

# SEPARABILITY OF EMBEDDED SURFACES IN 3-MANIFOLDS

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ABSTRACT. We prove that if  $S$  is a properly embedded  $\pi_1$ -injective surface in a compact 3-manifold  $M$ , then  $\pi_1 S$  is separable in  $\pi_1 M$ .

## 1. INTRODUCTION

A subgroup  $H \subset G$  is *separable* if  $H$  equals the intersection of finite index subgroups of  $G$  containing  $H$ . Scott proved that if  $G = \pi_1 M$  for a manifold  $M$  with universal cover  $\widetilde{M}$ , then  $H$  is separable if and only if each compact subset of  $H \backslash \widetilde{M}$  embeds in an intermediate finite cover of  $M$  [Sco78, Lem 1.4]. Thus, if  $H = \pi_1 S$  for a compact surface  $S \subset H \backslash \widetilde{M}$ , then separability of  $H$  implies that  $S$  embeds in a finite cover of  $M$ . Rubinstein–Wang found a properly immersed  $\pi_1$ -injective surface  $S \looparrowright M$  in a graph manifold such that  $S$  does not lift to an embedding in a finite cover of  $M$ , and they deduced that  $\pi_1 S \subset \pi_1 M$  is not separable [RW98, Ex 2.6].

The objective of this paper is to prove:

**Theorem 1.1.** *Let  $M$  be a compact connected 3-manifold and let  $S \subset M$  be a properly embedded connected  $\pi_1$ -injective surface. Then  $\pi_1 S$  is separable in  $\pi_1 M$ .*

The problem of separability of an embedded surface subgroup was raised for instance by Silver–Williams — see [SW09] and the references therein to their earlier works. The Silver–Williams conjecture was resolved recently by Friedl–Vidussi in [FV12], who proved that  $\pi_1 S$  can be separated from some element in  $[\pi_1 M, \pi_1 M] - \pi_1 S$  whenever  $\pi_1 S$  is not a fiber.

We proved Theorem 1.1 when  $M$  is a graph manifold in [PW11, Thm 1.1]. Theorem 1.1 was also proven when  $M$  is hyperbolic [Wis11]. In fact, every finitely generated subgroup of  $\pi_1 M$  is separable for hyperbolic  $M$ , by [Wis11] in the case  $\partial M \neq \emptyset$  and by Agol’s theorem [Ago12] for  $M$  closed.

**Overview:** In Section 2 we introduce the basic notation and reduce to studying irreducible  $M$  that is *simple* in the sense that its Seifert-fibred components are

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products with base surfaces of sufficient complexity. In Section 3 we prove a topological result establishing separability of finite *semicovers* of  $M$ , i.e. maps required to be covers only over the interior of the blocks of the JSJ decomposition. This requires an omnipotence result for hyperbolic manifolds with boundary [Wis11, Cor 16.15] coming from virtual specialness.

To prove Theorem 1.1 we enhance the strategy employed in [PW11, Thm 1.1] for graph manifolds. Its main element was [PW11, Constr 4.13] which produced  $S$ -*injective* covers of  $M^g$ , which are covers  $\overline{M^g}$  to which  $S$  lifts and, among other properties, such that the intersection with  $S$  is connected for each JSJ torus or component of  $\overline{M^g}$ . We extend the construction of  $S$ -injective semicovers to all compact 3-manifolds in Section 4. We use the double coset separability of relatively quasiconvex subgroups of  $\pi_1$  of hyperbolic 3-manifolds with boundary [Wis11, Thm 16.23] and separability of double cosets of embedded surface subgroups of  $\pi_1$  of graph manifolds [PW11, Thm 1.2].

We conclude with the proof of Theorem 1.1 in Section 5.

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## 2. FRAMEWORK AND REDUCTIONS

**Separability:** We have the following finite index maneuverability: If  $[H : H'] < \infty$  and  $H' \subset G$  is separable, then  $H \subset G$  is separable. Moreover, if  $[G : G'] < \infty$  then a subgroup  $H' \subset G'$  is separable if and only if  $H' \subset G$  is separable. Finally,  $H \subset G$  is separable if and only if for each  $g \in G - H$  there is a finite quotient  $\phi: G \rightarrow F$  with  $\phi(g) \notin \phi(H)$ . Thus  $G$  is *residually finite* when  $\{1_G\}$  is separable. We will freely employ these statements.

