SYSTOLIZING BUILDINGS

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Abstract. We introduce a construction turning some Coxeter complexes and Davis complexes into systolic complexes. Consequently groups acting geometrically on buildings of triangle types distinct from $(2, 4, 4)$, $(2, 4, 5)$, $(2, 5, 5)$, and various rank 4 types are systolic.

1. Introduction

A flag simplicial complex is *systolic* if it is simply-connected and all of its vertex links are 6–*large*, that is all cycles of length 4 or 5 have diagonals. A group is *systolic* if it acts geometrically (i.e. properly and cocompactly by automorphisms) on a systolic complex. Systolic complexes and groups were introduced by Januszkiewicz and Świątkowski [JŚ06], and independently by Haglund [Hag03] although their 1–skeleta were studied earlier by Chepoi and others under the name of *bridged graphs* (see [BC08]). They constructed examples in high dimensions, established analogies with CAT(0) spaces, and studied their exotic filling properties.

In this article we systolize the Coxeter and Davis complexes of buildings of given Coxeter types. We say that a systolic complex $\hat{X}$ is a *systolization* of its subcomplex X, if the action of the group of type preserving automorphisms of X extends to $\hat{X}$, and $X \subset \hat{X}$ is a quasi-isometry.

We assume that all buildings have finite thickness. By the *Davis complex* we mean the subcomplex of the barycentric subdivision of the Coxeter complex obtained by removing the open stars of vertices of infinite type. Note that this is the barycentric subdivision of what is usually called the Davis complex.

Theorem 1.1. *Let W be a triangle Coxeter group with finite exponents distinct from $(2, 4, 4), (2, 4, 5)$ and $(2, 5, 5)$. Then the Coxeter complex and the Davis complex of a building of type W admit a systolization. Consequently the group W and any group acting geometrically by type preserving automorphisms on a building of type W is systolic.*

Theorem 1.2. *The Coxeter group of type $(2, 4, 4)$ is not systolic.*

We believe that Coxeter groups of the other two excluded types are also not systolic.

Theorem 1.3. *Let W be a Coxeter group of rank 4 with finite exponents. Assume that all of its special rank 3 subgroups are infinite and not of type $(2, 4, 4), (2, 4, 5)$ or $(2, 5, 5)$. Moreover, assume that there is at most one*

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exponent 2. *Then the Coxeter complex and the Davis complex of a building of type W admit a systolization. Consequently the group W and any group acting geometrically by type preserving automorphisms on the Davis complex of a building of type W is systolic.*

The groups in Theorem 1.3 have cohomological dimension 2. However, we also discuss a possible application pointed out by Januszkiewicz that should give rise to new systolic groups of cohomological dimension 3.

Organization. In Section 2 we list basic lemmas allowing to identify systolic complexes. In Section 3 we prove Theorem 1.2, that the Coxeter group of type $(2, 4, 4)$ is not systolic. Section 4 illustrates our systolizing method for the Coxeter complex of the triangle group $(2, 3, 6)$. We prove Theorem 1.1, up to discussing the Davis complex, in Section 5. In Section 6 we describe the systolization of the Coxeter complex in Theorem 1.3, prove that it is simply-connected and has simply-connected vertex links. To prove that it is systolic it remains to verify 6–largeness of the edge links, which we do in Section 7. We systolize the Davis complex in Section 8. Finally, in Section 9 we describe Januszkiewicz's construction.

Discussion of assumptions. The groups acting geometrically on spherical buildings are finite, hence systolic, since they act geometrically on a point. That is why we consider only the infinite case. Infinite buildings of rank 2 are trees, hence systolic, so the smallest interesting rank is 3.

Let W be a Coxeter group of rank 3. If one of the exponents of W is infinite, then the Davis complexes of buildings of type W retract to trees, and that is why we focus on the case where all the exponents are finite. In that case the Davis complex coincides with the barycentric subdivision of the Coxeter complex.

Let now W be a Coxeter group of rank 4. We exclude special subgroups of type $(2, 4, 4), (2, 4, 5)$ and $(2, 5, 5)$, since a finitely presented subgroup of a systolic group is systolic [Zad13, HMP13]. Consider first the case where there is an infinite exponent. If the defining graph of W is a cycle of length 4, then the triangles of the Davis complex of type W building are arranged into squares forming a $\mathcal{VH}$–complex. Groups acting geometrically on $\mathcal{VH}$–complexes were shown to be systolic in [EP13].

If there is an infinite exponent and the defining graph of W is a not cycle of length 4, then $W = W_1 *_{W_3} W_2$, where W_1, W_2 are triangle groups and W_3 is finite. The Davis complex of a building of type W is equivariantly homotopy equivalent to a tree of Davis complexes of buildings of type W_1, W_2. Systolizing these gives a tree of systolic complexes, which is systolic. Hence a group acting geometrically on a building of type W is systolic. That is why in Theorem 1.3 we focus on the case where all the exponents are finite.

If all of the special rank 3 subgroups of W are finite, then W acts geometrically on $\mathbb{R}^3$ or $\mathbb{H}^3$, thus it is not systolic [JŚ07]. Since the cases where some but not all of the special rank 3 subgroups are finite are difficult to handle, we assume that all of them are infinite.

The complication in the problem comes from the exponents 2, and that is why we study the simplest case, that is the case where there is at most one exponent 2. If all the exponents are ≥ 3, then the Coxeter complex is systolic to begin with. To systolize the Davis complex consider the face complex of the Coxeter complex and remove open stars of original vertices (see Section 8 for details).

2. Simplicial nonpositive curvature

In this section we recall basic definitions and notation used to study systolic complexes. We also give criteria for a complex to be systolic.

Definition 2.1. A simplicial complex X is *flag* if every clique (a set of vertices pairwise connected by edges) spans a simplex. A subcomplex Y of X is *full* if any simplex of X spanned by vertices in Y is in Y. A *cycle* in X is a subcomplex of X that is a subdivision of the circle. A flag simplicial complex is *k–large*, for $4 \leq k \leq \infty$, if it has no full cycle of length $< k$.

In other words a flag simplicial complex X is k–large if every cycle of length $4 \leq l < k$ has a *diagonal*, that is an edge in X between a pair of non-consecutive vertices of the cycle. Consequently, every closed edge-path γ in X of length $< k$ bounds a disc diagram with no interior vertices. In particular, if γ is locally embedded, then it has three consecutive vertices u, v, w such that uw is an edge.

Definition 2.2. A simplicial complex is *systolic* if it is connected, simply-connected, and all its vertex links are 6–large. A group is *systolic* if it acts geometrically on a systolic complex.

While in [JŚ06] authors require in the definition of a systolic complex that the links of all the simplices are 6–large, this is trivially equivalent with our definition. Note that by [JŚ06, Prop 1.4] a systolic complex is itself 6–large. Moreover, by [JŚ06, Thm 4.1(1)] a systolic complex is contractible.

Lemma 2.3. *Suppose that a simplicial complex X is obtained from two flag simplicial complexes A and B by gluing them along a simplex. Then X is k–large if and only if A and B are k–large.*

Proof. This follows from the fact that every cycle in X that is not contained in A or B must have two non-consecutive vertices in $A \cap B$ □

Corollary 2.4. *The join of a simplex with a discrete set is ∞–large.*

Here is another criterion for k–largeness.

Lemma 2.5. *Let $f\colon A \to B$ be a simplicial map from a flag simplicial complex A onto a flag simplicial complex B. Suppose that vertices a, a' of A are adjacent if and only if the vertices $f(a), f(a')$ are adjacent or equal. Then A is k–large if and only if B is k–large.*

Note that f is a homotopy equivalence.

Proof. It suffices to show that the lengths of shortest full cycles in A and B are equal. A cycle β in B lifts to A. If β has no diagonals, then neither does its lift. Conversely, suppose that we have a full cycle α in A. Then for non-consecutive vertices a, a'' of α the vertices $f(a), f(a'')$ are neither equal nor adjacent. In particular, if a, a', a'' are consecutive, then $f(a), f(a'')$ are not adjacent, hence $f(a) \neq f(a')$. Therefore $f(\alpha)$ is a cycle and has no diagonals. □

Here is another criterion.

