Canonical Decompositions in Monadically Stable and Bounded Shrubdepth Graph Classes

Pierre Ohlmann, Michał Pilipczuk, Wojciech Przybyszewski, Szymon Toruńczyk

ICALP 2023

Splitter Game

The radius-r Splitter game is played on a graph G_1. In round i

1. Splitter chooses a vertex v to delete
2. Localizer chooses G_{i+1} as a radius-r ball in $G_i - v$.

Splitter wins once G_i has size 1.

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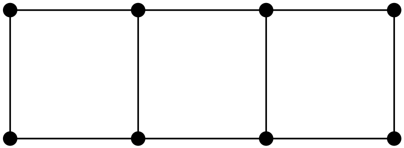

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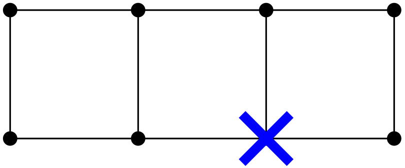

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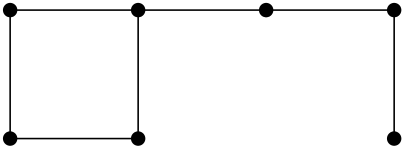

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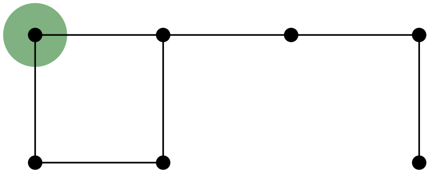

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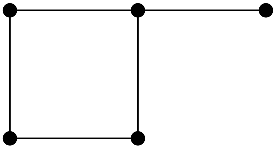

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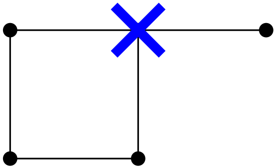

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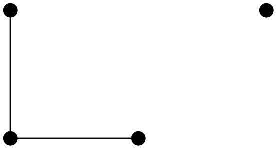

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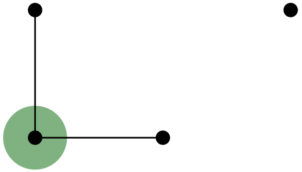

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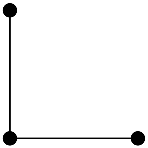

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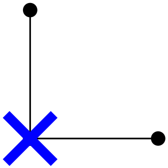

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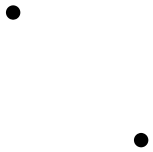

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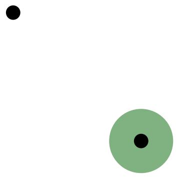

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Observation: We don't need to assume that G is a finite graph for this definition to make sense.

Progressing moves in the Splitter game

Theorem. [Ohlmann, Pilipczuk, P., Toruńczyk, 2023]

For every nowhere dense class of graphs $\mathscr{C}$ and every $r \in \mathbb{N}$ there exists a constant c such that for every $G \in \mathscr{C}$ Splitter has at most c progressing moves in the radius-r Splitter game on G.

Progressing moves in infinite graphs

Lemma.

For every infinite graph G and $r \in \mathbb{N}$ if Splitter can win the radius-r Splitter game on G in d rounds, then he has finitely many progressing moves.

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Assume statement doesn't hold. Denote $V(G) = \{v_i : i \in I\}$.

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- v_∞ is a progressing move.

