Combinatorial characterization of forking independence in monadically stable graphs

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Joint work with Szymon Toruńczyk

Definitions

- (Colored) Graph – a structure G over a signature consisting of the binary edge relation symbol E and a number of unary predicates, such that E is symmetric and antireflexive in G. We refer to the elements of the domain of G as vertices.

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- Stable graph – a graph G such that no formula has the order property in G.

Order property

Formula $\varphi(\bar{x}; \bar{y})$ with two tuples of variables has the order property in G if there exist $\bar{a}_1, \bar{a}_2, \ldots$ and $\bar{b}_1, \bar{b}_2, \ldots$ such that

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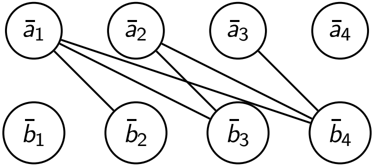

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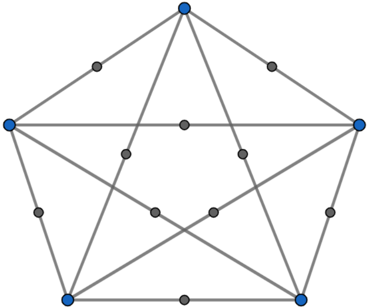
- Monadically stable graph – a graph G such that if we add in any way a number of new unary predicates to it thus obtaining a graph $\hat{G}$, then $\hat{G}$ is stable.

Example

Consider an infinite 1-subdivided clique.



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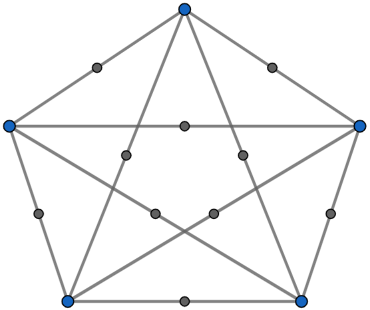

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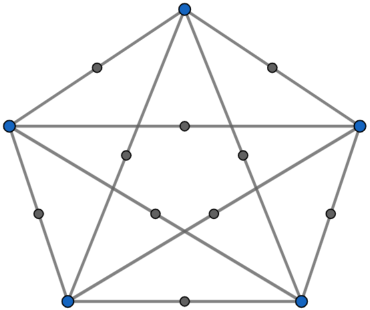

Is it stable?
Is it monadically stable?

Flips

Let G be a graph and X, Y be two subsets of its vertices. The (X, Y) flip of G is the graph $\bar{G}$ with the same domain as G, such that for any two vertices u, v:

$$\bar{G} \models E(u, v) \iff [G \models E(u, v)] \text{ xor } [(u, v) \in X \times Y \cup Y \times X].$$

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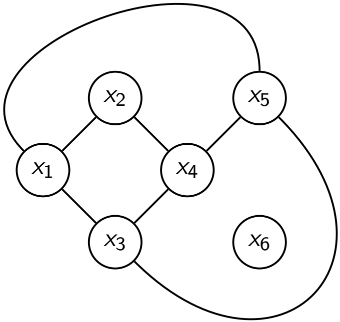

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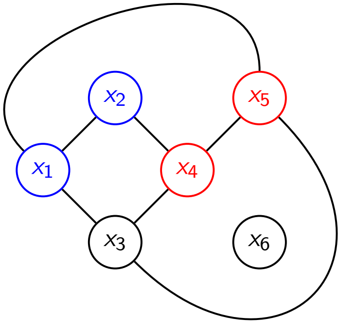

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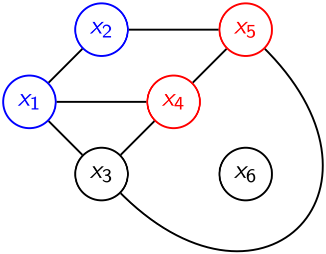

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Let G be a graph and S be a subset of its vertices. An S flip of G is any graph $\bar{G}$ obtained from G by performing a sequence of flips between sets with the same atomic type on S.

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Example below.

Elementary extension

Let G and H be two graphs with $G \subseteq H$. Then G is an elementary substructure of H, written $G \prec H$, if for every formula $\varphi(\bar{x})$ (without parameters) and tuple $\bar{v} \in G^{|\bar{x}|}$ the following equivalence holds:

$$G \models \varphi(\bar{v}) \iff H \models \varphi(\bar{v}).$$

Definability of types

Theorem

Let $G \prec H$ be stable graphs, $\varphi(\bar{x};\bar{y})$ be a formula and $\bar{v} \in H^{|x|}$ be a tuple of vertices. Then there are tuples $\bar{a}_1, \ldots, \bar{a}_k \in G^{|x|}$ such that for any $\bar{u} \in G^{|y|}$ whether $\varphi(\bar{v};\bar{u})$ hold in H can be expressed by a Boolean combination of the truth values of $\varphi(\bar{a}_i;\bar{u})$.
We denote this Boolean combination by $\psi_{\varphi;\bar{v}}(\bar{y})$.

