SEPARABILITY OF EMBEDDED SURFACES IN 3–MANIFOLDS

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Abstract. We prove that if $S$ is a properly embedded $\pi_1$–injective surface in a compact 3–manifold $M$, then $\pi_1S$ is separable in $\pi_1M$.

1. Introduction

A subgroup $H \subset G$ is separable if $H$ equals the intersection of finite index subgroups of $G$ containing $H$. Scott proved that if $G = \pi_1M$ for a manifold $M$ with universal cover $\tilde{M}$, then $H$ is separable if and only if each compact subset of $H \setminus \tilde{M}$ embeds in an intermediate finite cover of $M$ [Sco78, Lem 1.4]. Thus, if $H = \pi_1S$ for a compact surface $S \subset H \setminus \tilde{M}$, then separability of $H$ implies that $S$ embeds in a finite cover of $M$. Rubinstein–Wang found a properly immersed $\pi_1$–injective surface $S \looparrowright M$ in a graph manifold such that $S$ does not lift to an embedding in a finite cover of $M$, and they deduced that $\pi_1S \subset \pi_1M$ is not separable [RW98, Ex 2.6].

The objective of this paper is to prove:

**Theorem 1.1.** Let $M$ be a compact connected 3–manifold and let $S \subset M$ be a properly embedded connected $\pi_1$–injective surface. Then $\pi_1S$ is separable in $\pi_1M$.

The problem of separability of an embedded surface subgroup was raised for instance by Silver–Williams — see [SW09] and the references therein to their earlier works. The Silver–Williams conjecture was resolved recently by Friedl–Vidussi in [FV12], who proved that $\pi_1S$ can be separated from some element in $[\pi_1M, \pi_1M] - \pi_1S$ whenever $\pi_1S$ is not a fiber.

We proved Theorem 1.1 when $M$ is a graph manifold in [PW11, Thm 1.1]. Theorem 1.1 was also proven when $M$ is hyperbolic [Wis11]. In fact, every finitely generated subgroup of $\pi_1M$ is separable for hyperbolic $M$, by [Wis11] in the case $\partial M \neq \emptyset$ and by Agol’s theorem [Ago12] for $M$ closed.

**Overview:** In Section 2 we introduce the basic notation and reduce to studying irreducible $M$ that is simple in the sense that its Seifert-fibred components are

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products with base surfaces of sufficient complexity. In Section 3 we prove a topological result establishing separability of finite semicovers of $M$, i.e. maps required to be covers only over the interior of the blocks of the JSJ decomposition. This requires an omnipotence result for hyperbolic manifolds with boundary [Wis11, Cor 16.15] coming from virtual specialness.

To prove Theorem 1.1 we enhance the strategy employed in [PW11, Thm 1.1] for graph manifolds. Its main element was [PW11, Constr 4.13] which produced $S$–injective covers of $M^g$, which are covers $\overline{M^g}$ to which $S$ lifts and, among other properties, such that the intersection with $S$ is connected for each JSJ torus or component of $M^g$. We extend the construction of $S$–injective semicovers to all compact 3–manifolds in Section 4. We use the double coset separability of relatively quasiconvex subgroups of $\pi_1$ of hyperbolic 3–manifolds with boundary [Wis11, Thm 16.23] and separability of double cosets of embedded surface subgroups of $\pi_1$ of graph manifolds [PW11, Thm 1.2].

We conclude with the proof of Theorem 1.1 in Section 5.

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2. Framework and Reductions

Separability: We have the following finite index maneuverability: If $[H : H'] < \infty$ and $H' \subset G$ is separable, then $H \subset G$ is separable. Moreover, if $[G : G'] < \infty$ then a subgroup $H' \subset G'$ is separable if and only if $H' \subset G$ is separable. Finally, $H \subset G$ is separable if and only if for each $g \in G - H$ there is a finite quotient $\phi: G \to F$ with $\phi(g) \notin \phi(H)$. Thus $G$ is residually finite when $\{1_G\}$ is separable. We will freely employ these statements.