Definition 2.6. Let Γ be a graph whose maximal cliques (with respect to inclusion) intersect only along vertices. Denote by V, M the sets of vertices and maximal cliques of Γ. Consider the following graph Γ^* with vertex set $V \cup M$. We connect $v, v' \in V$ by an edge in Γ^* if they are connected by an edge in Γ. We connect $m, m' \in M$ if $m \cap m' \neq \emptyset$, and we connect $v \in V, m \in M$ if $v \in m$. See Figure 1.

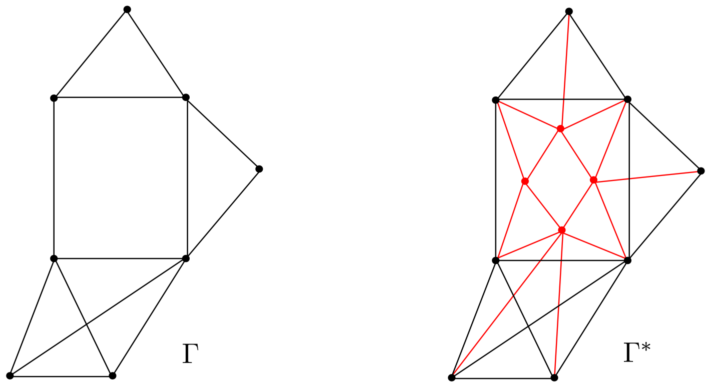

FIGURE 1. An example of a graph Γ and associated Γ^*.

Lemma 2.7. *Let Γ be a graph whose maximal cliques intersect only along vertices. The flag complex spanned on Γ^* is k–large if and only if the flag complex spanned on Γ is k–large.*

Proof. The 'only if' part is obvious, since the complex spanned on Γ is a full subcomplex of the complex spanned on Γ^*. For the converse, let γ^* be a shortest full cycle in Γ^* with vertices $\rho_0, \rho_1, \ldots, \rho_n = \rho_0$. Assume by contradiction $n < k$. We denote the relevant vertices of ρ_i in the following way. If ρ_i is a vertex, then let $v_{2i} = v_{2i+1} = \rho_i$. If ρ_i is a clique, then let $v_{2i} = \rho_i \cap \rho_{i-1}, v_{2i+1} = \rho_i \cap \rho_{i+1}$ (understood cyclically). Note that in this case $v_{2i} \neq v_{2i+1}$, since γ^* has no diagonals.

Let γ be the closed edge-path formed by $(v_i)_{i=0}^{2n-1}$, after removing consecutive repeating vertices. Since γ^* has no diagonals, there are combinatorially only three possibilities for a triple of consecutive vertices u, v, w of γ, up to interchanging u with w: Either $u = \rho_{i-1}, v = \rho_i, w = \rho_{i+1}$ for some i, or $v = \rho_i \cap \rho_{i+1}$ with $u \in \rho_i, w \in \rho_{i+1}$, or else $v = \rho_i \in \rho_{i+1}$ with $u = \rho_{i-1}$. We claim that in all three cases $u \neq w$ and moreover u and w are not connected by an edge. In the first case this follows from the fact that $\rho_{i-1}\rho_{i+1}$ is not a diagonal of γ^*. In the other two cases the clique ρ_{i+1} would have at least vw in common with the triangle uvw, so it would have to be equal to the maximal clique containing uvw. In the third case this contradicts the fact that $\rho_{i-1}\rho_{i+1}$ is not a diagonal of γ^*. In the second case we get that ρ_i is also the maximal clique containing uvw, so it coincides with ρ_{i+1}, contradiction. This proves the claim, so that in particular γ is locally embedded.

Let $g\colon \gamma^* \to \gamma$ be the map defined in the following way. If ρ_i is a vertex, then let $g(\rho_i) = \rho_i$. Otherwise, let $g(\rho_i)$ be the barycenter of the edge

$v_{2i}v_{2i+1}$. This extends uniquely to a simplicial (possibly degenerate) map g between the barycentric subdivisions of γ^* and γ. Hence $|\gamma| \leq |\gamma^*| = n < k$. Since Γ is k–large, γ can be triangulated by consecutively adding diagonals. This yields three consecutive vertices u, v, w on γ such that uw is an edge, and contradicts the claim. □

Finally, we have the following criterion.

Definition 2.8. Let $\Gamma = (V, E)$ be a graph of girth ≥ 4. Let $\widetilde{\Gamma}$ be the following graph whose vertices are pairs (v, σ), where $v \in V$, $\sigma \in V \cup E$ and $v \subset \sigma$. Vertices $(v, \sigma), (v', \sigma')$ are connected by an edge if $v = v'$ or v and v' are adjacent and $\sigma \in \{v, vv'\}, \sigma' \in \{v', vv'\}$ (see Figure 2).

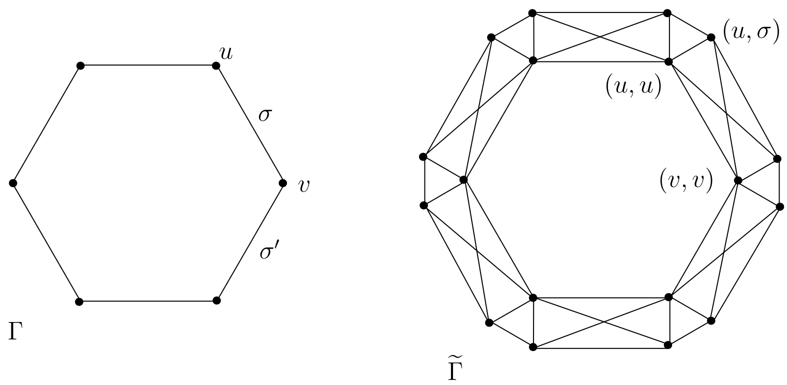

FIGURE 2. An example of a graph Γ and associated $\widetilde{\Gamma}$.

Lemma 2.9. *Let Γ be a graph of girth ≥ 4. The flag complex spanned on $\widetilde{\Gamma}$ is k–large if and only if Γ has girth $\geq k$.*

Proof. The 'only if' part is again obvious, since Γ embeds as a full subcomplex of the complex spanned on $\widetilde{\Gamma}$ under the map $v \to (v, v)$. For the converse, let $g \colon \widetilde{\Gamma} \to \Gamma$ be the simplicial map mapping each (v, σ) to v. Note that for each vertex v of Γ, its preimage $g^{-1}(v)$ is a clique. The preimage $g^{-1}(e^\circ)$ of each open edge e° of Γ is also contained in a clique (on 4 vertices). If γ is a cycle in $\widetilde{\Gamma}$ of length $< k$, then $g(\gamma)$ is homotopically trivial. Hence $g(\gamma)$ backtracks in the sense that there are three consecutive vertices $vv'v''$ of $g(\gamma)$ with $v = v''$ or four consecutive vertices $vv'v''v'''$ with $v = v''', v' = v''$. Considering $g^{-1}(v)$ in the first case and $g^{-1}(vv'^\circ)$ in the other produces a diagonal of γ. □

3. THE COXETER GROUP OF TYPE $(2, 4, 4)$ IS NOT SYSTOLIC

Proof of Theorem 1.2. Let W be the $(2, 4, 4)$ triangle group with Coxeter generating set s, t, r, where $sr = rs$. Let $g = rtst$, $h = tsrts$. Then t conjugates g^2 and h^2. The subgroup $W' = \langle g, h \rangle$ is the Klein bottle group with relation $gh = hg^{-1}$. In particular, W' is a torsion-free group that is virtually $\mathbb{Z}^2$.

We now follow the proof of [EP13, Thm 4.1]. A *systolic flat* $\mathbb{E}^2_\triangle$ is the systolic complex that is the equilateral triangulation of the Euclidean plane. Suppose that W acts geometrically on a systolic complex X. By the systolic flat torus theorem [Els09, Cor 6.2(1) and Thm 5.4], the torsion-free subgroup W' acts properly on a systolic flat $\mathbb{E}^2_\triangle \subset X$. If the Klein bottle group acts properly by isometries on the Euclidean plane, then h acts as a glide reflection and g acts as a translation in the direction perpendicular to the glide reflection axis. Since $\mathbb{E}^2_\triangle$ is equipped additionally with a combinatorial structure, there are only two possibilities for the axes of g, h. Exactly one of them is *quasi-convex* in the sense that a 1–skeleton geodesic starting and terminating at the axis is contained in its uniform neighborhood [Els10, Prop 3.11]. By [Els10, Prop 3.12], this contradicts the fact that g^2, h^2 are conjugate. See the proof of [EP13, Thm 4.1] for details. □

4. Coxeter complexes of type $(2, 3, 6)$

To illustrate the method used in the proof of Theorem 1.1, we first show how to systolize the Coxeter complex Σ of type $(2, 3, 6)$.