□

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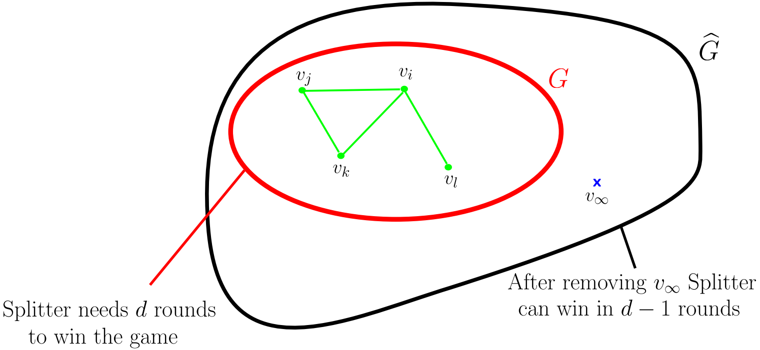

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Theorem. [Ohlmann, Pilipczuk, P., Toruńczyk, 2023]

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Assume the statement is not true. Consider the following theory:

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Compactness yields a model that contradicts the previous lemma. □

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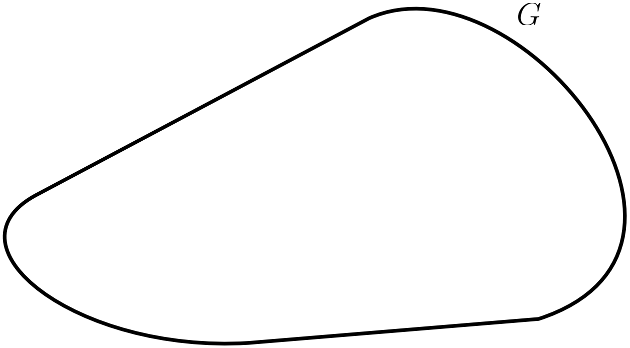
Fact.

There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph G of treedepth d there exists at most $f(d)$ vertices v such that treedepth of every connected component of $G - \{v\}$ is at most $d - 1$.

Cannonical decomoposition of graphs of bounded treedepth



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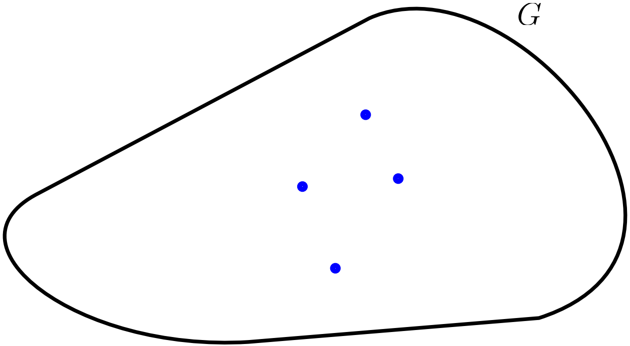

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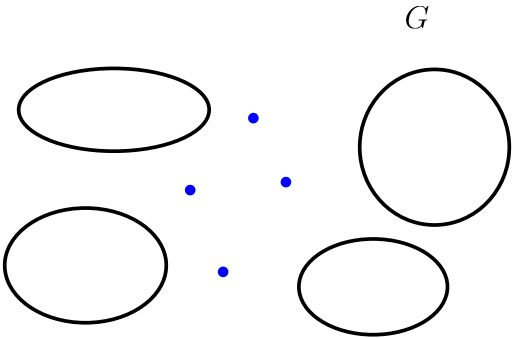

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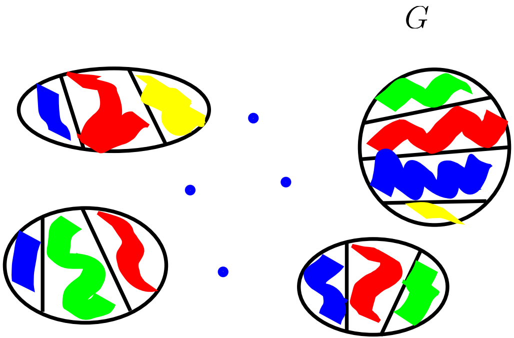

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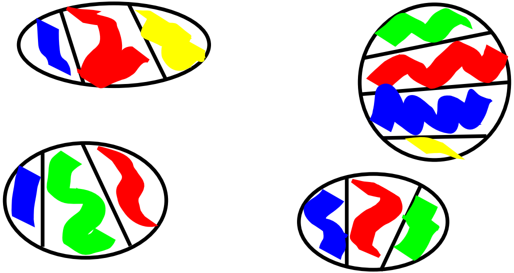

