Selected Topics in Combinatorics, tutorial 14

Permutations

June 15, 2023

Problem 1. Let $\sigma \in S_n$ satisfy $312 \not\preceq \sigma$. Show that σ can be obtained from id_n using a stack.

Problem 2. For a permutation $\sigma \in S_n$ by $\mathrm{rev}(\sigma)$ we denote its reverse permutation, i.e. the permutation obtained from σ by reversing its string representation (e.g. $\mathrm{rev}(52413) = 31425$). Show that $|\operatorname{Av}_m(\sigma)| = |\operatorname{Av}_m(\mathrm{rev}(\sigma))|$ for any $\sigma \in S_n$.

Problem 3. Show that $|\operatorname{Av}_m(\sigma)| = |\operatorname{Av}_m(\sigma^{-1})|$ for any $\sigma \in S_n$.

Problem 4. Show that $|\operatorname{Av}_m(123)| = C_m$, where C_m denotes m'th Catalan number.

Theorem 1 (Knuth-MacMahon)**.** For every permutation $\sigma \in S_3$ and every $m \in \mathbb{N}$ we have $\operatorname{Av}_m(\sigma) = C_m$.

Problem 5. Deduce Theorem 1 from previous exercises.

Theorem 2 (Erdös-Szekeres)**.** Let $r, s, n \in N$ satisfy $n \geq (r-1)(s-1)+1$. Show that for every $\sigma \in S_n$ either $\mathrm{id}_r \preceq \sigma$ or $\mathrm{rev}(\mathrm{id}_s) \preceq \sigma$.

Problem 6. Prove Theorem 2.

Definition 1. Let $\mathcal{C}$ be a class of graphs. We say that $\mathcal{C}$ has bounded sparse twin-width if there is a constant t such that for every $G \in \mathcal{C}$ there is an order $\preceq$ on $V(G)$ such that $M_{\preceq}(G)$ does not contain t-grid-minor.

Problem 7. Let $\mathcal{C}$ be a class of graphs of bounded sparse twin-width. Show that $\mathcal{C}$ has a linear neighbourhood complexity, i.e. there is a constant c such that for every $G \in \mathcal{C}$ and every $A \subseteq V(G)$ vertices of G have at most $c|A|$ different neighbourhoods in A.