Selected Topics in Combinatorics, tutorial 6

Sample compression schemes, minimax theorem, (p, q)-theorem

April 13, 2023

Problem 1. Let $\mathcal{F}$ be a set system with sample compression scheme of size k. Prove that $\dim(\mathcal{F}) < \infty$.

Problem 2. For a graph G define $L(G) = \max_H \frac{E(H)}{V(H)}$, where H ranges over all subgraphs of G (thus $L(G)$ is a half of the maximim values of the average degree of a subgraph of G). Prove that $L(G) \leq d$ if and only if we can orient edges of G so that every vertex has outdegree at most d.

Lemma 1 (Farkas lemma)**.** Consider systems of linear inequalities of the form

$$P_1(\bar{x}) \geq 0, \ldots, P_m(\bar{x}) \geq 0$$

where $\bar{x} \in \mathbb{R}^n$ and $P_1, \ldots, P_m : \mathbb{R}^n \to \mathbb{R}$ are linear functions. Then exactly one of the statements is true:

1. there is a solution $\bar{x} \in \mathbb{R}^n$ to the system; or

2. there exist non-negative reals $q_1, \ldots q_n$ such that $q_1 P_1 + \ldots q_m P_m = -1$.

Problem 3. Prove Farkas lemma.
Hint: Proceed by induction on n.

Theorem 1 (Minimax Theorem)**.** Let $E \subseteq A \times B$ be a binary relation with A, B finite, and let $\alpha \in \mathbb{R}$. Then exactly one of the following cases holds:

1. there is some probability distribution ν on B such that $\nu(E(a)) \geq \alpha$ holds for every $a \in A$; or

2. there is some probability distribution μ on A such that $\mu(E^*(b)) < \alpha$ holds for every $b \in B$.

Problem 4. Prove Minimax Theorem

Problem 5. Prove that if $\mathcal{F}$ is a finite family of disks in the plane such that every two members of $\mathcal{F}$ intersect, then $\tau(\mathcal{F})$ is bounded by a constant.

Problem 6. Show that for every $p \geq 2$ there is p' such that if a family of disks in the plane has the $(p, 2)$-property then it has the $(p', 3)$-property.
Hint: Use Problem 5 and Ramsey theorem.

Problem 7. Show that $\tau(\mathcal{F}) \leq \tau^*(\mathcal{F}) \cdot \ln(|\mathcal{F}| + 1)$ for all finite set systems $\mathcal{F}$.
Hint: Choose transversal as a random sample.