set of n lines in the plane in general position. We already know that they split the plane into set $\mathcal{R}$ of $\binom{n+1}{2} + 1$ regions. Define a function $d : \mathcal{R} \times \mathcal{R} \to \mathbb{R}_{\ge 0}$ in the following way. For any two regions $R_1, R_2 \in \mathcal{R}$ take points $x, y \in \mathbb{R}^2$ that belong to R_1 and R_2 respectively. Set $d(R_1, R_2)$ as the number of lines from L crossed by the open interval (x, y). Show that d is well defined and that it is a metric on $\mathcal{R}$.

Problem 6. Let L be a set of n lines in the plane in general position, let $x \in \mathbb{R}^2$ be a point, and let $r < \frac{n}{2}$ be a number. Show that the number of intersections of the lines of L lying at distance at most r from x (that is, intersections v such that the open interval (v, x) is intersected by at most r lines of L) is at least cr^2, with an absolute constant $c > 0$.

Problem 7. Let $(\mathbb{R}^2, \mathcal{F})$ be a set system of n half-planes in the general position. Prove Haussler's packing theorem for $\mathcal{F}^*$.