Assumptions on $M$ and $S$: Throughout this article $M$ is a compact connected 3–manifold and might have nonempty boundary. We will make additional assumptions arising from the following reductions:

We can assume that $S$ is not a sphere or a disc, since otherwise Theorem 1.1 follows from Hempel’s residual finiteness of Haken 3–manifolds [Hem87] and Perelman’s hyperbolization. By passing to a double cover we can assume that $M$ is oriented. Furthermore, if $S$ is not orientable, then the boundary $\overline{S}$ of its tubular neighborhood is an oriented $\pi_1$–injective surface. As $[\pi_1S : \pi_1\overline{S}] = 2$, the separability of $\pi_1\overline{S}$ implies separability of $\pi_1S$. Hence we can assume that $S$ is oriented. In the presence of our assumptions, the $\pi_1$–injectivity of $S$ is equivalent to saying that $S$ is incompressible and we will stay with this term.

Decomposition of $M$ into blocks: An incompressible surface $S$ in a reducible manifold can be homotoped into one of its prime factors, say $M_0$. Observe that there is a retraction $\pi_1M \to \pi_1M_0$ that kills the other factors. Consequently, if $g \in \pi_1M_0 - \pi_1S$, and we can separate $g$ from $\pi_1S$ in a finite quotient of $\pi_1M_0$, then we can separate $g$ from $\pi_1S$ in a finite quotient of $\pi_1M$. If $g \in \pi_1M - \pi_1M_0$,
then applying [Hem87] to the factors we can find a finite cover $M'$ of $M$ where all the terms of the normal form of $g$ lie outside factor subgroups. Then the path representing $g$ is nontrivial in the graph dual to the prime decomposition of $M'$, and it suffices to use the residual finiteness of free groups. Hence we can assume that $M$ is irreducible (though possibly $\partial$-reducible).

We will employ the JSJ decomposition of $M$, which is the minimal collection of incompressible tori (up to isotopy) each of whose complementary components is Seifert-fibred or atoroidal. If $M$ is a single Seifert-fibred manifold, then all finitely generated subgroups of $\pi_1 M$ are separable [Sco78], so we can assume that $M$ is not Seifert-fibred.

By passing to a double cover we can assume that there are no $\pi_1$-injective Klein bottles in $M$. We can also assume that $M$ is not a torus bundle over the circle, since then the only embedded surfaces are the fibers. Now a complementary component of JSJ tori cannot be simultaneously Seifert-fibred and algebraically atoroidal. Algebraically atoroidal components are hyperbolic by hyperbolization, in other words, their interior carries a geometrically finite hyperbolic structure (possibly of infinite volume if there are non-toroidal boundary components, as in a handlebody). We will call these complementary components hyperbolic blocks. The other complementary components are Seifert-fibred and we assemble adjacent Seifert-fibred components into graph manifold blocks. The JSJ tori that are adjacent to at least one hyperbolic block are called transitional. An incompressible surface can be homotoped so that its intersection with each block is incompressible. Moreover, we can assume that $S$ intersects each Seifert-fibred component along a horizontal or vertical surface, unless $S$ is a $\partial$-parallel annulus. In the latter case separability follows easily from separability of the boundary torus group (since it is a maximal abelian subgroup) and from a variant of Lemma 3.1 with $T^*$ in the boundary.

**The $m$–characteristic covers and simplicity:** For a manifold $E$ let $E_{[m]}$ denote the $m$–characteristic cover of $E$, which is the regular cover corresponding to the intersection of all subgroups of index $m$ in $\pi_1E$. In particular, if $T$ is a torus, then $T_{[m]}$ is the cover corresponding to the subgroup $m\mathbb{Z} \times m\mathbb{Z} \subset \mathbb{Z} \times \mathbb{Z} = \pi_1T$. A Seifert-fibred manifold $E$ is simple if it is the product of the circle with a surface of genus $\geq 1$ that has at least 2 boundary components. This boundary hypothesis (not required in [PW11]) ensures that there is a retraction onto each boundary component. Consequently, $E_{[m]}$ restricts to $m$–characteristic covers on boundary tori. An irreducible 3–manifold $M$ is simple if its Seifert-fibred components are simple. We will pass to a simple finite cover of $M$ in Lemma 3.1.

