Selected Topics in Combinatorics, tutorial 10

Range minimum queires, Szemerédi regularity lemma

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Problem 1. Give a data structure that for a given tree T on n vertices can be constructed in time $\mathcal{O}(n)$ and then answers queries about lowest common ancestor of any given two vertices $u, v \in V(T)$ in time $\mathcal{O}(1)$.

Problem 2. Give a recurrence relation for the number of different Cartesian trees for arrays of length n.

Problem 3. Let G be a graph and $A, B \subseteq V(G)$ be two disjoint subsets of its vertices. Suppose for every $X \subseteq A, Y \subseteq B, X, Y \neq \emptyset$ we have

$$|d(A,B) - d(X,Y)| \leq 0$$

(something like 0-regularity). What can you say about the graph induced by $A \cup B$?

Problem 4. Let G be a graph and let (A, B) be an ε-regular pair in G for some $0 < \varepsilon \leq 1$. Let $B_0 \subseteq B$ have size at least $\varepsilon|B|$. Then:

- the number of vertices of A with more than $(d(A,B)+\varepsilon)|B_0|$ neighbors in B_0 is less than $\varepsilon|A|$, and
- the number of vertices of A with less than $(d(A,B)-\varepsilon)|B_0|$ neighbors in B_0 is less than $\varepsilon|A|$.

Problem 5. Let (A, B) be an ε-regular pair with $d(A, B) = d$, and let $\alpha > \varepsilon$. If $X \subseteq A, |X| \geq \alpha|A|$, and $Y \subseteq B, |Y| \geq \alpha|B|$, then (X, Y) is an ε'-regular pair, where $\varepsilon' = \max(\frac{\varepsilon}{\alpha}, 2\varepsilon)$, and $d(X, Y) = d'$ for some d' with $|d' - d| \leq \varepsilon$.

Problem 6. Let $|A| = |B| = |C| = n$, let (A, B), (B, C), (C, A) be three ε-regular pairs, for some $\varepsilon \in (0, 1/2]$. Assume that $d(A, B), d(B, C), d(C, A) \geq 2\varepsilon$. Let $t = t(A, B, C)$ be the number of triangles with one vertex in A, another in B and the third in C. Then

$$t \geq (1-2\varepsilon)(d(A,B)-\varepsilon)(d(B,C)-\varepsilon)(d(C,A)-\varepsilon)n^3.$$

Theorem 1 (Triangle removal lemma)**.** For every $0 < \alpha \leq 1$, there exists $\beta > 0$ and n_0 such that if G is a graph with $n \geq n_0$ vertices, then either

- G contains at least βn^3 triangles, or
- there exists a set $X \subseteq E(G)$ such that $|X| \leq \alpha n^2$ and $G - X$ contains no triangles.

Problem 7. Prove Theorem 1.
Hint: Apply Szemerédi regularity lemma to G. Then delete small number of edges so either all triangles are removed or we can see a configuration as in Problem 6.