We say that a vertex is of *type* **k** if its stabilizer is of order $2k$. The links of vertices of type **2** are squares preventing the complex from being systolic. In order to systolize Σ we will add diagonals to all these squares, and verify that no new short full cycles are created.

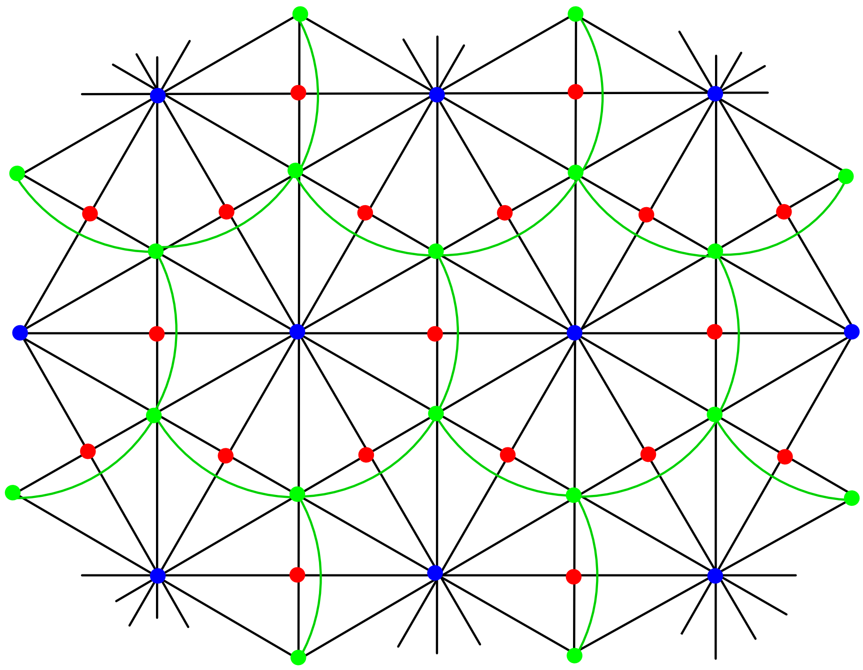

FIGURE 3. Systolization of the Coxeter complex of type $(2, 3, 6)$.

The *systolization* $\hat{\Sigma}$ of Σ is the flag simplicial complex spanned on the following 1–skeleton. The vertex set of $\hat{\Sigma}$ is the same as the vertex set of Σ. A pair of vertices is connected by an edge in $\hat{\Sigma}$ if it is either connected by an edge in Σ or it is a pair of vertices of type **3** that are adjacent to the same vertex of type **2**.

Figure 3 shows the 1–skeleton of $\hat{\Sigma}$. Vertices of type **2** are pictured in red, vertices of type **3** in green, vertices of type **6** in blue. The new edges (green) are diagonals of the squares that are the links of the red vertices.

We now prove that $\hat{\Sigma}$ is indeed systolic. First observe that $\hat{\Sigma}$ is simply-connected, since each loop in the 1–skeleton of $\hat{\Sigma}$ can be homotoped to a loop in the 1–skeleton of Σ. It remains to show that the vertex links in $\hat{\Sigma}$ are 6–large. The link of a vertex of type **2** in $\hat{\Sigma}$ is a pair of triangles glued along an edge, which is obviously ∞–large (since it is a join of an edge and a pair of vertices, it is a special case of Corollary 2.4).

The link Σ_6 of a (blue) vertex of type **6** in Σ is a cycle of length 12. Each of its edges joins a (green) vertex of type **3** and a (red) vertex of type **2**. In the link $\hat{\Sigma}_6$ of a type **6** vertex in $\hat{\Sigma}$, additional edges appear between pairs of green vertices originally at distance two in Σ_6. See the left side of Figure 4. The new edges create a cycle of length 6 in $\hat{\Sigma}_6$. Observe that $\hat{\Sigma}_6$ is obtained from that cycle by gluing six triangles along edges. Hence to show that $\hat{\Sigma}_6$ is 6–large, it suffices to apply six times Lemma 2.3.

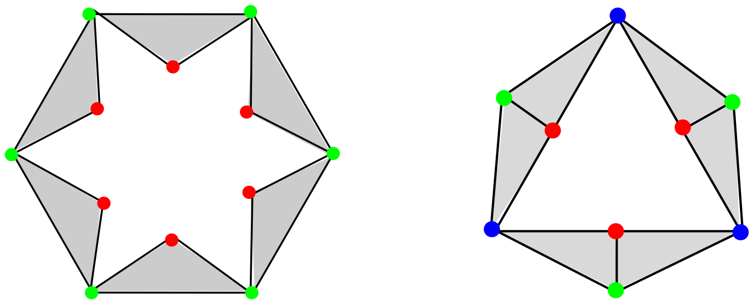

FIGURE 4. On the left the link of a vertex of type **6** in the systolization of the Coxeter complex of type $(2, 3, 6)$. On the right the link of a type **3** vertex.

The link Σ_3 of a (green) vertex of type **3** in Σ is a blue-red cycle of length 6. This is the only vertex type whose link $\hat{\Sigma}_3$ in $\hat{\Sigma}$ contains new vertices. There is in $\hat{\Sigma}_3$ one additional (green) vertex of type **3** for each type **2** vertex, coning off the star of the latter, which is a blue-red-blue edge-path of length 2. See the right side of Figure 4. There is a map $f\colon \hat{\Sigma}_3 \to \Sigma_3$ satisfying the hypothesis of Lemma 2.5. Since Σ_3 is 6–large, $\hat{\Sigma}_3$ is 6–large as well.

This concludes the proof that $\hat{\Sigma}$ is systolic and completes the discussion of our example. We could have also made Σ systolic by removing vertices of type **2** and edges of type **26**, i.e. by merging pairs of chambers along edges of type **26**. This approach might seem easier at first but does not generalize to buildings.

5. Rank three

In this section we construct a systolization of the building in Theorem 1.1. We are postponing the discussion of a systolization of the Davis complex till Section 8. As buildings of triangle type (l, k, m) with $l, k, m \geq 3$ are themselves systolic, we only need to study triangle groups of type $(2, k, m)$ with $k \geq 3$ and $m \geq 6$.

Recall that a vertex is of *type* **k** if its stabilizer is of order $2k$. Even if $m = k$ we will distinguish these two types and will refer to corresponding vertices as of type **m** or **k**, respectively.

Construction 5.1. Let X be a building of triangle type $(2, k, m)$ with $k \geq 3$ and $m \geq 6$. The *systolization* $\hat{X}$ of X is the flag simplicial complex spanned on the following 1–skeleton. The vertex set of $\hat{X}$ is the same as the vertex set of X. A pair of vertices is connected by an edge in $\hat{X}$ if it is either connected by an edge in X or it is a pair of vertices of type **k** that are adjacent to the same vertex of type **2**.

Note that the inclusion $X \subset \hat{X}$ is obviously a quasi-isometry. Moreover, the action of the group of type preserving automorphisms of X extends to $\hat{X}$. Before we prove that $\hat{X}$ is systolic, we need a handful of lemmas. We consider the CAT(0) metric on X in which the apartments are isometric to $\mathbb{E}^2$ or $\mathbb{H}^2$. Unless mentioned otherwise, all stars are closed.

Lemma 5.2. *Stars of vertices in X are convex.*

Proof. All angles in the triangle of type $(2, k, m)$ are $\leq \frac{\pi}{2}$. This implies that vertex stars in X are locally convex, hence convex. □

Let v, v' be two vertices of type **k** adjacent to a vertex w of type **2**. Then the concatenation of the edges vw and wv' is a geodesic. Consequently, w is the unique vertex of type **2** adjacent to both v and v'. Moreover, we have the following.

Corollary 5.3. *Let v, v' be two vertices of type* **k** *both adjacent to a vertex w of type* **2** *and to a vertex u of type* **m**. *Then w and u are adjacent.*

Proof. By Lemma 5.2 the geodesic vv' is contained in the star of u. Since $w \in vv'$, we have that w lies in the star of u. □

The final preparatory lemma involves triples of vertices of type **k**.

