# What's in a flip?

Highlights 2025

Wojciech Przybyszewski and Szymon Toruńczyk

University of Warsaw

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**Main settings:**

- Sparse case: $\mathcal{C}$ is monotone $=$ $\mathcal{C}$ is closed under removing vertices and edges.
- Dense case: $\mathcal{C}$ is hereditary $=$ $\mathcal{C}$ is closed under removing vertices.

# Sparse case

**Dividing line:** Nowhere denseness

Theorem [Grohe, Kreutzer, Siebertz, '14]

Let $\mathcal{C}$ be a monotone class of graphs. Under the assumption $\mathrm{FPT} \neq \mathrm{AW}[*]$, first-order model checking is tractable on $\mathcal{C}$ if and only if $\mathcal{C}$ is nowhere dense.

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**Definition** [Podewski, Ziegler, '78] [Nešetřil, Ossona de Mendez, '10]

A class of graphs $\mathcal{C}$ is *nowhere dense* if for every $r \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that no graph in $\mathcal{C}$ contains the $r$-subdivision of the clique $K_m$ as a subgraph.

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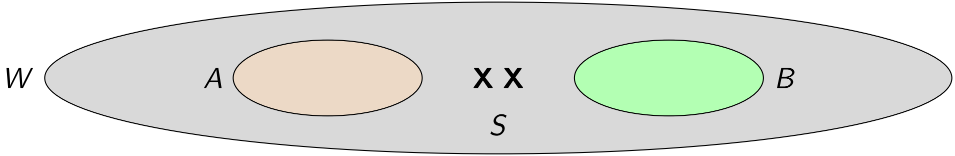
**Theorem** [Dreier, Mählmann, Toruńczyk, 2024]

A monotone class of graphs $\mathcal{C}$ is nowhere dense if and only if it is deletion-breakable: $\forall r \geqslant 1 \;\; \exists k_r \geqslant 1 \;\; \exists U_r : \mathbb{N} \to \mathbb{N}$ unbounded function such that

$$\forall G \in \mathcal{C} \quad \forall W \subseteq V(G) \quad \exists A, B \subseteq W,\, |A| = |B| \geqslant U_r(|W|) \quad \exists S \subseteq V(G),\, |S| \leqslant k_r$$

$$\mathrm{dist}(A, B) \geqslant r \text{ in } G - S.$$



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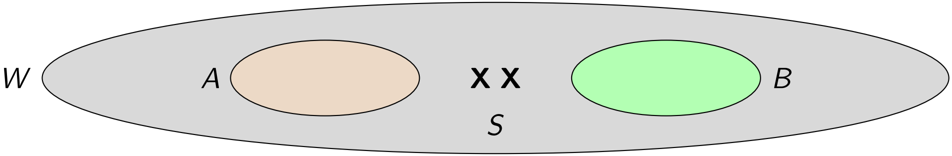


**Key property of each characterization:** Delete a bounded number of vertices to make remaining vertices generally far apart so you can apply FO locality (Gaifman Theorem).

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Conjecture [2016]

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A class of graphs is *monadically dependent* if it does not FO-transduce the class of all graphs.

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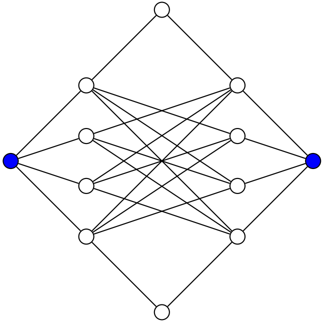
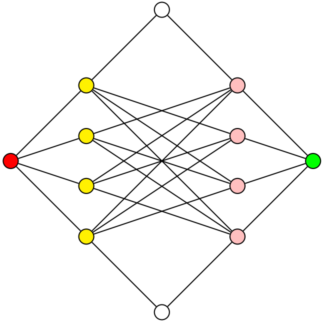
**Equivalent combinatorial definitions:** ?

# (Definable) flips



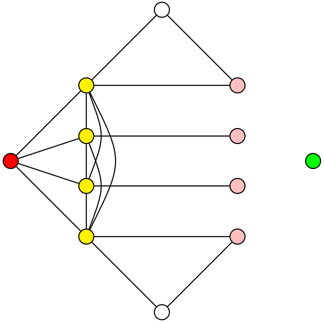
We start with a graph and a subset $S$ of its vertices (in blue).

# (Definable) flips



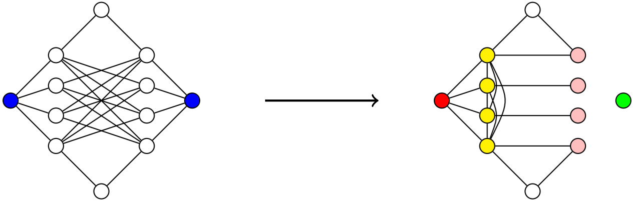
We partition the vertices of $V(G)$ by their neighborhoods on $S$.
Let's flip the edges between these pairs of colors: (●, ●), (●, ●), (●, ●).

# (Definable) flips



The result of the flip.

# (Definable) flips



**Definition**

Fix a graph $G$ and a set $S \subseteq V(G)$. An $S$-flip of $G$ is obtained as follows:
1. Partition $V(G)$ by their neighborhoods in $S$;
2. For each pair or parts $(P_1, P_2)$ (possibly $P_1 = P_2$) either keep the edges between $P_1$ and $P_2$ or complement them.