**Assumptions on  $M$  and  $S$ :** Throughout this article  $M$  is a compact connected 3-manifold and might have nonempty boundary. We will make additional assumptions arising from the following reductions:

We can assume that  $S$  is not a sphere or a disc, since otherwise Theorem 1.1 follows from Hempel's residual finiteness of Haken 3-manifolds [Hem87] and Perelman's hyperbolization. By passing to a double cover we can assume that  $M$  is oriented. Furthermore, if  $S$  is not orientable, then the boundary  $\widehat{S}$  of its tubular neighborhood is an oriented  $\pi_1$ -injective surface. As  $[\pi_1 S : \pi_1 \widehat{S}] = 2$ , the separability of  $\pi_1 \widehat{S}$  implies separability of  $\pi_1 S$ . Hence we can assume that  $S$  is oriented. In the presence of our assumptions, the  $\pi_1$ -injectivity of  $S$  is equivalent to saying that  $S$  is *incompressible* and we will stay with this term.

**Decomposition of  $M$  into blocks:** An incompressible surface  $S$  in a reducible manifold can be homotoped into one of its prime factors, say  $M_0$ . Observe that there is a retraction  $\pi_1 M \rightarrow \pi_1 M_0$  that kills the other factors. Consequently, if  $g \in \pi_1 M_0 - \pi_1 S$ , and we can separate  $g$  from  $\pi_1 S$  in a finite quotient of  $\pi_1 M_0$ , then we can separate  $g$  from  $\pi_1 S$  in a finite quotient of  $\pi_1 M$ . If  $g \in \pi_1 M - \pi_1 M_0$ ,

then applying [Hem87] to the factors we can find a finite cover  $M'$  of  $M$  where all the terms of the normal form of  $g$  lie outside factor subgroups. Then the path representing  $g$  is nontrivial in the graph dual to the prime decomposition of  $M'$ , and it suffices to use the residual finiteness of free groups. Hence we can assume that  $M$  is irreducible (though possibly  $\partial$ -reducible).

We will employ the *JSJ decomposition* of  $M$ , which is the minimal collection of incompressible tori (up to isotopy) each of whose complementary components is Seifert-fibred or atoroidal. If  $M$  is a single Seifert-fibred manifold, then all finitely generated subgroups of  $\pi_1 M$  are separable [Sco78], so we can assume that  $M$  is not Seifert-fibred.

By passing to a double cover we can assume that there are no  $\pi_1$ -injective Klein bottles in  $M$ . We can also assume that  $M$  is not a torus bundle over the circle, since then the only embedded surfaces are the fibers. Now a complementary component of JSJ tori cannot be simultaneously Seifert-fibred and algebraically atoroidal. Algebraically atoroidal components are *hyperbolic* by hyperbolization, in other words, their interior carries a geometrically finite hyperbolic structure (possibly of infinite volume if there are non-toroidal boundary components, as in a handlebody). We will call these complementary components *hyperbolic blocks*. The other complementary components are Seifert-fibred and we assemble adjacent Seifert-fibred components into *graph manifold blocks*. The JSJ tori that are adjacent to at least one hyperbolic block are called *transitional*. An incompressible surface can be homotoped so that its intersection with each block is incompressible. Moreover, we can assume that  $S$  intersects each Seifert-fibred component along a horizontal or vertical surface, unless  $S$  is a  $\partial$ -parallel annulus. In the latter case separability follows easily from separability of the boundary torus group (since it is a maximal abelian subgroup) and from a variant of Lemma 3.1 with  $T^*$  in the boundary.

**The  $m$ -characteristic covers and simplicity:** For a manifold  $E$  let  $E_{[m]}$  denote the  $m$ -characteristic cover of  $E$ , which is the regular cover corresponding to the intersection of all subgroups of index  $m$  in  $\pi_1 E$ . In particular, if  $T$  is a torus, then  $T_{[m]}$  is the cover corresponding to the subgroup  $m\mathbb{Z} \times m\mathbb{Z} \subset \mathbb{Z} \times \mathbb{Z} = \pi_1 T$ . A Seifert-fibred manifold  $E$  is *simple* if it is the product of the circle with a surface of genus  $\geq 1$  that has at least 2 boundary components. This boundary hypothesis (not required in [PW11]) ensures that there is a retraction onto each boundary component. Consequently,  $E_{[m]}$  restricts to  $m$ -characteristic covers on boundary tori. An irreducible 3-manifold  $M$  is *simple* if its Seifert-fibred components are simple. We will pass to a simple finite cover of  $M$  in Lemma 3.1.

