Distal combinatorial tools for graphs of bounded twin-width

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Twin-width

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We will define a different parameter, *mixed-width*, which is functionally equivalent to twin-width.

Mixed width of a matrix

Horizontal matrix – all rows are equal:

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Mixed width of a matrix

Horizontal matrix – all rows are equal:

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Vertical matrix – all collumns are equal:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Mixed width of a matrix

Mixed matrix – a matrix that is neither vertical nor horizontal:

$$\begin{pmatrix} 1 & 1 & \color{red}{0} & \color{red}{1} & 1 \\ 1 & 1 & \color{red}{0} & \color{red}{0} & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

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A matrix is mixed if and only if it has a 2×2 mixed submatrix (corner).

Mixed width of a matrix

$$\left[\begin{array}{cc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

Figure: A 3-mixed minor on a matrix: no zone is horizontal or vertical. (Example from *Bonnet et al. Twin-width I: tractable FO model checking*)

Mixed width of a matrix

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Mixed-width of a matrix M is the smallest k such that M is k-mixed free.

Mixed width of a graph

For a given graph G and a total order σ on its vertices we denote by $M_\sigma(G)$ adjacency matrix of G in order σ.

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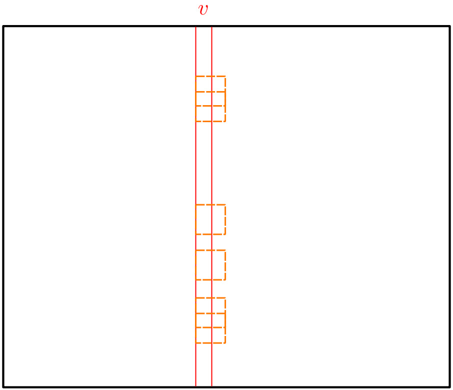
Theorem (Bonnet, Kim, Thomassé, Watrigant, '20)

If G is a graph of twin-width less than t, then there is a total ordering on its vertices σ such that $M_\sigma(G)$ is $(2t+2)$-mixed free. On the other hand, if G is a graph and σ is a total ordering on its vertices such that $M_\sigma(G)$ is k-mixed free, then $\mathrm{tww}(G) = 2^{2^{O(k)}}$.

Encoding vertices with many corners

Assume that $M_\sigma(G)$ is t-mixed free and in the column of some vertex v there are at least $2t$ corners.



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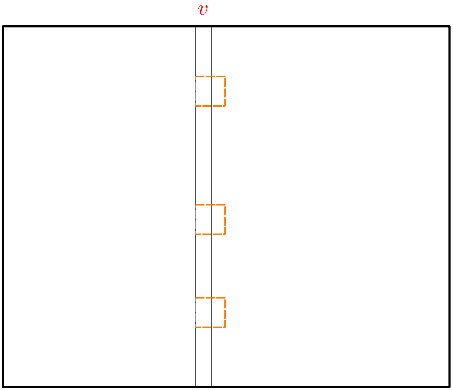

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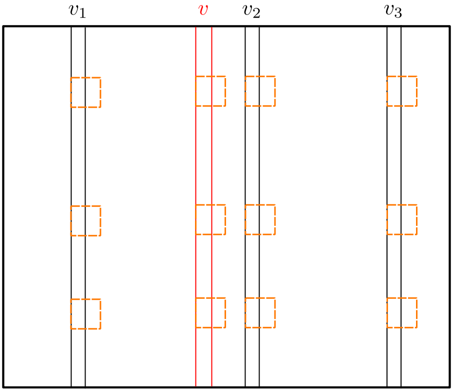

Edge-distality

Definition (Very informal)

A class of structures $\mathcal{C}$ is *edge-distal* if for every $G \in \mathcal{C}$ and every $A \subseteq V(G)$ every neighbourhood class with respect to A (i.e. set of vertices with the same closed neighbourhood in A) can be defined by a small first-order formula that can use a small number of parameters from A.

Main theorem

Theorem (P., '23)

Let $\mathcal{C}$ be a class of graphs of twin-width bounded by t. Then we can add a total order to every graph in $\mathcal{C}$ thus obtaining a class of ordered graphs $\hat{\mathcal{C}}$ which is edge-distal and it is witnessed by a set of small formulas that use $\mathcal{O}(t)$ parameters from A.

Distal tools

Using previous results of Chernikov, Simon and Starchenko on the notion of distality we can derive from the encoding theorem strong combinatorial tools for graphs of bounded twin-width.

Distal tools

Theorem (Distal cutting lemma)

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Let G be a graph of twin-width at most t, $A \subseteq V(G)$ be a subset of vertices of G of size n and take any real $1 \leq r \leq n$.

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We can partition the vertices of $V(G)$ into at most $Cr^{O(t)}$ sets $X_1, \ldots, X_l$ such that the vertices in every X_i have almost the same neighborhood in A.

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More precisely, there are at most $\frac{n}{r}$ vertices $a \in A$ for which there are $u, v \in X_i$ with $(u, a) \in E(G)$ and $(v, a) \notin E(G)$.

Distal tools

Theorem (Distal regularity lemma)

For every integer t there exists a constant $c = c(t)$ such that: for every $\varepsilon > 0$ and for every graph $G = (V, E)$ of twin-width at most t, there exists a partition $V = V_1 \cup \ldots \cup V_k$ into non-empty sets, and a set $\Sigma \subseteq [k] \times [k]$ with the following properties.

1. *Polynomially bounded size of the partition: $k = O((\frac{1}{\varepsilon})^c)$.*
2. *Few exceptions: $|\bigcup_{(i,j)\in\Sigma} V_i \times V_j| \geq (1-\varepsilon)|V|^2$.*
3. *$0-1$-regularity: for all $(i,j) \in \Sigma$ there are either all edges between V_i and V_j or no edge at all.*

Thank you for your attention!