Distal combinatorial tools for graphs of bounded twin-width

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Based on *VC-density and abstract cell decomposition for edge relation in graphs of bounded twin-width*

Mixed width of a matrix

Horizontal matrix – all rows are equal:

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Mixed width of a matrix

Horizontal matrix – all rows are equal:

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Vertical matrix – all collumns are equal:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Mixed width of a matrix

Mixed matrix – a matrix that is neither vertical nor horizontal:

$$\begin{pmatrix} 1 & 1 & \color{red}{0} & \color{red}{1} & 1 \\ 1 & 1 & \color{red}{0} & \color{red}{0} & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

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A matrix is mixed if and only if it has a 2×2 mixed submatrix (corner).

Mixed width of a matrix

$$\left[\begin{array}{cc|ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Figure: A 3-mixed minor on a matrix: no zone is horizontal or vertical. (Example from *Bonnet et al. Twin-width I: tractable FO model checking*)

Mixed width of a matrix

$$\left[\begin{array}{cc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

Figure: A 3-mixed minor on a matrix: no zone is horizontal or vertical. (Example from *Bonnet et al. Twin-width I: tractable FO model checking*)

Mixed-width of a matrix M is the smallest k such that M is k-mixed free.

Mixed width of a graph

For a given graph G and a total order σ on its vertices we define mixed-width of (G, σ) as the mixed width of the adjacency matrix of G where the rows and columns are ordered according to σ (we denote it by $M_\sigma(G)$).

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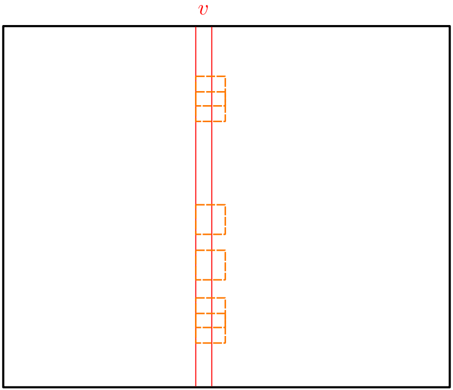
Theorem

If G is a graph of twin-width less than t, then there is a total ordering on its vertices σ such that $M_\sigma(G)$ is $(2t+2)$-mixed free. On the other hand, if G is a graph and σ is a total ordering on its vertices such that $M_\sigma(G)$ is k-mixed free, then $\mathrm{tww}(G) = 2^{2^{O(k)}}$.

Encoding vertices with many corners

Assume that $M_\sigma(G)$ is t-mixed free and in the column of some vertex v there are at least $2t$ corners.



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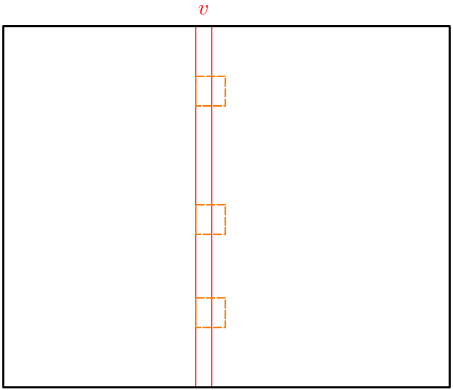

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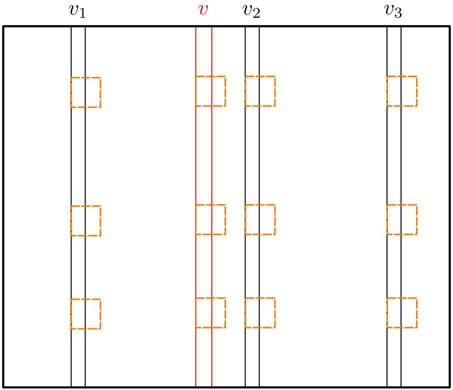

Distal tools

Using previous results of Chernikov, Simon and Starchenko on the notion of distality (which originated in model theory) we can derive from the encoding theorem strong combinatorial tools for graphs of bounded twin-width.

Distal tools

Theorem (Distal cutting lemma)

For any t there is a constant C depending only on t such that following holds. For any graph G of twin-width at most t, any $A \subseteq V(G)$ of size n and any real $1 \leq r \leq n$ we can partition the vertices of $V(G)$ into at most $Cr^{O(t)}$ sets $X_1, \ldots, X_l$ such that the vertices in every X_i have almost the same neighborhood in A. More precisely, there are at most $\frac{n}{r}$ vertices $a \in A$ for which there are $u, v \in X_i$ with $(u, a) \in E(G)$ and $(v, a) \notin E(G)$.

Distal tools

Theorem (Distal regularity lemma)

For every integer t there exists a constant $c = c(t)$ such that: for every $\varepsilon > 0$ and for every graph $G = (V, E)$ of twin-width at most t, there exists a partition $V = V_1 \cup \ldots \cup V_k$ into non-empty sets, and a set $\Sigma \subseteq [k] \times [k]$ with the following properties.

1. *Polynomially bounded size of the partition:* $k = O((\frac{1}{\varepsilon})^c)$.
2. *Few exceptions:* $|\bigcup_{(i,j)\in\Sigma} V_i \times V_j| \geq (1 - \varepsilon)|V|^2$.
3. $0 - 1$*-regularity: for all* $(i, j) \in \Sigma$ *there are either all edges between* V_i *and* V_j *or no edge at all.*