Selected Topics in Combinatorics, tutorial 11

Szemerédi regularity lemma, property testing

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Problem 1. Let (A, B) be a pair of disjoint subsets of vertices of a graph G. Show that if (A, B) is ε-homogenous (i.e. $d(A, B) \leq \varepsilon$ or $d(A, B) \geq 1 - \varepsilon$) then (A, B) is ε^3-regular.

Theorem 1 (Szemerédi regularity lemma)**.** For any positive integer m_0 and real number $\varepsilon > 0$, there exists an integer $M \geq m_0$ such that the following holds. Every graph G with at least m_0 vertices has a partition $V_0, V_1, \ldots, V_m$ of $V(G)$ with $m_0 \leq m \leq M$ such that:

- $|V_0| \leq \varepsilon |V(G)|$;
- $|V_1| = |V_2| = \ldots = |V_m|$;
- for all but at most εm^2 values of $1 \leq i < j \leq m$, the pair (V_i, V_j) is not ε-regular.

Theorem 2 (Triangle removal lemma)**.** For every $0 < \alpha \leq 1$, there exists $\beta > 0$ and n_0 such that if G is a graph with $n \geq n_0$ vertices, then either

- G contains at least βn^3 triangles, or
- there exists a set $X \subseteq E(G)$ such that $|X| \leq \alpha n^2$ and $G - X$ contains no triangles.

Theorem 3 (Mantel)**.** If a graph G on n vertices contains no triangle then it contains at most $\frac{n^2}{4}$ edges.

Problem 2. Prove Theorem 3.

Problem 3. Let $\delta > 0$. Show that there exists n_0 such that, for $n \geq n_0$, any subset A of $[n]^2$ with at least δn^2 elements must contain a triple of the form $(x, y), (x + d, y), (x, y + d)$ with $d \neq 0$.

Problem 4. Solve Problem 3 with the additional requirement, that $d > 0$.

Theorem 4 (Roth)**.** For all $\delta > 0$ there exists n_0 such that, for $n \geq n_0$, any subset A of $[n]$ with at least δn elements contains an arithmetic progression of length 3.

Problem 5. Prove Theorem 4.

Problem 6. Prove that the class of half-graphs does not satisfy Theorem 1 if we additionally want all the pairs (V_i, V_j) for $1 \leq i < j \leq m$ to be ε-regular.

Definition 1. A list of length n is ε-close to sorted if one can remove εn its elements to obtain a sorted list.

Problem 7. Give an algorithm that works in time $\mathcal{O}(\varepsilon^{-1} \log n)$ and decides if a given list is sorted with one-side error for lists that are not ε-close to sorted. More precisely, the algorithm should correctly identify sorted lists and if the list is not ε-close to sorted the algorithm should identify that with probability at least 3/4.

Definition 2. A graph on n vertices is ε-close to triangle-free if one can remove εn^2 its edges to obtain a triangle-free graph.

Problem 8. Give an algorithm that works in time $\mathcal{O}_\varepsilon(1)$ and decides if a given graph is triangle-free with one-side error for graphs that are not ε-closed to traingle-free.