Finally, by separability of the JSJ tori subgroups in $\pi_1M$, we can assume that $S \subset M$ is straight. This means that $S$ does not intersect a Seifert-fibred component $E$ of $M$ along a vertical annulus with both boundary circles in the same boundary torus of $E$. 


3. Extending semicovers to covers

We begin this section with the following additional simplification:

**Lemma 3.1.** Let $M$ be an irreducible 3–manifold that is not Seifert-fibred. Then $M$ has a finite cover $M'$ that is simple. Moreover, given covers $\{T^*\}$ of the transitional tori $\{T\}$ in $M$, we can assume that all the tori of $M'$ covering $T$ are isomorphic and factor through $T^*$.

A key element of the proof employs the following omnipotence result for hyperbolic 3–manifolds with boundary.

**Lemma 3.2** ([Wis11, Cor 16.15]). Let $M^h$ be a hyperbolic 3–manifold with boundary tori $\{T\}$. There exist finite covers $\{\hat{T}\}$ such that for any further finite covers $\{T'\}$ there exists a finite cover $M^{h'}$ of $M^h$ that restricts on boundary tori to $\{T'\}$.

By passing to a further cover we can assume that $M^{h'} \to M^h$ is regular.

**Proof of Lemma 3.1.** Luecke and Wu proved in [LW97, Prop 4.4] that every graph manifold block $M^g$ of $M$ has a finite cover $M^g'$ that is simple. Without loss of generality we can assume that $M^g' \to M^g$ is regular.

Choose $m$ such that

(i) for any $M^g$ adjacent along a torus $T$ to a hyperbolic block $M^h$, the cover $T'_{[m]}$ of the torus $T' \subset \partial M^{g'}$ covering $T$ factors through $\hat{T}$ of Lemma 3.2 and through $T^*$.

(ii) for a transitional or boundary torus $T \subset M$ adjacent to a hyperbolic block $M^h$ but not to a graph manifold block, the cover $T_{[m]}$ factors through $\hat{T}$ of Lemma 3.2 and through $T^*$, if $T$ is transitional.

By Lemma 3.2, each hyperbolic block $M^h$ of $M$ has a finite regular cover $M^{h'}$ restricting on the boundary to $\{T'_{[m]}\}$ of (i) or $\{T_{[m]}\}$ of (ii). For a Seifert-fibred component $E$ of one of the simple graph manifolds $M^{g'}$, as $E$ is simple its retractive property ensures that the cover $E_{[m]}$ restricts to $m$–characteristic covers on its boundary tori. Gluing appropriately many copies of the various $E_{[m]}$ and $M^{h'}$ together provides the desired simple cover $M'$ of $M$. $\square$

Henceforth we always assume that $M$ is simple (and irreducible as assumed in Section 2).

**Definition 3.3.** A semicover $\overline{M}$ of $M$ w.r.t. transitional tori is a local embedding $\overline{M} \to M$ that restricts to a covering map over each transitional torus and over each open block. Thus $\overline{M}$ can only fail to be a covering map at a component of $\partial M$ that covers a transitional torus $T \subset M$. We say that $\overline{M} \to M$ is finite if $\overline{M}$ is compact.

We can now prove the main result of this section.
Proposition 3.4. Any finite semicover $\overline{M}$ of $M$ has a finite cover $\overline{M}' \to \overline{M}$ that embeds in a finite cover $M'$ of $M$.

Proof of Proposition 3.4. By Lemma 3.1, there is a cover $\hat{M}$ of $M$ such that for each transitional torus $T$ of $M$ all of the tori $\hat{T} \subset \hat{M}$ covering $T$ are isomorphic and factor through all the covers of $T$ in $\overline{M}$.

Let $\overline{M}' \to \overline{M}$ be the pullback of the cover $\hat{M} \to \hat{M}$ via the semicover $M' \to M$. Then on all of its boundary tori the semicover $\overline{M}' \to \overline{M}$ restricts to the corresponding tori in $\hat{M}$. Gluing $\overline{M}'$ with appropriately many copies of the components of $\hat{M}$ extends $\overline{M}'$ to a cover $M'$ of $M$. □

4. Surface-injective semicovers

In this section we construct a family of semicovers of $M$ to which a given surface $S \subset M$ lifts. We keep the assumptions from Section 2.

