Selected Topics in Combinatorics, tutorial 7

Haussler's packing theorem, Szemerédi-Trotter theorem

April 20, 2023

Problem 1. Let $\mathcal{F}$ be a set system and let d be its VC-dimension. Prove that the Hamming graph $H_{\mathcal{F}}$ of $\mathcal{F}$ has edge density at most d.

Problem 2. Let L be a set of n lines in the plane in general position. We already know that they split the plane into set $\mathcal{R}$ of $\binom{n+1}{2}+1$ regions. Define a function $d : \mathcal{R} \times \mathcal{R} \to \mathbb{R}_{\geq 0}$ in the following way. For any two regions $R_1, R_2 \in \mathcal{R}$ take points $x, y \in \mathbb{R}^2$ that belong to R_1 and R_2 respectively. Set $d(R_1, R_2)$ as the number of lines from L crossed by the open interval (x, y). Show that d is well defined and that it is a metric on $\mathcal{R}$.

Problem 3. Let L be a set of n lines in the plane in general position, let $x \in \mathbb{R}^2$ be a point, and let $r < \frac{n}{2}$ be a number. Show that the number of intersections of the lines of L lying at distance at most r from x (that is, intersections v such that the open interval (v, x) is intersected by at most r lines of L) is at least cr^2, with an absolute constant $c > 0$.

Problem 4. Let $(\mathbb{R}^2, \mathcal{F})$ be a set system of n half-planes in the general position. Prove Haussler's packing theorem for $\mathcal{F}^*$.

Definition 1. Let G be a graph together with a planar embedding ρ. The *crossing number* of G and ρ (denoted $\mathrm{cr}(G, \rho)$) is the number of pairs of edges of G such that their images with ρ cross. The crossing number of a graph G (denoted $\mathrm{cr}(G)$) is the minimum crossing number of G together with a planar embedding.

Problem 5. Show that the crossing number of any simple graph $G = (V, E)$ is at least $|E| - 3|V|$.

Problem 6. Let $G = (V, E)$ be a simple graph. Show that:

$$\mathrm{cr}(G) \geq \frac{1}{64} \cdot \frac{|E|^3}{|V|^2} - |V|.$$

Hint: Fix $p \in (0, 1)$. Take a random subgraph of G by selecting every vertex with probability p and use Problem 5.

Theorem 1 (Szemerédi-Trotter)**.** For a set of lines L and a set of points P denote by $I(L, P)$ the number of pairs $(\ell, p) \in L \times P$ such that $p \in \ell$. For two integers n, m denote by $I(n, m)$ the maximum value of $I(L, P)$ where L is a set of n lines and P is a set of m points. Then $I(n, m) = \mathcal{O}(m^{2/3}n^{2/3} + m + n)$.

Problem 7. Prove Theorem 1.