Proving combinatorial properties of graphs using model theory

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University of Warsaw

Highlights 2023

Treedepth game

The treedepth game is played on a graph G_1. In round i

1. Splitter chooses a vertex v to delete
2. Connector chooses G_{i+1} as a connected component in $G_i - v$.

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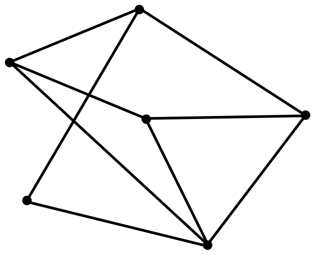

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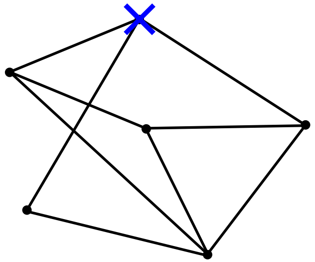

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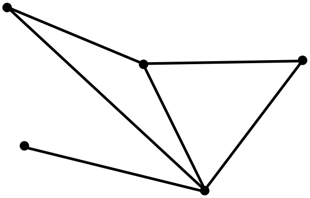

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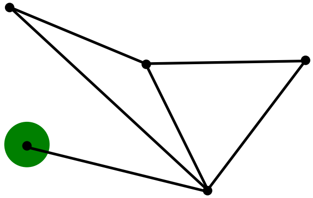

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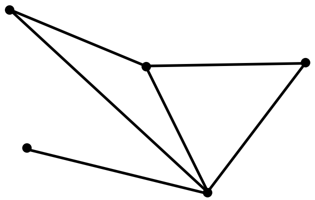

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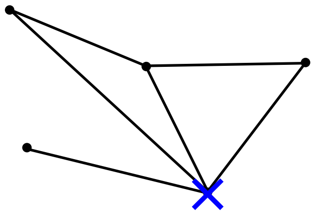

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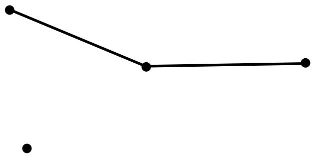

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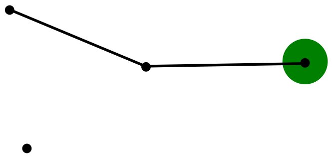

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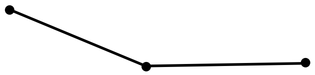

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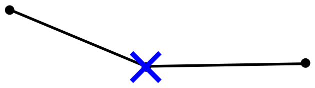

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A treedepth of a graph G is the minimum number of rounds that are enough for Splitter to always win the treedepth game, no matter how Connector is playing.

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Observation: We don't need to assume that G is a finite graph for this definition to make sense.

Progressing moves in the treedepth game

Theorem. [Ohlmann, Pilipczuk, P., Toruńczyk, 2023]

There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that if a graph G has treedepth d then Splitter has at most $f(d)$ progressing moves[1].

[1] A vertex v is a progressing move for Splitter if every connected component C of $G - \{v\}$ has strictly smaller treedepth than G.

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- $E(v_i, v_j)$ for every $(v_i, v_j) \in E(G)$ and $\neg E(v_i, v_j)$ for every $(v_i, v_j) \notin E(G)$;

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- v_∞ is a progressing move.

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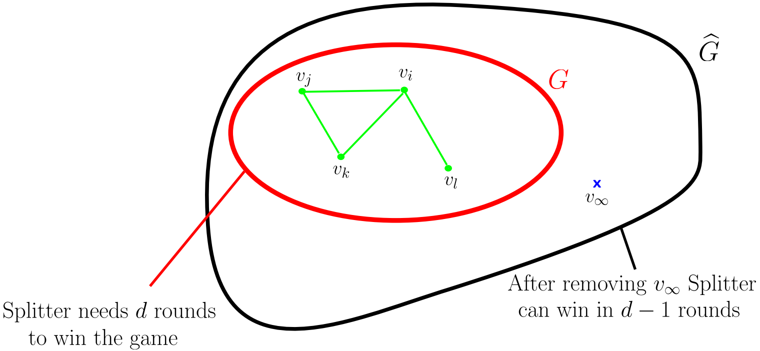

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Compactness yields a model that contradicts the previous lemma. □

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- Game characterization of monadically stable classes of graphs. [Gajarský, Mählmann, McCarty, Ohlmann, Pilipczuk, P., Siebertz, Sokołowski, Toruńczyk, '23]

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Thank you for your attention!