# Selected Topics in Combinatorics, tutorial 4

Neural nets, PAC-learning, transversals, packings

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**Problem 1.** Consider multilayered feedforward neural nets with a single output which use sgn(x) as an activation function (i.e. every neuron computes a linear threshold function of it inputs). Prove that any fixed architecture with $n_0$ inputs and $\omega$ parameters the class $\mathcal{C}$ of function that can be obtained by instantiating these parameters satisifies:

$$\mathrm{VC}(\mathcal{C}) \leq 2\omega \log_2(e\omega).$$

**Problem 2.** Consider the space $\mathbb{R}^2$ and the concept class $\mathcal{C}$ of all the rectangles in the plane with sides parallel to the coordinate axes. Prove that $\mathcal{C}$ is PAC-learnable with the following learner: given a sample $S$ of points in $\mathbb{R}^2$ output the smallest rectangle from $\mathcal{C}$ that is consistent with $S$.

**Problem 3.** Prove that if the concept class $\mathcal{C}$ has infinite VC-dimension then it is not PAC-learnable.

**Hint:** Let $X = \{1, \ldots, 2m\}$ and $\mathcal{C} = 2^X$. Show that $\mathcal{C}$ is not PAC-learnable for $\varepsilon = \delta = 0.1$ using $m$ samples.

**Problem 4.** Let $\mathcal{F}$ be a set system of finitely many closed intervals on the real line. Prove that $\tau(\mathcal{F}) = \nu(\mathcal{F})$.

**Theorem 1** (Hall's theorem)**.** If $G$ is a bipartite graph with parts $A$ and $B$ such that every subset $S \subseteq A$ has at least $|S|$ neighbors in $B$ then there is a matching in $G$ containing all vertices of $A$.

**Theorem 2** (König's theorem)**.** Let $G$ b a bipartite graph and let $\mathcal{F}$ be the set system of edges of $G$. Then $\tau(\mathcal{F}) = \nu(\mathcal{F})$.

**Problem 5.** Derive König's theorem from Hall's theorem (and reversely).

**Problem 6.**

1. Let $X = [m]$ and $\mathcal{F} = \binom{[m]}{n}$. Show that $\tau = m - n + 1$ while $\tau^* = m/n$. Thus for $m = 2n$ we have $\tau = n + 1$ while $\tau^* = 2$.
2. Similarly, find a family of set systems where $\nu$ is bounded by a constant, while $\nu^*$ is arbitrarily large.

**Problem 7.** Show that $\tau(\mathcal{F}) \leq \tau^*(\mathcal{F}) \cdot \ln(|\mathcal{F}| + 1)$ for all finite set systems $\mathcal{F}$.

**Hint:** Choose transversal as a random sample.