Lemma 5.4. *In X there is no cycle of length 6 whose vertices have alternating types* **2** *and* **k**.

Proof. Assume that there is such a cycle $vwv'w'v''w''v$, where v is of type **k** and w is of type **2**. Let S be the union of the star $\mathrm{st}_X(v)$ of v in X and the stars of the vertices of type **2** in $\mathrm{st}_X(v)$. We claim that S is locally convex, hence convex. At a vertex of type **k** in S that is distinct from v, all the triangles of S have a common edge, so S is locally convex as in Lemma 5.2. At a vertex of type **m**, there are fans of four triangles of S, but their corresponding angle equals $\frac{\pi}{m}$. Hence the total angle is $\frac{4\pi}{m} < \pi$, so S is locally convex at such a vertex as well, justifying the claim. Thus the geodesic $v'v''$ lies in S, whence $w' \in S$. All vertices of type **2** in S are adjacent to v. Thus $w' = w = w''$, contradiction. □

We split the proof of Theorem 1.1 into two steps. We first prove that $\hat{X}$ is simply-connected and then that its vertex links are 6–large.

Lemma 5.5. *The complex $\hat{X}$ is simply-connected.*

Proof. The fundamental group of $\hat{X}$ is carried by its 1–skeleton $\hat{X}^{(1)}$, we therefore only need to look at loops in $\hat{X}^{(1)}$. For each edge $e = vv'$ in $\hat{X}$ connecting two vertices of type $\mathbf{k}$, there is an edge-path γ in $X^{(1)}$ connecting v, v' of combinatorial length two via a vertex of type $\mathbf{2}$. The concatenation of γ and e bounds a triangle in $\hat{X}$, hence e is homotopic to γ. Thus any loop in $\hat{X}^{(1)}$ is homotopic to a loop in $X^{(1)}$. Since X is simply-connected, the complex $\hat{X}$ is simply-connected. □

Lemma 5.6. *Links of vertices in $\hat{X}$ are 6–large.*

Proof. First consider the link $\hat{X}_2$ of a vertex of type $\mathbf{2}$. The vertices of type $\mathbf{k}$ in $\hat{X}_2$ are pairwise connected by edges and hence span a simplex. Thus $\hat{X}_2$ is a join of that simplex with a discrete set of vertices of type $\mathbf{m}$, and is hence ∞–large by Corollary 2.4.

Now consider the link $\hat{X}_m$ of a vertex of type $\mathbf{m}$. The star of each type $\mathbf{2}$ vertex in $\hat{X}_m$ is a cone over the simplex formed by its adjacent vertices of type $\mathbf{k}$. Hence $\hat{X}_m$ is glued out of such cones and the subcomplex Y spanned by the vertices of type $\mathbf{k}$. By Lemma 2.3 it suffices to show that Y is 6–large. By Corollary 5.3, vertices of Y are connected by an edge if and only if they are at distance 2 in the link X_m in X. Now assume that there is a full cycle γ in Y of length 4 or 5. For any pair of consecutive vertices v, v' in γ there is a unique vertex w in X_m of type $\mathbf{2}$ adjacent to both v and v'. For any triple of consecutive vertices v, v', v'' in γ, consider the corresponding vertices w, w' in X_m of type $\mathbf{2}$ forming an edge-path $vwv'w'v''$. Since γ does not have diagonals, we have $w \neq w'$. Thus γ gives rise to a locally embedded closed edge-path in X_m of length 8 or 10. This contradicts the fact that the girth of X_m is $2m$. Thus Y and $\hat{X}_m$ are 6–large.

The link $\hat{X}_k$ of a vertex of type $\mathbf{k}$ contains all three types of vertices. Each vertex v of type $\mathbf{k}$ in $\hat{X}_k$ is adjacent to a unique vertex $w = w(v)$ of type $\mathbf{2}$ in $\hat{X}_k$ as well as to all type $\mathbf{m}$ neighbors of w. By Corollary 5.3, the vertex v is not adjacent to any other vertices of type $\mathbf{m}$ in $\hat{X}_k$. By Lemma 5.4, two vertices v, v' of type $\mathbf{k}$ in $\hat{X}_k$ are adjacent if and only if $w(v) = w(v')$. Hence the retraction $f\colon \hat{X}_k \to X_k$ assigning $f(v) = w(v)$ satisfies the hypothesis of Lemma 2.5. Since X_k is of girth $2k$, it is 6–large, thus $\hat{X}_k$ is 6–large as well. □

Thus the systolization $\hat{X}$ of the building X from Construction 5.1 is indeed systolic, as required in Theorem 1.1.

6. Rank four

In this section we construct a systolization of the 3–dimensional building required in Theorem 1.3. Let $T = \mathbf{abcd}$ be the tetrahedron that is the base chamber of a building X whose Coxeter group has rank 4. We label the edges of T by the exponents in the Coxeter presentation. We suppose as in Theorem 1.3 that all the exponents are finite and that all special subgroups of rank 3 are infinite triangle groups not of type $(2, 4, 4)$, $(2, 4, 5)$ or $(2, 5, 5)$. If all the exponents are ≥ 3, then X is systolic to begin with. Thus we further assume that there is precisely one edge labeled by 2, say **ab**. Without loss of generality we can also assume that the edge **ac** is labeled by $m \geq 6$ and the edge **ad** is labeled by $k \geq 3$. We have two possible labelings of **bc** and **bd**, see Figure 5.

Case I. The edge **bc** is labeled by $k' \geq 3$ and the edge **bd** is labeled by $m' \geq 6$.

Case II. The edge **bc** is labeled by $m' \geq 6$ and the edge **bd** is labeled by $k' \geq 3$.

The edge **cd** is labeled by $l \geq 3$. Note that if, keeping the other conditions, one allowed more edges labeled by 2, then it could only be the edge **cd**, and the rest of the edges would be labeled as in case I. However, in our article we only allow the edge **ab** to be labeled by 2.

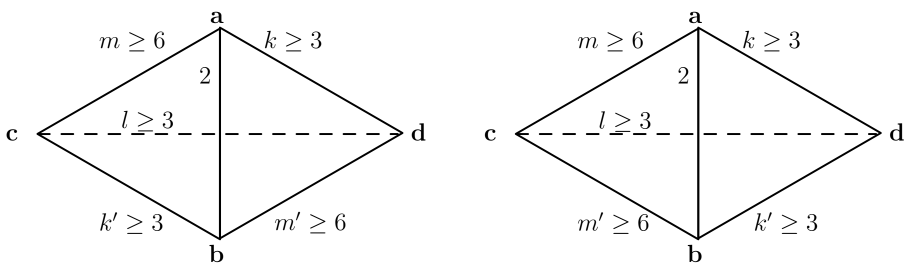

FIGURE 5. The two cases of admissible tetrahedral types.

A simplex in X is of *type* $I \subset$ **abcd** if it maps to I under the retraction to T. Unless otherwise mentioned we will denote vertices of type **a** by a, a', a'' etc. We say that two simplices are *adjacent* if they span a simplex.

We now proceed with the construction of a systolization.

Definition 6.1. Two vertices of the same type **c** or **d** adjacent to a common edge of type **ab** are *friends*. Two vertices of the same type **c** or **d** that are not friends but adjacent to a common edge of type labeled by k or k' are *acquaintances*.

Note that the link of an edge of type **ab** is a complete bipartite graph. Hence friends are also adjacent to common edges of types labeled by k or k', except for friends of type **d** in case II.

Construction 6.2. Let X be a building of the type described in case I or case II. The *systolization* $\hat{X}$ of X is the flag simplicial complex spanned on the following 1–skeleton. The vertex set of $\hat{X}$ is the same as the vertex set of X. Two vertices are adjacent in $\hat{X}$ if they are either adjacent in X or are friends or acquaintances. More explicitly, that means:

Case I: vertices of type **c** adjacent to a common edge of type **ad**, or vertices of type **d** adjacent to a common edge of type **bc**.

Case II: vertices of type **c** adjacent to a common edge of type **ad** or **bd**, or vertices of type **d** adjacent to a common edge of type **ab**.

