Selected Topics in Combinatorics, tutorial 13

Stability

June 1, 2023

Definitions (partial types). A *partial type* τ in a graph G is a pair of two disjoint subsets (τ_+, τ_-) of $V(G)$. By $|\tau|$ we denote $|\tau_+| + |\tau_-|$.

We say that a partial type σ is a *subtype* of a partial type τ, and write $\sigma \subseteq \tau$, if $\sigma_- \subseteq \tau_-$ and $\sigma_+ \subseteq \tau_+$. Partial types are ordered by the relation $\subseteq$, and we can therefore say that a partial type σ' is a *successor* of σ if σ' is a least partial type such that $\sigma \subsetneq \sigma'$.

A partial type τ such that τ_+ and τ_- form a partition of $V(G)$ is a *complete type*. The *complete type of an element* $a \in V(G)$ is the complete type τ such that $\tau_+ = \{b \in V(G) \colon ab \in E(G)\}$. If ℓ is a number, we say that a partial type τ is *ℓ-consistent* if every subtype $\sigma \subseteq \tau$ with $|\sigma| < \ell$, is a subtype of a type of some $a \in V(G)$.

If σ is a partial type and $b \in V(G)$, then by $\sigma + b$ we denote the partial type $(\sigma_+ \cup \{b\}, \sigma_-)$, which is only defined when $b \notin \sigma_-$, and by $\sigma - b$ we denote the partial type $(\sigma_+, \sigma_- \cup \{b\})$, which is only defined when $b \notin \sigma_+$.

Fix a parameter $\ell \in \mathbb{N}$. The *ℓ-rank of a partial type* τ (the graph G is implicit), denoted $R_\ell(\tau)$, is defined inductively:

- $R_\ell(\tau) = -1$ if and only if τ is not ℓ-consistent;
- $R_\ell(\tau) \geq 0$ if and only if τ is ℓ-consistent;
- $R_\ell(\tau) \geq n+1$ if and only if there is some $b \in V(G)$ such that $R_\ell(\tau + b) \geq n$ and $R_\ell(\tau - b) \geq n$.

By $R_\ell(G)$ we denote $R_\ell(\tau)$ where τ is the empty type (i.e. $\tau = (\emptyset, \emptyset)$). If $\mathcal{C}$ is a class of graphs then by $R_\ell(\mathcal{C})$ we denote the supremum of all values $R_\ell(G)$, for all $G \in \mathcal{C}$.

For an ℓ-consistent partial type τ we call a subtype $\sigma \subseteq \tau$ *robust* (with respect to τ) if for every subtype σ' which is a successor of σ and is a subtype of τ, we have $R_\ell(\sigma') = R_\ell(\sigma)$.

Problem 1. Show that if $\tau \subseteq \sigma$ then $R_\ell(\tau) \geq R_\ell(\sigma)$.

Problem 2. Show that for given ℓ, n and $\rho_+, \rho_- \subseteq V(G)$ the property $R_\ell(\rho) \geq n$ is expressible by an FO formula that uses vertices from ρ_+, ρ_- as parameters.

Problem 3. Let $\mathcal{C}$ be a class of graphs with ladder index smaller than ℓ. Show that $R_\ell(\mathcal{C}) \leq f(\ell)$ for some function f.

Problem 4. Let $G \in \mathcal{C}$ such that the ladder index of $\mathcal{C}$ is smaller than ℓ. Show that if τ is an ℓ-consistent partial type then there is a robust partial type $\rho \subseteq \tau$ with $|\rho| \leq R(G)$.

Problem 5. Let τ be a complete, ℓ-consistent type and let $\rho \subseteq \tau$ be a robust partial type. Show that $b \in \tau_+$ if and only if $R_\ell(\rho + b) = R_\ell(\rho)$.

Problem 6 (Definability of types)**.** Let $\mathcal{C}$ be a class of graphs with ladder index less than ℓ. Show that for every graph $G \in \mathcal{C}$ and every complete, ℓ-consistent type τ there exists a formula $\varphi(y)$ with at most ℓ parameters from $V(G)$ such that for any $b \in V(G)$ we have $b \in \tau_+$ if and only if $G \models \varphi(b)$.

Problem 7. Let G be a graph with ladder index ℓ. Show that for any two sets $A, B \subseteq V(G)$ at least one of the followings holds:

- there is a subset $|A_0| \subseteq A$ of size at most ℓ which dominates B;
- there is a subset $|B_0| \subseteq B$ of size at most ℓ which antidominates A.

Problem 8. Show that a Boolean combination of stable edge relations is stable.

Definition (Flip). We say that a graph G' is a *flip* of a graph G if there is a subset $A \subseteq V(G)$ and $E(G') = E(G)$ xor $A \times A$.

Problem 9. Show that a flip of stable graph is stable.