We will use the following case of a theorem of Martínez-Pedroza:

Theorem 4.1 ([MP09, Thm 1.1]). Let $S_0 \subset M^h$ be an incompressible geometrically finite surface properly embedded in a hyperbolic manifold $M^h$. Let $\partial S_0 = C_1 \sqcup \ldots \sqcup C_k$ and suppose these circles are contained in boundary tori $T_1, \ldots, T_k$ of $M^h$ (some $T_i$ may coincide). Then for almost all cyclic covers $T_i'$ of $T_i$ to which $C_i$ lift, the graph of spaces obtained by amalgamating $S_0$ with $T_i'$ along $C_i$ maps $\pi_1$–injectively into $M^h$ and the image of its $\pi_1$ in $\pi_1 M^h$ is relatively quasiconvex.

The separability of double cosets of relatively quasiconvex subgroups of $\pi_1$ of a hyperbolic 3–manifold with boundary was established in [Wis11, Thm 16.23]. Consequently, we have:

Corollary 4.2. For almost all cyclic covers $T_i'$ described in Theorem 4.1, the group $\pi_1(S_0 \sqcup_{\{C_i\}} \{T_i'\})$ is separable in $\pi_1 M^h$.

Corollary 4.3. The subgroup $\pi_1 S_0$ as well as the double cosets $\pi_1 S_0 \pi_1 T_i$ are separable in $\pi_1 M^h$.

Definition 4.4. Let $S \subset M$ be an incompressible surface. A semicover $\overline{M} \to M$ to which $S$ lifts is $S$–injective if for each hyperbolic or graph manifold block $B$ of $\overline{M}$ the intersection $S \cap B$ is connected. We allow $S$ itself to be disconnected.

Lemma 4.5 ([PW11, Constr 4.13]). Let $S \subset M^g$ be a possibly disconnected straight incompressible surface in a simple graph manifold. Suppose $n$ is an integer divisible by all of the degrees of (possibly disconnected) covers $S \cap E \to F$, where $E \subset M^g$ is a Seifert-fibred component with base surface $F$, and $S \cap E$ is horizontal. Then there is a finite cover $\overline{M}^g$ of $M^g$ to which $S$ lifts such that for each torus $\overline{T} \subset \partial \overline{M}^g$ intersecting $S$:

• $S \cap \overline{T}$ is connected,
• \( T \) maps to a torus \( T \subset \partial M^g \) with degree \( \frac{n}{|S \cap T|} \).

Here \( |S \cap T| \) denotes the number of components in the intersection of the surface \( S \) with the torus \( T \).

**Proposition 4.6.** Let \( S \subset M \) be an incompressible surface. Let \( S_0 \) be a component of intersection of \( S \) with a hyperbolic or graph manifold block \( M_0 \) of \( M \). Let \( T_i \) be the (possibly repeating) tori of \( \partial M_0 \) intersected by \( S_0 \). Let \( g \in \pi_1 M_0 - \pi_1 S_0 \) (resp. \( g_i \in \pi_1 M_0 - \pi_1 S_0 \pi_1 T_i \) for each \( i \)). Then there is a finite \( S \)-injective semicover \( \overline{M} \) with \( g \notin \pi_1 \overline{M}_0 \) (resp. \( g_i \notin \pi_1 \overline{M}_0 \pi_1 T_i \)), where \( \overline{M}_0 \) is the block of \( \overline{M} \) containing the lift of \( S_0 \).

To make sense of the double cosets \( \pi_1 S_0 \pi_1 T_i \) inside \( \pi_1 M_0 \), pick basepoints \( x_i \) of \( M_0 \) in \( C_i \) and interpret \( \pi_1 S_0 \), \( \pi_1 T_i \) as subgroups of \( \pi_1 M_0 \) determined by loops based at \( x_i \) staying in \( S_0, T_i \), respectively.