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**Theorem** [Dreier, Mählmann, Toruńczyk, 2024]

A monotone class of graphs $\mathcal{C}$ is nowhere dense if and only if it is deletion-breakable:
$\forall r \geqslant 1 \;\; \exists k_r \geqslant 1 \;\; \exists U_r : \mathbb{N} \to \mathbb{N}$ unbounded function such that

$$\forall G \in \mathcal{C} \quad \forall W \subseteq V(G) \quad \exists A, B \subseteq W,\ |A| = |B| \geqslant U_r(|W|) \quad \exists S \subseteq V(G),\ |S| \leqslant k_r$$

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**Equivalent combinatorial definitions:** flip-breakability, flip-separability

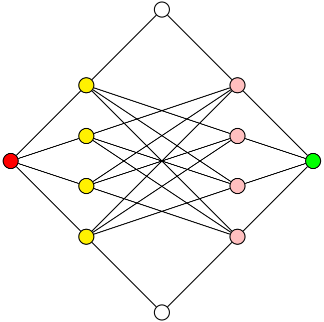
Theorem [Dreier, Mählmann, Toruńczyk, 2024]

A hereditary class of graphs $\mathcal{C}$ is mon. dependent if and only if it is flip-breakable:
$\forall r \geqslant 1 \;\; \exists k_r \geqslant 1 \;\; \exists U_r : \mathbb{N} \to \mathbb{N}$ unbounded function such that

$$\forall G \in \mathcal{C} \quad \forall W \subseteq V(G) \quad \exists A, B \subseteq W,\ |A| = |B| \geqslant U_r(|W|) \quad \exists S \subseteq V(G),\ |S| \leqslant k_r$$

$$\mathrm{dist}(A, B) \geqslant r \text{ in an } S\text{-flip } G' \text{ of } G.$$

# What are flips?



**Observation**

The edges of an $S$-flip can be defined by a quantifier free formula $\varphi(x, y)$ with parameters from $S$.

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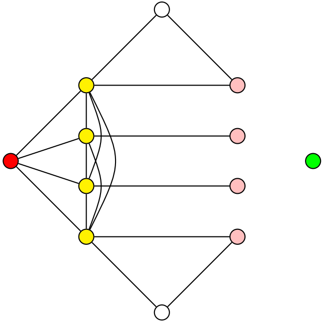

Flips are reversible – the edges of the original graph can be defined in a flip by a quantifier free formula $\psi(x, y)$ with parameters from $S$.

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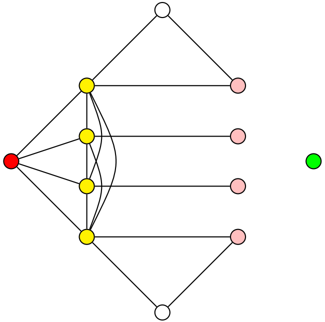

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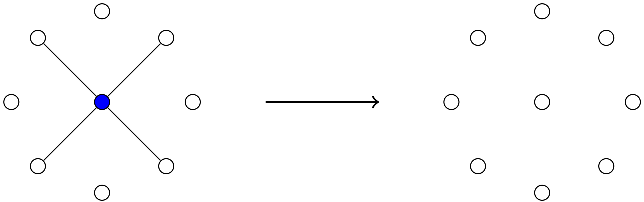

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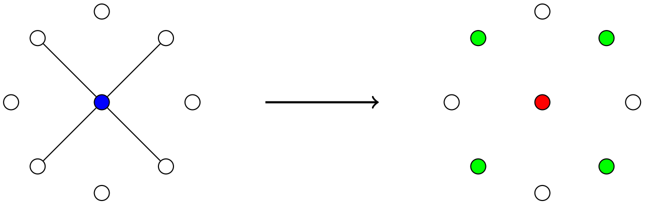

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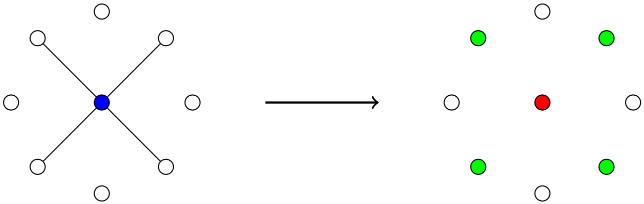

Flips are reversible – the edges of the original graph can be defined in a flip by a quantifier free formula $\psi(x, y)$ with parameters from $S$.
We need the access to the neighborhood classes in the original graph. Luckily, they are also definable by a quantifier free formula!

# Flips of relational structures

**Definition** [P., Toruńczyk]

Fix a $\Gamma$-structure $\mathbf{N}$ and a $\Sigma$-structure $\mathbf{M}$ on the same universe. We say that $\mathbf{N}$ is a flip of $\mathbf{M}$ if they are quantifier-free bi-interpretable with parameters:

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For each $R(\bar{x}) \in \Gamma$ there exists $\varphi_R(\bar{x})$ a quantifier-free Σ-formula with parameters s.t.

$$\mathbf{N} \models R(\bar{a}) \iff \mathbf{M} \models \varphi_R(\bar{a}) \quad \text{for every } \bar{a} \in M^{\bar{x}}$$

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and for each $T(\bar{y}) \in \Sigma$ there exists $\psi_T(\bar{y})$ a q.f. $\Gamma$-formula with parameters s.t.

$$\mathbf{M} \models T(\bar{b}) \iff \mathbf{N} \models \psi_T(\bar{b}) \quad \text{for every } \bar{b} \in M^{\bar{y}}.$$

# Meaning of flips

**Corollary**

Let **N** be a flip of **M**. For every formula $\varphi(\bar{x})$ in the language of **M** there exists a formula with parameters $\psi(\bar{x})$ in the language of **N** of the same quantifier rank such that

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**Meaning of flips:** Flips are a way to apply Gaifman locality theorem in structures with dense Gaifman graphs.

# Our results

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Thank you!