In fact, if $k \geq 6$ or $k' \geq 6$, then fewer new edges would have done the job of systolizing X. To make the argument uniform we chose Construction 6.2 over a “minimal” one.

Our systolization of X induces systolizations of its 2–dimensional vertex links which are slightly thicker than the ones defined in Construction 5.1. For example if X has a vertex link that is the Coxeter complex of type $(2,3,6)$, then in Construction 6.2 we also add edges between pairs of vertices of type **6** adjacent to the same vertex of type **2**, and not only between such pairs of vertices of type **3** as in Construction 5.1.

Before we dive into the proof of Theorem 1.3, we first establish some preliminary lemmas on the combinatorial structure of the building X. As in Section 5, we find it convenient to use a metric argument. The tetrahedron T admits a Euclidean $\widetilde{A}_3$ metric, in which the dihedral angles at edges **ab** and **cd** are $\frac{\pi}{2}$, and the remaining dihedral angles are $\frac{\pi}{3}$. In particular, the dihedral angle at an edge labeled by an exponent i is $\geq \frac{\pi}{i}$. This equips X with a complete CAT(0) metric.

Lemma 6.3. *Vertex stars in X are convex.*

Proof. Let v be a vertex of X and e an edge in $\mathrm{lk}_X(v)$ of type labeled by an exponent i. It suffices to prove that $\mathrm{st}_X(v)$ is locally convex at e. The fans at e of simplices outside $\mathrm{st}_X(v)$ have length $2i-2$, with dihedral angles $\geq \frac{\pi}{i}$. Hence the total angle of such a fan is bounded below by $(2i-2)\frac{\pi}{i} \geq \pi$. □

Let c, c' be friends of type **c** adjacent to a common edge ab of type **ab** in X. The angle between the triangles abc and abc' is π, hence the geodesic cc' passes through ab. Thus c and c' determine the edge ab uniquely. We have the following strengthening.

Corollary 6.4. *Let c, c' be friends of type* **c**. *There are unique vertices a, b of types* **a**, **b** *adjacent to both c, c'. Any vertex of type* **d** *adjacent to both c, c' is adjacent to these a and b. The same holds if we interchange type* **c** *with* **d**.

Proof. If there is another vertex a' of type **a** with this property, then by Lemma 6.3 the geodesic cc' is contained in $\mathrm{st}_X(a')$. Thus $a \in \mathrm{st}_X(a')$, which is a contradiction. Remaining assertions follow in same way. □

Let c, c' be acquaintances of type **c** in the link of an edge ad of type **ad** in X. The angle between the triangles adc and adc' is then $\geq 4\frac{\pi}{3} > \pi$. Thus the geodesic cc' passes through ad, and the edge ad is determined uniquely by c, c'. Lemma 6.3 implies as before the following.

Corollary 6.5. *Let c, c' be acquaintances of type* **c** *in the link of an edge ad of type* **ad** *in X. The vertices a, d are unique of types* **a**, **d** *adjacent to both c, c'. There is no vertex of type* **b** *adjacent to both c, c'. The same holds if we interchange in case I types* **ad** *with* **bc** *and in case II type* **a** *with* **b**.

We need the following classification of triples and quadruples of friends and acquaintances.

Lemma 6.6. *If c, c', c'' are pairwise friends or acquaintances of type* **c**, *then there is an edge of type* **ad** *(or possibly* **bd** *in case II) adjacent to all of c, c', c''. This edge is unique, unless c, c', c'' are friends adjacent to a common edge of type* **ab**.

The same holds for a set of four vertices of type **c**. *Moreover, the same holds if we interchange types* **ad** *with* **bc** *in case I. In case II sets of friends of type* **d** *are adjacent to a common edge of type* **ab**.

Proof. Let S be the union of the star $\mathrm{st}_X(c)$ of c in X and the stars of all the edges of types labeled by k, k' in $\mathrm{lk}_X(c)$. We claim that S is locally convex, hence convex. The claim is proved by inspection. At each boundary edge of S, we look at a longest fan of tetrahedra in S. All the fans are of length 1 or 2, except possibly at an edge of a simplex $abc'd$ with abd in $\mathrm{lk}_X(c)$. There are fans of length 4 at ac', and also at bd in case I, and at bc' in case II, but their label is ≥ 6. In case II there are fans of length 3 at $c'd$ but its label is ≥ 3. As in Lemma 6.3, this proves the claim.

We first focus on case I. The geodesic $c'c''$ lies in S, whence the unique vertex a from Corollary 6.4 or 6.5 adjacent to both c', c'' lies in S. All vertices of type $\mathbf{a}$ in S lie in $\mathrm{lk}_X(c)$, thus a is adjacent to c, as desired. If c', c'' are acquaintances, then we obtain the vertex d adjacent to a in the same way. Otherwise, if c', c'' are friends, then let b be the unique vertex from Corollary 6.4 adjacent to both c', c''. If b is adjacent to c, then any vertex d adjacent to ab satisfies the lemma. Otherwise, since b lies in S, there is a unique vertex d in $\mathrm{lk}_X(c)$ such that bad is a triangle. Such d satisfies the lemma. By symmetry we can interchange types $\mathbf{ad}$ with $\mathbf{bc}$.

Now assume that there is a fourth vertex c^* of type $\mathbf{c}$ that is a friend or acquaintance of all c, c', c''. The vertex a is adjacent to c^*, as before. If any pair of the vertices of type c are acquaintances, then then the geodesic joining them passes through required ad. So now we assume that all the vertices of type c are pairwise friends. Let d', d'' be vertices of type $\mathbf{d}$ adjacent to the triples c, c', c^* and c, c'', c^*. Denote the edges from Corollary 6.4 for pairs cc^*, cc', cc'' by ab, ab', ab''. If the vertices d, d', d'' are distinct, then they satisfy the hypothesis of the current lemma with the roles of types $\mathbf{ad}$ and $\mathbf{bc}$ interchanged. Hence $b = b' = b''$, so that all c, c', c'', c^* are adjacent to b, whence to d.

We now consider case II. If c', c'' are friends, then the edge ab from Corollary 6.4 lies in S. If ab is adjacent to c, then there is nothing to prove. Otherwise, there is a unique triangle bad with ad or bd in $\mathrm{lk}_X(c)$, as desired. By symmetry, we can now assume that all c, c', c'' are pairwise acquaintances. Let $x'd', x''d''$ be the edges from Corollary 6.5 applied to the pairs c, c' and c, c'', where x', x'' are of types $\mathbf{a}$ or $\mathbf{b}$. We then inspect that the union of $\mathrm{st}_X(c)$ with the triangles $c'x'd', c''x''d''$ is convex. Hence the geodesic $c'c''$ passes through both $x'd'$ and $x''d''$. Thus $d' = d'', x' = x''$, as desired. For the last assertion, if we consider friends $d, d', d'', \ldots$ of type $\mathbf{d}$, then the edge ab from Corollary 6.4 lies in $\mathrm{lk}_X(d'')$.

Now assume that there is a fourth vertex c^* of type $\mathbf{c}$ that is a friend or acquaintance of all c, c', c''. If any pair of the vertices of type c are acquaintances, then the geodesic joining them passes through required xd. If all the vertices of type c are pairwise friends, then as in case I we consider the triple of vertices of type $\mathbf{d}$ for triples $c, c', c''; c, c', c^*$ and c, c'', c^*. If they are distinct, then by the previous paragraph all c, c', c'', c^* are adjacent to the same edge ab of type $\mathbf{ab}$, and any d adjacent to ab satisfies the lemma. □

To prove Theorem 1.3 we need to show that $\hat{X}$ is simply-connected and that all its vertex links are 6–large. To prove that vertex links are 6–large it suffices to prove that they are systolic. Equivalently, they are simply-connected and edge links of $\hat{X}$ are 6–large. We first prove simple-connectivity of $\hat{X}$ and its vertex links and then embark on proving 6–largeness of the edge links in Section 7.