**Proof.** In the case where we assume \( g \notin \pi_1 S_0 \), we use that \( \pi_1 S_0 \) is separable in \( \pi_1 M_0 \). If \( M_0 \) is hyperbolic, this follows from Corollary 4.3. If \( M_0 \) is a graph manifold, we use separability of embedded surfaces in graph manifolds [PW11, Thm 1.1]. Hence there is a finite cover \( \overline{M}_0^\ast \to M_0 \) to which \( S_0 \) lifts with \( g \notin \pi_1 \overline{M}_0^\ast \).

In the case where we assume \( g_i \notin \pi_1 S_0 \pi_1 T_i \) for all \( i \), we use that each double coset \( \pi_1 S_0 \pi_1 T_i \) is separable in \( \pi_1 M_0 \). This follows from Corollary 4.3 and [PW11, Thm 1.2]. Hence there exists a cover \( \overline{M}_0^\ast \to M_0 \) to which \( S_0 \) lifts with \( g_i \notin \pi_1 \overline{M}_0^\ast \pi_1 T_i \). Let \( n_i \) be the degree of the restriction of \( \overline{M}_0^\ast \to M_0 \) to the torus intersecting (the lift of) \( S_0 \) along (the lift of) \( C_i \).

Choose \( n \) so that it is divisible by the numbers in (a)–(c), and also satisfies (d):

(a) every \( |S \cap T| \), where \( T \) is a transitional or boundary torus,
(b) the degrees of (possibly disconnected) covers \( S \cap E \to F \), where \( E \subset M \) is a Seifert-fibred component with base surface \( F \), and \( S \cap E \) is horizontal,
(c) each \( n_i |S \cap T_i| \) as above.
(d) We also require \( \frac{n}{|S \cap T|} \) to be the degree of one of the covers \( T' \to T \) given by Theorem 4.1 for a geometrically finite component of \( S \cap M^h \) in a hyperbolic block \( M^h \) of \( M \).

We construct the semicover \( \overline{M} \) in the following way. Start with a copy \( \overline{S} \) of \( S \). Let \( T \) be a transitional or boundary torus of \( M \). For each component of \( S \cap T \) we attach along the corresponding circle in \( \overline{S} \) the degree \( \frac{n}{|S \cap T|} \) cyclic cover \( \overline{T} \) of \( T \). The value \( \frac{n}{|S \cap T|} \) is an integer by (a).

For each graph manifold block \( M^g \) of \( M \) consider the finite (possibly disconnected) cover \( \overline{M}^g \) from Lemma 4.5 applied to the surface \( S \cap M^g \). The boundary components of \( \overline{M}^g \) intersecting \( S \) coincide with the \( \overline{T} \) attached to \( \overline{S} \) above.

Consider now a hyperbolic block \( M^h \) of \( M \) such that \( S \cap M^h \) is a union of fibers. In this case we choose \( \overline{M}^h \) to be the union of \( \frac{n}{|S \cap M^h|} \) copies of degree \( \frac{1}{|S \cap M^h|} \)
cyclic covers of \( M^h \) to which components of \( S \cap M^h \) lift. Again, components of \( \partial M^h \) coincide with \( \overline{T} \), so that we can consistently attach the \( M^h \) to \( \overline{S} \).

Finally, if \( S \cap M^h \) is not a union of fibers, then \( \pi_1 \) of each of its components is relatively quasiconvex in \( \pi_1 M^h \), so by (d) and Corollary 4.2, there is a finite cover \( \overline{M}^h \) extending \( S \cap M^h \cup \{ \overline{T} \} \), and we consistently attach the \( \overline{M}^h \) to \( \overline{S} \).

At this point we have constructed a finite \( S \)-injective semicover \( \overline{M} \), without yet separating \( g \) (resp. \( g_i \)). Now we replace the block \( \overline{M}_0 \) with its fiber product with \( \overline{M}_0 \). (Algebraically \( \pi_1 \) of the fiber product is \( \pi_1 \overline{M}_0 \cap \pi_1 \overline{M}_0 \subset \pi_1 M_0 \).) This is possible by (c) which guarantees that the fiber product agrees with \( \overline{M}_0 \) on its boundary components intersecting \( S_0 \). After this replacement, \( \overline{M} \) satisfies the requirement on \( g \) (resp. \( g_i \)), by definition of \( \overline{M}_0 \).

\[ \square \]

5. Separability

**Proof of Theorem 1.1.** Choose a basepoint of \( M \) in \( S \) outside all JSJ and boundary tori. Let \( f \in \pi_1 M - \pi_1 S \). Consider the based cover \( M^S \) of \( M \) with fundamental group \( \pi_1 S \). Let \( \gamma^S \) be a path in \( M^S \) starting at the basepoint and representing \( f \). Then \( \gamma^S \) does not terminate on \( S \). Assume that \( \gamma^S \) is chosen so that it does not backtrack, i.e. its image in \( M \) intersects the transitional tori a minimal number of times.

