Selected Topics in Combinatorics, tutorial 1

VC-dimension

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Definition 1. Let $(X, \mathcal{F})$ be a set system. The dual set system to $(X, \mathcal{F})$ is defined as follows: The ground set is $Y = \{Y_S \colon S \in \mathcal{F}\}$, where the Y_S are pairwise distinct points, and for each $x \in X$ we have the set $\mathcal{F}^* = \{Y_S \colon S \in \mathcal{F}, x \in S\}$ (the same set may be obtained for several different x, but this does not matter for the VC-dimension).

Problem 1. Let $(X, \mathcal{F})$ be a set system and let $\mathcal{F}^*$ be its dual set system. Prove that $\mathrm{vc}(\mathcal{F}^*) < 2^{\mathrm{vc}(\mathcal{F})+1}$. Is this bound optimal?

Problem 2. Let $\mathcal{F}$ be the family of half-planes in $\mathbb{R}^2$ and let $\mathcal{F}^*$ be its dual set system. Prove that $\pi_{\mathcal{F}^*}(m) = \binom{m+1}{2} + 1$.

Definition 2. For a graph G, let $\mathcal{N}(G) = \{N_G(v) \colon v \in V(G)\}$ be the system of vertex neighborhoods (where $N_G(v) = \{u \in V(G) \colon (u, v) \in E(G)\}$).

Problem 3. Prove that there is a constant d_0 such that $\mathrm{vc}(\mathcal{N}(G)) \leq d_0$ for all planar G.

Problem 4. Let G be a $K_{t,t}$-free graph (i.e. a graph that excludes $K_{t,t}$ as a subgraph) for some $t > 1$. Prove that $\pi_{\mathcal{N}(G)}(s) \leq s^t$ for $s > 4$.

Theorem 1 (Radon's theorem)**.** Let A be a set of $d+2$ points in $\mathbb{R}^d$. Then there exist two disjoint subsets $A_1, A_2 \subset A$ such that

$$\mathrm{conv}(A_1) \cap \mathrm{conv}(A_2) \neq \emptyset.$$

Problem 5. Prove Radon's theorem.

Problem 6. Show that the VC-dimension of the set system of all half-spaces in $\mathbb{R}^d$ equals $d+1$.

Problem 7. Let $\mathbb{R}[x_1, x_2, \ldots, x_d]_{\leq D}$ denote the set of all real polynomials in d variables of degree at most D, and let

$$\mathcal{P}_{d,D} = \{\{x \in \mathbb{R}^d \colon p(x) \geq 0\} \colon p \in \mathbb{R}[x_1, x_2, \ldots, x_d]_{\leq D}\}.$$

Show that $\mathrm{vc}(\mathcal{P}_{d,D}) \leq \binom{d+D}{d}$.

Hint: Consider the following trick known as *Veronese mapping*: Let M be the set of all possible non-constant monomials of degree at most D in $x_1, x_2, \ldots, x_d$. For example, for $D = d = 2$, we have $M = \{x_1, x_2, x_1x_2, x_1^2, x_2^2\}$. Let $m = |M|$ and consider the map $\varphi : \mathbb{R}^d \to \mathbb{R}^m$ which evaluates every point in $\mathbb{R}^d$ on all the monomials from M. For example, for $D = d = 2$, the map is

$$\mathbb{R}^2 \ni (x_1, x_2) \mapsto (x_1, x_2, x_1x_2, x_1^2, x_2^2) \in \mathbb{R}^5.$$

Show that if $A \subset \mathbb{R}^d$ is shattered by $\mathcal{P}_{d,D}$, then $\varphi(A)$ is shattered by half-spaces in $\mathbb{R}^m$.

Problem 8. Show that the family of balls in $\mathbb{R}^d$ has VC-dimension at most $d+2$.

Problem 9. Show that the unit square cannot be expressed as $\{(x, y) \in \mathbb{R}^2 : p(x, y) \geq 0\}$ for any polynomial $p(x, y)$.

Problem 10. Let $F(X_1, X_2, \ldots, X_k)$ be a fixed set-theoretic expression (using the operations of union, intersection, and difference) with variables $X_1, X_2, \ldots, X_k$ standing for sets; for instance,

$$F(X_1, X_2, X_3) = (X_1 \cup X_2 \cup X_3) \setminus (X_1 \cap X_2 \cap X_3).$$

Let $\mathcal{S}$ be a set system on a ground set X with $\mathrm{vc}(\mathcal{S}) = d < \infty$. Let

$$\mathcal{T} = \{F(S_1, S_2, \ldots, S_k) : S_1, S_2, \ldots, S_k \in \mathcal{S}\}.$$

Prove that $\mathrm{vc}(\mathcal{T}) = \mathcal{O}(kd \log k)$.

Hint: Show that for any set A we have $F(S_1, S_2, \ldots S_k) \cap A = F(S_1 \cap A, S_2 \cap A, \ldots, S_k \cap A)$.