Lemma 6.7. *The complex $\hat{X}$ and its vertex links are simply-connected.*

Proof. All the edges that we add to the 1–skeleton of X to obtain the 1–skeleton of $\hat{X}$ connect vertices at distance 2 in X. Hence $\hat{X}$ is simply-connected by the same argument as in the proof of Lemma 5.5: all loops can be homotoped into X that is simply-connected. Similarly, a link in $\hat{X}$ of a vertex of type **a** or **b** is obtained from its link in X by adding edges between some vertices at distance 2, by Corollaries 6.4 and 6.5. The link $\mathrm{lk}_{\hat{X}}(c)$ of a vertex c of type **c** is obtained from $\mathrm{lk}_X(c)$ in the following way. We add edges as before between some vertices at distance 2. But we also add vertices of type **c** and some incident edges. However, by Lemma 6.6, an edge between type **c** vertices is homotopic in $\mathrm{lk}_{\hat{X}}(c)$ to an edge-path of length 2 through a vertex of type **a** (or possibly **b** in case II). Moreover, each edge-path of length two in $\mathrm{lk}_{\hat{X}}(c)$ whose only vertex of type **c** is the middle vertex, is homotopic by Corollaries 6.4 and 6.5 to an edge-path of length 2 with the middle vertex replaced by a vertex of type **a** (or possibly **b**). Hence any loop in $\mathrm{lk}_{\hat{X}}(c)$ can be homotoped into $lk_X(c)$, which is simply-connected. Analogously, the link of a vertex of type **d** in $\hat{X}$ is simply-connected as well. □

7. Edge links

We will prove that the links of edges of types $\mathbf{ab}, \mathbf{ac}, \mathbf{ad}$ and **cd** in $\hat{X}$ as well as links of edges of friends and acquaintances are 6–large. Using symmetries of T it will follow that the links of edges of types **bd** and **bc** are also 6–large.

Lemma 7.1. *The link of an edge of type* **ab** *is a simplex.*

Proof. The link of an edge ab of type **ab** in X is a complete bipartite graph on vertices of types **c** and **d**. The link $\mathrm{lk}_{\hat{X}}(ab)$ of ab in $\hat{X}$ has the same set of vertices. All the vertices of type **c** in $\mathrm{lk}_{\hat{X}}(ab)$ are friends, and the same holds for all the vertices of type **d**. □

Proposition 7.2. *The link of an edge of type* **ad** *is ∞–large.*

Proof. The link $\mathrm{lk}_X(ad)$ of an edge ad of type **ad** is a bipartite graph of girth $2k$ on vertices of types **b** and **c**. We will now describe how one obtains $\mathrm{lk}_{\hat{X}}(ad)$ from $\mathrm{lk}_X(ad)$. By Construction 6.2, all the vertices of type **c** become connected by an edge. Each vertex d' of type **d** that appears in $\mathrm{lk}_{\hat{X}}(ad)$ is a friend of d, by Corollary 6.5. By Corollary 6.4, there is a unique vertex $b(d')$ of type **b** in $\mathrm{lk}_{\hat{X}}(ad)$ adjacent to d'. Moreover, the vertices of type **c** in $\mathrm{lk}_{\hat{X}}(ad)$ that are adjacent to d' are exactly the ones adjacent to $b(d')$. Finally, by Lemma 6.6, vertices d', d'' in $\mathrm{lk}_{\hat{X}}(ad)$ are adjacent if and only if $b(d') = b(d'')$. See Figure 6.

Let $B \subset \mathrm{lk}_{\hat{X}}(ad)$ be the subcomplex spanned by the vertices of type **b** and **c**. Then the retraction $f\colon \mathrm{lk}_{\hat{X}}(ad) \to B$ defined by $f(d') = b(d')$ satisfies the hypothesis of Lemma 2.5. Hence it suffices to prove that B is ∞–large. Observe that B is glued out of simplices as follows. There is one base-simplex spanned by all the vertices of type **c** in B. Each vertex of type **b** cones off

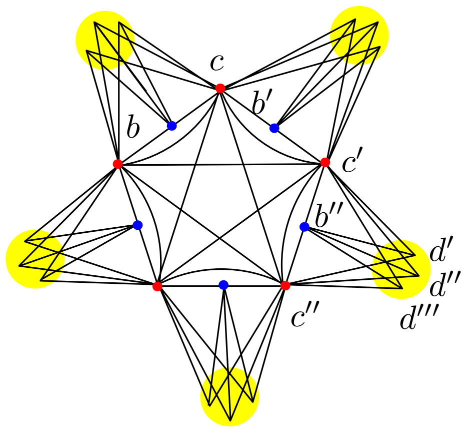

FIGURE 6. $\mathrm{lk}_{\hat{X}}(ad)$: yellow areas mark cliques of vertices d' with common $b(d')$.

a subsimplex of that base-simplex. Therefore by repeated application of Lemma 2.3, the complex B is ∞–large. □

Proposition 7.3. *The link of an edge of friends is ∞–large.*

Proof. We only verify the proposition for a pair of friends of type **c**, since the proof for type **d** friends is analogous. Let c, c' be friends. Let a, b be the vertices from Corollary 6.4, which are unique of type **a**, **b** in $\mathrm{lk}_{\hat{X}}(cc')$. Then the set D consisting of a, b and all the vertices of type **d** in $\mathrm{lk}_{\hat{X}}(cc')$ spans a simplex. This simplex is contained in another obtained by adding the set C of vertices of type **c** adjacent to ab. Consider a vertex c'' of type **c** in $\mathrm{lk}_{\hat{X}}(cc') - C$. By Lemma 6.6, there are unique $x \in \{a, b\}$ and d of type **d** in $\mathrm{lk}_{\hat{X}}(cc')$ such that c'' is adjacent to xd but not to $\{a, b\} - \{x\}$ or to other vertices of type **d**. Note that c'' is adjacent to all the vertices of C. Now suppose that c^* is another vertex of type **c** in $\mathrm{lk}_{\hat{X}}(cc') - C$. By Lemma 6.6 for quadruples, vertices c'' and c^* are adjacent if and only if c^* is also adjacent to xd. Thus $\mathrm{lk}_{\hat{X}}(cc')$ is obtained from the simplex spanned by $D \cup C$ by amalgamating with simplices corresponding to edges xd along subsimplices spanned by $\{x, d\} \cup C$. See Figure 7. By Lemma 2.3, the link $\mathrm{lk}_{\hat{X}}(cc')$ is ∞–large. □

Lemma 7.4. *The link of an edge of acquaintances is a simplex.*

Proof. Without loss of generality, we can assume that the acquaintances are of type **c**, denote them by c and c'. Again without loss of generality, let a, d be the vertices from Corollary 6.5. Then the link $\mathrm{lk}_{\hat{X}}(cc')$ contains ad but no other vertex of type **a** or **d** or a vertex of type **b**. Moreover, by Lemma 6.6, all of the vertices of type **c** in $\mathrm{lk}_{\hat{X}}(cc')$ are connected to ad, hence to each other. Thus $\mathrm{lk}_{\hat{X}}(cc')$ is a simplex. □

Proposition 7.5. *The link of an edge of type* **ac** *is 6–large.*

Proof. The link $\mathrm{lk}_X(ac)$ of an edge ac of type **ac** is a bipartite graph of girth $2m$ on vertices of types **b** and **d**. When we pass to $\mathrm{lk}_{\hat{X}}(ac)$, two vertices of

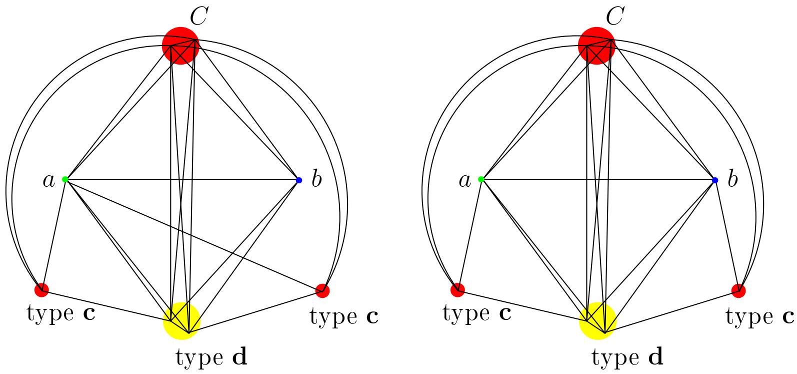

FIGURE 7. On the left the link $\operatorname{lk}_{\hat{X}}(cc')$ for friends c, c' in case I, on the right in case II.

type **d** are connected by an edge if and only if they have a common neighbor of type **b**, by Corollaries 6.4 and 6.5. Denote by $\Gamma \subset \operatorname{lk}_{\hat{X}}(ac)$ the subgraph spanned by the vertices of type **d**.