Firstly, consider the case where \( \gamma^S \) terminates in a block \( M^S_0 \subset M^S \) that intersects the lift of \( S \). Denote \( S_0 = S \cap M^S_0 \) and let \( M_0 \subset M \) be the block covered by \( M^S_0 \). In the case where \( S_0 \) contains the basepoint, let \( g \in \pi_1 M_0 \) be an element represented by a path in \( M^S_0 \) from the basepoint to the endpoint of \( \gamma^S \).

By Proposition 4.6 there is a finite \( S \)-injective semicover \( \overline{M} \) of \( M \) with \( g \notin \pi_1 \overline{M}_0 \). Thus \( \gamma^S \) projects to a path \( \overline{\gamma} \) in \( \overline{M} \) that ends in \( \overline{M}_0 \) outside the lift of \( S_0 \). By Proposition 3.4 the semicover \( \overline{M} \) has a finite cover \( \overline{M'} \) that extends to a finite cover \( M' \) of \( M \). Since the endpoint of the lift of \( \overline{\gamma} \) to \( M' \), which lies in \( \overline{M'} \), does not terminate on the based connected component of \( \preimage \) of \( S_0 \), we have \( f \notin \pi_1 M' \pi_1 S \), as desired.

Secondly, consider the case where \( \gamma^S \) terminates in a block of \( M^S \) disjoint from the lift of \( S \). Let \( T^S \subset M^S \) be then the first connected component of \( \preimage \) of a transitional torus \( T \subset M \) crossed by \( \gamma^S \) and disjoint from \( S \). Let \( M^S_0 \) be the last block that \( \gamma^S \) travels through before it hits \( T^S \). Let \( S_0 = S \cap M^S_0 \) and let \( M_0 \subset M \) be the block covered by \( M^S_0 \). If \( T \) coincides with one of the tori \( T_i \subset M_0 \) crossed by \( S_0 \) along \( C_i \), then let \( x_i \in C_i \) be a basepoint for \( M_0 \). Let \( x'_i \) be a lift of \( x_i \) in \( T^S \). We keep the notation \( x_i \) for the lift of \( x_i \) to \( S_0 \subset M^S_0 \). Let \( g_i \in \pi_1 M_0 \) be an element represented by a path in \( M^S_0 \) from \( x_i \) to \( x'_i \).

Since \( T^S \) is disjoint from \( S_0 \), we have \( g_i \notin \pi_1 S_0 \pi_1 T_i \). By Proposition 4.6 there is a finite \( S \)-injective semicover \( \overline{M} \) of \( M \) with \( g_i \notin \pi_1 \overline{M}_0 \pi_1 T_i \) for all \( i \). In other words, \( \overline{\gamma} \) leaves \( \overline{M}_0 \) through a torus disjoint from \( S_0 \).
By Proposition 3.4 the semicover $\overline{M}$ has a finite cover $\overline{M}'$ that extends to a finite cover $M'$ of $M$. By separability of the transitional tori groups (since they are maximal abelian) and residual finiteness of the free group (dual to transitional tori), by replacing $M'$ with a further cover we can assume that the lift of $\gamma$ to $M'$ does not pass twice through the same transitional torus.

Let $T' \subset M'$ be the projection of $T^S$. Consider the double cover $M''$ obtained by taking two copies of $M'$, cutting along $T'$, and regluing. Then the based connected component of the preimage of $S$ lies in one copy of (the cut) $M'$ in $M''$, while the endpoint of the lift of $\gamma$ lies in the other copy. Hence $f \notin \pi_1 M'' \pi_1 S$, as desired. □

References


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