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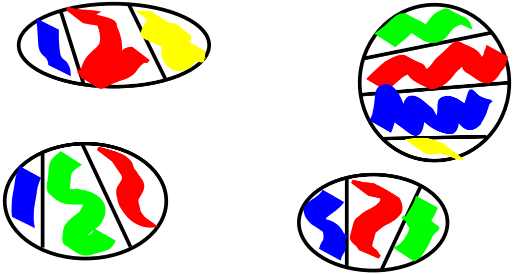

Observation: This yields a decomposition algorithm working in time $f(d) \cdot n^2$ on graphs of treedepth at most d.

Graph isomorphism for bounded treedepth

Theorem. [Bouland, Dawar, Kopczyński, 2012]

Graph isomorphism can be solved on graphs of treedepth at most d in time $f(d) \cdot n^3 \cdot \log n$.

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Remark: The running time can be further improved to $f(d) \cdot n \cdot \log^2 n$.

Beyond sparsity

1. We found canonical moves for Splitter in the Splitter game
2. and applied them for the radius-∞ Splitter game
3. to obtain canonical decompositions and a graph isomorphism algorithm for graphs of bounded treedepth.

Beyond sparsity

1. We want to find canonical moves for Flipper in the Flipper game
2. and apply them for the radius-∞ Flipper game
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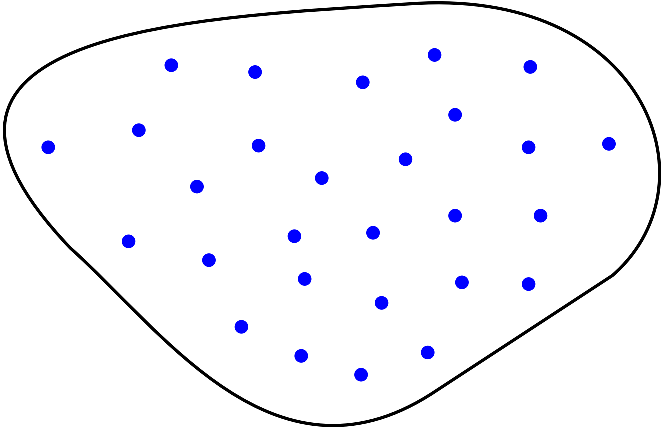
Defintion. [Ganian, Hliněný, Nešetřil, Obdržálek, Ossona de Mendez, 2017]

A graph G has *shrubdepth* at most d if Flipper can win the radius-∞ game on G in at most d rounds.

Finitary Substitute Lemma

$G \in \mathcal{C}$ $\phi(x_1, x_2; y_1, \ldots, y_k)$ ψ



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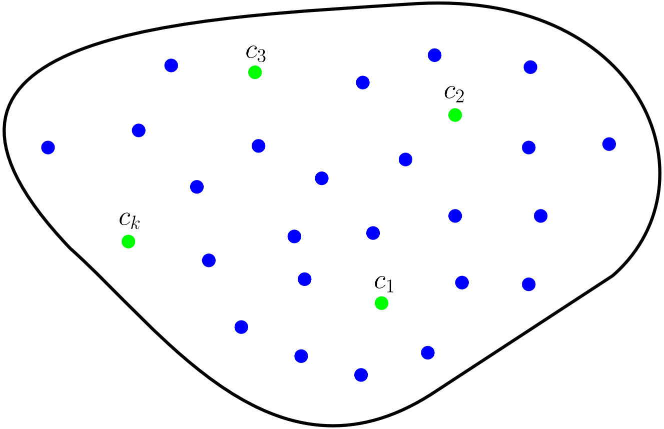