There are two classes of vertices of type **c** in $\operatorname{lk}_{\hat{X}}(ac)$ coming from friends respectively acquaintances of c. If c' in $\operatorname{lk}_{\hat{X}}(ac)$ is a friend of c, then by Corollary 6.4 it is adjacent to a unique vertex $b(c')$ of type **b** in $\operatorname{lk}_{\hat{X}}(ac)$ and to all its neighbors of type **d**. Moreover, c' is not adjacent to any other vertex of type **d** in $\operatorname{lk}_{\hat{X}}(ac)$. All c' with common $b = b(c')$ span a clique, which will be called the *b–clique*.

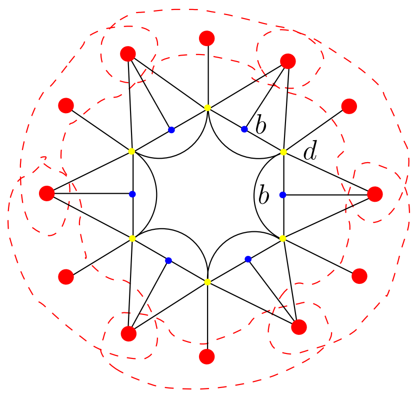

FIGURE 8. $\operatorname{lk}_{\hat{X}}(ac)$ in cases I and II. The big red dots stand for b–cliques and d–cliques. The edges connecting these cliques to other vertices represent multiple edges.

Assume now that c' in $\operatorname{lk}_{\hat{X}}(ac)$ is an acquaintance of c. By Corollary 6.5, the vertex c' is adjacent to a unique vertex $d(c')$ of type **d** in $\operatorname{lk}_{\hat{X}}(ac)$, and to none of the vertices of type **b**. All c' with common $d = d(c')$ span a clique, which we call the *d–clique*. By Lemma 6.6, two vertices c', c'' of type **c** in $\operatorname{lk}_{\hat{X}}(ac)$ are adjacent if and only if they have a common neighbor d. Thus if they are not in the same b–clique or d–clique, they are adjacent if and only if: either both c', c'' are friends of c and the stars of $b(c'), b(c'')$ in Γ intersect, or if c' is friend of c and c'' is an acquaintance of c and $d(c'')$ is adjacent to $b(c')$, or vice-versa. See Figure 8.

The stars of all the vertices of type **b**, and of acquaintances of c in $\mathrm{lk}_{\hat{X}}(ac)$ are simplices. By Lemma 2.3, to prove that the link $\mathrm{lk}_{\hat{X}}(ac)$ is 6–large it suffices to prove that the complex obtained by removing these vertices is 6–large. We remove them, and denote by Γ^* be the graph obtained from the 1–skeleton of the resulting complex by collapsing each b–clique to a vertex. By Lemma 2.5, it suffices to verify that Γ^* is 6–large. The maximal cliques of Γ correspond to b–cliques, thus Γ^* is obtained from Γ as in Definition 2.6. As in the proof of Lemma 5.6, the flag complex spanned on Γ is 6–large. Hence by Lemma 2.7, the flag complex spanned on Γ^* is 6–large. □

Proposition 7.6. *The link of an edge of type* **cd** *is 6–large.*

Proof. The link Γ of an edge cd in X of type **cd** is a bipartite graph on vertices of type **a** and **b** of girth $2l \geq 6$. We will now describe how one obtains $\mathrm{lk}_{\hat{X}}(cd)$ from Γ. If c' is a vertex of type **c** in $\mathrm{lk}_{\hat{X}}(cd)$ that is a friend of c, then by Corollary 6.4 it is adjacent to unique adjacent $a(c'), b(c')$ of type **a**, **b** in $\mathrm{lk}_{\hat{X}}(cd)$. If c' is an acquaintance of c, then by Corollary 6.5 it is adjacent to a unique $x = x(c')$ in $\mathrm{lk}_{\hat{X}}(cd)$ of type **a**, or possibly **b** in case II.

Let c', c'' be of type **c** in $\mathrm{lk}_{\hat{X}}(cd)$. In case I the vertices c', c'' are adjacent if and only if $a(c') = a(c'')$, by Lemma 6.6. In case II the vertices c', c'' are adjacent also if $b(c') = b(c'')$. In the same way we describe vertices of type **d** in $\mathrm{lk}_{\hat{X}}(cd)$. If vertices c', d' of types **c**, **d** in $\mathrm{lk}_{\hat{X}}(cd)$ are adjacent, then by Corollary 6.5, the vertices c, c' are friends, d, d' are friends and $a(c') = a(d')$, $b(c') = b(d')$. Conversely, if we have latter equalities, then c', d' are adjacent.

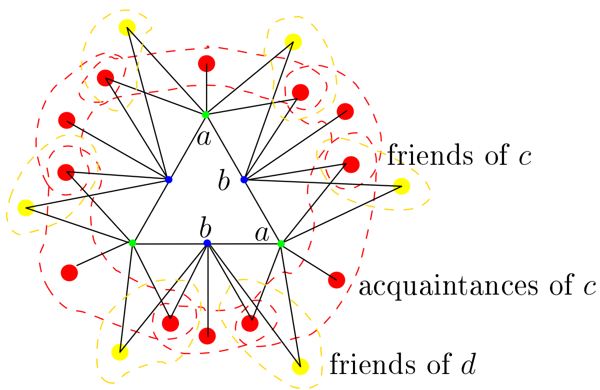

FIGURE 9. $\mathrm{lk}_{\hat{X}}(cd)$ in case II. The big red and yellow dots represent cliques of vertices of type **c**, respectively **d**. The encircled dots form even larger cliques.

Assume first that we are in case II. All the vertices of type **d** in $\mathrm{lk}_{\hat{X}}(cd)$ are then friends of d, and their stars are simplices. The stars of the acquaintances of c are also simplices. See Figure 9. By Lemma 2.3, to prove that $\mathrm{lk}_{\hat{X}}(cd)$ is 6–large it suffices to prove that the complex obtained by removing the vertices of type **d** and acquaintances of c is 6–large. We collapse the resulting complex by identifying all the friends c' of c with common $a(c')$ and $b(c')$. The result Γ^* of the collapse is 6–large by Lemma 2.7, since Γ is 6–large.

Finally, consider case I, see Figure 10. As before, we remove from $\mathrm{lk}_{\hat{X}}(cd)$ all the acquaintances of c and d, using Lemma 2.3. Let $\widetilde{\Gamma}$ be the graph obtained by collapsing the vertices c' of type $\mathbf{c}$ with common $a(c')$ and $b(c')$, and the vertices d' of type $\mathbf{d}$ with common $a(d')$ and $b(d')$. Assign to each vertex x of type $\mathbf{a}$ or $\mathbf{b}$ of $\widetilde{\Gamma}$ the pair (x, x), and to each collapsed clique the pair $\big(a(c'), a(c')b(c')\big)$ for c' of type $\mathbf{c}$, and $\big(b(d'), a(d')b(d')\big)$ for d' of type $\mathbf{d}$. This shows that $\widetilde{\Gamma}$ has the form required in Definition 2.8. By Lemma 2.9, the complex spanned on $\widetilde{\Gamma}$ is 6–large, since Γ has girth ≥ 6. □

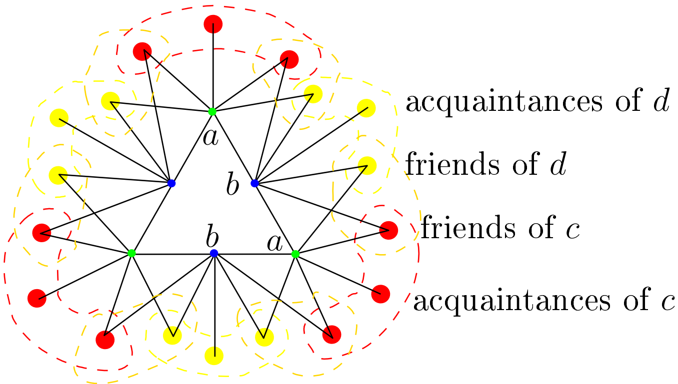

FIGURE 10. $\mathrm{lk}_{\hat{X}}(cd)$ in case I: The big red and yellow dots represent cliques of vertices of type $\mathbf{c}$, respectively $\mathbf{d}$. The encircled dots form cliques as well.