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Separating by flips

Theorem (Gajarský, Mählmann, McCarty, Ohlmann, Pilipczuk, P., Siebertz, Sokołowski, Toruńczyk)

Let $G \prec H$ be monadically stable graphs and r be an integer. Then for every $v \in H \setminus G$ there exists a finite subset $S \subseteq G$ and an S-flip $\bar{H}$ of H such that in $dist_{\bar{H}}(v, G) > r$.

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Example

The case $r = 1$ follows from definability of types. Indeed, take $a_1, \ldots, a_k$ defined as previously. Then for every equivalence class C of the same neighborhood in $\{a_1, \ldots, a_k\}$ the set $C \cap G$ consist only of neighbors or non-neighbors of v. Therefore we flip the equivalence class of v with the equivalence classes with its neighbors.

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Theorem (Baldwin and Shelah, simplified version)

Let $G \prec H$ be monadically stable graphs and $\bar{v}, \bar{u}$ be two tuples of vertices. Then:

- $\bar{v} \mathop{\smile\!\!\!\!\!\!|}_G \bar{u}$ *if and only if for every* $v \in \bar{v}, u \in \bar{u}$ *we have* $v \mathop{\smile\!\!\!\!\!\!|}_G u$;
- $\not\mathop{\smile\!\!\!\!\!\!|}_G$ *is an equivalence relation on the vertices in* $H \setminus G$.

When two vertices are not independent?

Lemma

Let $G \prec H$ and $v, u \in H \setminus G$. Assume that there is an integer r, a finite $S \subseteq G$, and an S-flip $\bar{H}$ of H such that $dist_{\bar{H}}(v, G) > r$ and $dist_{\bar{H}}(v, u) \leq r$. Then $u \not\mathrel{\perp\!\!\!\!\perp}_G v$.

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Take the formula $\varphi(x; y)$ with parameters from S which stipulates that after performing the S-flip there is a path between x and y of length at most r.

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Take the formula $\varphi(x; y)$ with parameters from S which stipulates that after performing the S-flip there is a path between x and y of length at most r.

Then $H \models \varphi(u; v)$, but we can't find any $u' \in G$ such that $H \models \varphi(u'; v)$. □

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Lemma

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Sketch of the proof.

Take any formula $\varphi(x; y)$ with parameters $\bar{m}$ from G, such that $H \models \varphi(u; v)$. Our goal is to find $u' \in G$ satisfying $H \models \varphi(u'; v)$.

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Denote by q the quantifier rank of φ. Consider the set S and an S-flip $\bar{H}$ of H such that $dist_{\bar{H}}(v, G) > r$ and $dist_{\bar{H}}(v, u) > r$ for $r := 2 \cdot 7^q$.

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Add a unary predicate to $\bar{H}$ which selects subsets of H that were flipped. We can write a formula $\varphi'(x, y)$ with parameters $\bar{m}$ of quantifier rank q with the following property:

$$H \models \varphi(x; y) \iff \bar{H} \models \varphi'(x; y).$$

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By Gaifman locality theorem there is a formula $\psi(x)$ with parameters $\bar{m}$ such that whenever a vertex u' is at distance at least r from $v\bar{H} \models \psi(u')$ then $\bar{H} \models \varphi'(u', v)$.

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Such vertex u' exists in $\bar{G}$ because $\bar{G}$ is an elementary substructure of $\bar{H}$. □

When two vertices are (not)independent?

Lemma

Let $G \prec H$ and $v, u \in H \setminus G$. Assume that there is an integer r, a finite $S \subseteq G$, and an S-flip $\bar{H}$ of H such that $dist_{\bar{H}}(v, G) > r$ and $dist_{\bar{H}}(v, u) \leq r$. Then $u \not\perp_G v$.

Lemma

Let $G \prec H$ and $v, u \in H \setminus G$. Assume that for every integer r there is a finite $S \subseteq G$ and an S-flip $\bar{H}$ of H such that $dist_{\bar{H}}(v, G) > r$ and $dist_{\bar{H}}(v, u) > r$. Then $u \perp_G v$.

Condition for radius 1 revisited

Let $G \prec H$ and $v, u \in H \setminus G$. Assume that there is a finite $S \subseteq G$ and an S-flip $\bar{H}$ of H such that $dist_{\bar{H}}(v, G) > 1$ and $dist_{\bar{H}}(v, u) = 1$.

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That means that if we take any equivalence class C of the same atomic type on S, then v is connected by an edge to every vertex in $C \cap G$ or it is not connected to every vertex in $C \cap G$.

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That means that if we take any equivalence class C of the same atomic type on S, then v is connected by an edge to every vertex in $C \cap G$ or it is not connected to every vertex in $C \cap G$.

Therefore, we had to flip the equivalence class of v with every equivalence class that contains neighbors of v, so

$$\bar{H} \models E(u, v) \iff [H \models E(u, v)] \text{ xor } [H \models \psi_{E,v}(u)].$$

Characterization of forking by connected components

Theorem (P., Toruńczyk)

Let $G \prec H$. Consider the following relation:

$$E'(u,v) \iff [u,v \in H \setminus G] \wedge [H \models E(u,v) \text{ xor } H \models \psi_v(u)].$$

Then $u \not\perp\!\!\!\perp_G v$ if and only if u and v are in the same connected component of E'.

Thank you!