Selected Topics in Combinatorics, tutorial 5

Transversals, packings, (p, q)-theorem

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Theorem 1 (Hall's theorem)**.** If G is a bipartite graph with parts A and B such that every subset $S \subseteq A$ has at least $|S|$ neighbors in B then there is a matching in G containing all vertices of A.

Theorem 2 (König's theorem)**.** Let G be a bipartite graph and let $\mathcal{F}$ be the set system of edges of G. Then $\tau(\mathcal{F}) = \nu(\mathcal{F})$.

Problem 1. Derive König's theorem from Hall's theorem (and reversely).

Problem 2.

1. Let $X = [m]$ and $\mathcal{F} = \binom{[m]}{n}$. Show that $\tau = m - n + 1$ while $\tau^* = m/n$. Thus for $m = 2n$ we have $\tau = n + 1$ while $\tau^* = 2$.
2. Similarly, find a family of set systems where ν is bounded by a constant, while ν^* is arbitrarily large.

Problem 3. Show that $\tau(\mathcal{F}) \leq \tau^*(\mathcal{F}) \cdot \ln(|\mathcal{F}| + 1)$ for all finite set systems $\mathcal{F}$.

Hint: Choose transversal as a random sample.

Problem 4. For which values of p and r does the following hold? Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$, and suppose that any subfamily consisting of at most p sets can be pierced by at most r points. Then $\mathcal{F}$ can be pierced by at most C points, for some $C = C(p, r)$.

Problem 5. Let $X \subset \mathbb{R}^2$ be a $(4k + 1)$-point set, and let $\mathcal{F} = \{\text{conv}(Y) \colon Y \subset X, |Y| = 2k + 1\}$. Verify that $\mathcal{F}$ has the $(4, 3)$-property.

Problem 6. Prove that if $\mathcal{F}$ is a finite family of disks in the plane such that every two members of $\mathcal{F}$ intersect, then $\tau(\mathcal{F})$ is bounded by a constant.

Problem 7. Show that for every $p \geq 2$ there is p' such that if a family of disks in the plane has the $(p, 2)$-property then it has the $(p', 3)$-property.

Hint: Use Problem 6 and Ramsey theorem.