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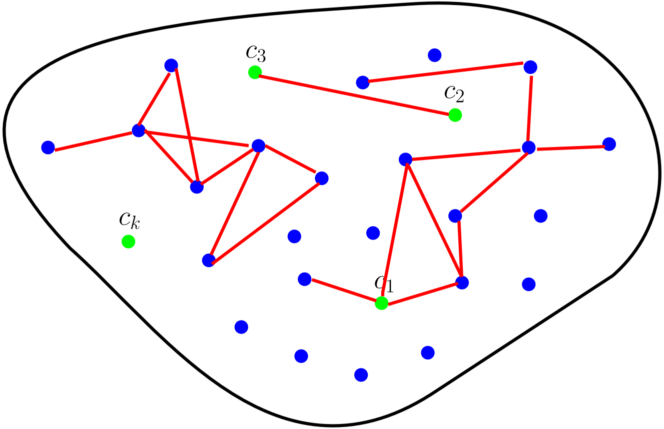

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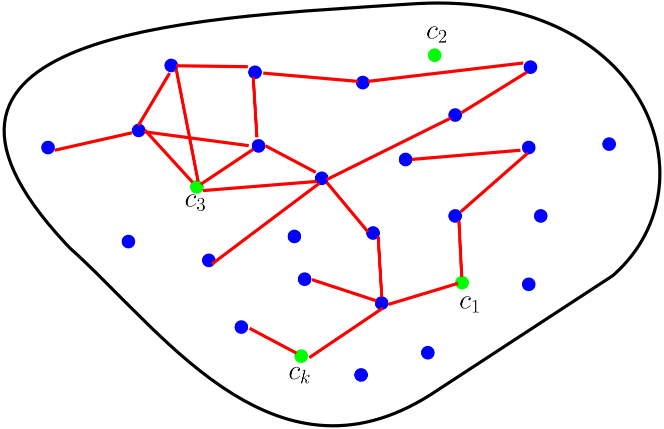

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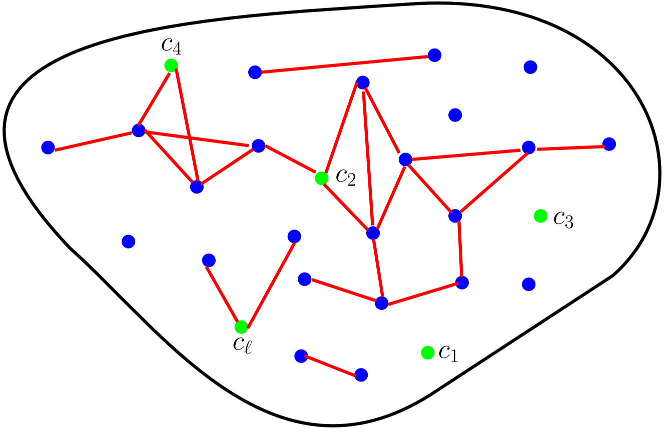

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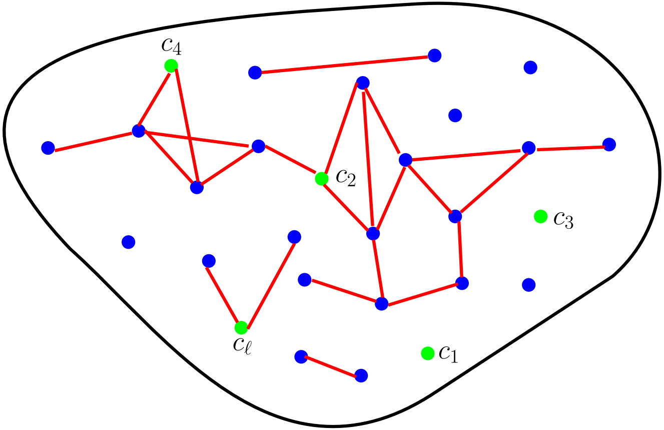

Proof uses a number of tools from stability theory [Shelah], most importantly properties of forking independence in stable theories.

Main theorem

Theorem. [Ohlmann, Pilipczuk, P., Toruńczyk, 2023]

Graph isomorphism can be solved on graphs of shrubdepth at most d in time $f(d) \cdot n^2$.