This concludes the proof that the systolization $\hat{X}$ in Construction 6.2 is indeed systolic, as required in Theorem 1.3.

8. Systolization of the Davis complex

In this Section we complete the proofs of Theorems 1.1 and 1.3, by systolizing the Davis complex.

Definition 8.1. Let X be a simplicial complex. The *face complex* X^f of X is the following simplicial complex. The vertex set of X^f is the set of simplices of X. A set of vertices of X^f spans a simplex of X^f, if the corresponding simplices of X are all contained in a common simplex of X.

Haglund observed the following.

Proposition 8.2 ([JŚ10, Prop B.1])**.** *If X is k–large, then its face complex X^f is k–large. Consequently, if X is systolic, then X^f is systolic, since it admits a deformation retraction to X.*

The systolization of the Davis complex in Theorem 1.1. Let X be a building of rank 3 type as in Theorem 1.1. The Davis complex of X is the barycentric subdivision X' of X. Let $\hat{X}$ be the systolization of X from Construction 5.1, except that in the case where all the exponents in the Coxeter presentation are ≥ 3, we take $\hat{X} = X$. Let $\hat{X}^f$ be the face complex of $\hat{X}$. The embedding $X \subset \hat{X}$ induces an embedding $X^f \subset \hat{X}^f$, and composing with $X' \subset X^f$ we obtain a natural embedding $X' \subset \hat{X}^f$, which is a quasi-isometry. By Proposition 8.2, the face complex $\hat{X}^f$ is a systolization of X'. □

The systolization of the Davis complex in Theorem 1.3. Let X be a building of rank 4 type as in Theorem 1.3. The Davis complex X_D of X is the 2–dimensional subcomplex of the barycentric subdivision X' of X obtained by removing the open stars of all the vertices of X' corresponding to the vertices of X. Let $\hat{X}$ be the systolization from Construction 6.2, except that in the case where all the exponents in Coxeter presentation are ≥ 3, we take $\hat{X} = X$. Let $\hat{X}^f$ be the face complex of $\hat{X}$, and let $\hat{X}^f_D$ be the subcomplex of $\hat{X}^f$ obtained by removing the open stars of all the vertices corresponding to the vertices of X. We will prove that $\hat{X}^f_D$ is a systolization of X_D.

By Proposition 8.2, the face complex $\hat{X}^f$ is systolic, whence all the vertex links of its full subcomplex $\hat{X}^f_D$ are 6–large. To prove that $\hat{X}^f_D$ is systolic it remains to prove that it is contractible. We first claim that for any simplex removed from $\hat{X}^f$ in the definition of $\hat{X}^f_D$ (in other words, for any simplex σ of X) its link in $\hat{X}^f$ is contractible. If σ is not a vertex, then its link in $\hat{X}^f$ is a cone, coned off by the vertex corresponding to σ, thus it is contractible. If σ is a vertex, then its link $\mathrm{lk}_{\hat{X}^f}(\sigma)$ can be collapsed as in Lemma 2.5 to $\big(\mathrm{lk}_{\hat{X}}(\sigma)\big)^f$. We showed in Lemma 6.7 that $\mathrm{lk}_{\hat{X}}(\sigma)$ is simply-connected, and since it is 6–large, it is systolic. Thus by Proposition 8.2 its face complex $\big(\mathrm{lk}_{\hat{X}}(\sigma)\big)^f$ is systolic, in particular contractible. Thus $\mathrm{lk}_{\hat{X}^f}(\sigma)$ is contractible, justifying the claim. In view of the claim, $\hat{X}^f_D$ is obtained from $\hat{X}^f$ by repeatedly removing open stars of simplices with contractible links (starting from maximal dimension). Thus $\hat{X}^f_D$ is homotopy equivalent to $\hat{X}^f$. This completes the proof of the fact that $\hat{X}^f_D$ is contractible, hence systolic.

The action of the group of type preserving automorphisms of X extends to $\hat{X}^f_D$. It remains to verify that the simplicial embedding $\psi\colon X_D \to \hat{X}^f_D$ is a quasi-isometry. To do that we construct the following quasi-inverse ϕ on the set E of these vertices of $\hat{X}^f_D$ that correspond to edges of $\hat{X}$. Note that the set E is 1–dense in the set of all the vertices of $\hat{X}^f_D$. Let $e \in E$ be an edge of $\hat{X}$. If e is an edge of X, then let $\phi(e)$ be the barycenter of e. Otherwise, if e is an edge of friends, then let $\phi(e)$ be the barycenter of the edge ab from Corollary 6.4. Finally, if e is an edge of acquaintances, then let $\phi(e)$ be the barycenter of the edge ad, or bd, from Corollary 6.5. It is easy to see that ψ and ϕ are quasi-inverses. It remains to verify that ϕ is Lipschitz. To do that it suffices to check that a pair of edges of triangle in $\hat{X}$ is sent to a pair of points at bounded distance. This follows from Lemma 6.6. □

9. Quotient construction

In this section we present a construction, suggested to us by Januszkiewicz, which might give rise to new systolic groups of cohomological dimension 3. In order to illustrate the method, we first recall the following. Let W be a Coxeter group of rank 4 with all exponents finite and ≥ 3. Then W is the fundamental group of the simplex of groups $\mathcal{W}$ over the tetrahedron T, where all the local groups $W_I,\ I \subsetneq T$ are special Coxeter subgroups of W. If I is a triangle, then $W_I = \mathbb{Z}_2$. If I is an edge, then W_I is a finite dihedral group. However, the vertex groups are infinite.

Let X_i be the Coxeter complex of the vertex group W_i. The complex X_i is systolic and for any normal finite index subgroup $W_i' \subset W_i$ avoiding the finite set (up to conjugacy) of elements with small translation length, the quotient $X_i' = W_i' \backslash X_i$ is 6–large and finite. Consider the simplex of groups $\mathcal{W}'$ over T obtained from $\mathcal{W}$ by replacing each W_i with W_i'. Then $\mathcal{W}'$ is *locally* 6*–large*, that is all vertex link developments are 6–large, since they coincide with X_i'. By [JŚ06, Thm 6.1], the simplex of groups $\mathcal{W}'$ is developable. This means that W_i/W_i' embed in the quotient W' of W by the normal closure of all the W_i'. The group W' acts geometrically on the development $T \times W'/\sim$, which is systolic. In particular W' is a systolic group of cohomological dimension 3.

We would like to repeat this construction for a Coxeter group W with an exponent 2 as in Theorem 1.3. The group W acts on the systolization $\hat{X}$ of the Coxeter complex X from Construction 6.2. The quotient complex $Y = W \backslash \hat{X}$ is equipped with the complex of groups structure $\mathcal{Y}$ coming from the stabilizers. The only infinite local groups are the vertex groups W_i at these vertices that come from the orbits of the original vertices of X. Each W_i acts on the systolic link $\hat{X}_i$ of the appropriate vertex in $\hat{X}$. Hence again for any normal finite index subgroup $W_i' \subset W_i$ avoiding a finite set of elements with small translation length, the quotient $\hat{X}_i' = W_i' \backslash \hat{X}_i$ is 6–large and finite. We form a complex of groups $\mathcal{Y}'$ over Y replacing all W_i by W_i'.

Question 9.1. *Is the fundamental group of $\mathcal{Y}'$ systolic?*

Observe that the problem is that the action of W on $\hat{X}$ has inversions, i.e. it stabilizes some simplices without fixing its vertices. Hence the complex Y has simplices that are not coming from the orbits of the original simplices of $\hat{X}$, but rather the simplices of its barycentric subdivision. Thus the combinatorial structure on the local developments differs from that of $\hat{X}_i'$. The property of being 6–large is not inherited under subdivision, thus we cannot apply [JŚ06, Thm 6.1].

Note that if $\mathcal{Y}'$ were developable, then we could combine appropriate simplices of the subdivision to turn the development into a systolic complex and prove that $\pi_1\mathcal{Y}'$ is systolic. Thus Question 9.1 reduces to the question of developability of what we could call *locally* 6*–large orbi-complexes of groups*. The proof should not differ much from the proof of [JŚ06, Thm 6.1], but would require a reworking that is outside the scope of the current article.

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