Finally, by separability of the JSJ tori subgroups in  $\pi_1 M$ , we can assume that  $S \subset M$  is *straight*. This means that  $S$  does not intersect a Seifert-fibred component  $E$  of  $M$  along a vertical annulus with both boundary circles in the same boundary torus of  $E$ .

## 3. EXTENDING SEMICOVERS TO COVERS

We begin this section with the following additional simplification:

**Lemma 3.1.** *Let  $M$  be an irreducible 3-manifold that is not Seifert-fibred. Then  $M$  has a finite cover  $M'$  that is simple. Moreover, given covers  $\{T^*\}$  of the transitional tori  $\{T\}$  in  $M$ , we can assume that all the tori of  $M'$  covering  $T$  are isomorphic and factor through  $T^*$ .*

A key element of the proof employs the following omnipotence result for hyperbolic 3-manifolds with boundary.

**Lemma 3.2** ([Wis11, Cor 16.15]). *Let  $M^h$  be a hyperbolic 3-manifold with boundary tori  $\{T\}$ . There exist finite covers  $\{\widehat{T}\}$  such that for any further finite covers  $\{T'\}$  there exists a finite cover  $M^{h'}$  of  $M^h$  that restricts on boundary tori to  $\{T'\}$ .*

By passing to a further cover we can assume that  $M^{h'} \rightarrow M^h$  is regular.

*Proof of Lemma 3.1.* Luecke and Wu proved in [LW97, Prop 4.4] that every graph manifold block  $M^g$  of  $M$  has a finite cover  $M^{g'}$  that is simple. Without loss of generality we can assume that  $M^{g'} \rightarrow M^g$  is regular.

Choose  $m$  such that

- (i) for any  $M^g$  adjacent along a torus  $T$  to a hyperbolic block  $M^h$ , the cover  $T'_{[m]}$  of the torus  $T' \subset \partial M^{g'}$  covering  $T$  factors through  $\widehat{T}$  of Lemma 3.2 and through  $T^*$ .
- (ii) for a transitional or boundary torus  $T \subset M$  adjacent to a hyperbolic block  $M^h$  but not to a graph manifold block, the cover  $T_{[m]}$  factors through  $\widehat{T}$  of Lemma 3.2 and through  $T^*$ , if  $T$  is transitional.

By Lemma 3.2, each hyperbolic block  $M^h$  of  $M$  has a finite regular cover  $M^{h'}$  restricting on the boundary to  $\{T'_{[m]}\}$  of (i) or  $\{T_{[m]}\}$  of (ii). For a Seifert-fibred component  $E$  of one of the simple graph manifolds  $M^{g'}$ , as  $E$  is simple its retractive property ensures that the cover  $E_{[m]}$  restricts to  $m$ -characteristic covers on its boundary tori. Gluing appropriately many copies of the various  $E_{[m]}$  and  $M^{h'}$  together provides the desired simple cover  $M'$  of  $M$ .  $\square$

Henceforth we **always** assume that  $M$  is simple (and irreducible as assumed in Section 2).

**Definition 3.3.** A *semicover*  $\overline{M}$  of  $M$  w.r.t. transitional tori is a local embedding  $\overline{M} \rightarrow M$  that restricts to a covering map over each transitional torus and over each open block. Thus  $\overline{M}$  can only fail to be a covering map at a component of  $\partial\overline{M}$  that covers a transitional torus  $T \subset M$ . We say that  $\overline{M} \rightarrow M$  is *finite* if  $\overline{M}$  is compact.

We can now prove the main result of this section.

**Proposition 3.4.** *Any finite semicover  $\overline{M}$  of  $M$  has a finite cover  $\overline{M}' \rightarrow \overline{M}$  that embeds in a finite cover  $M'$  of  $M$ .*

*Proof of Proposition 3.4.* By Lemma 3.1, there is a cover  $\widehat{M}$  of  $M$  such that for each transitional torus  $T$  of  $M$  all of the tori  $\widehat{T} \subset \widehat{M}$  covering  $T$  are isomorphic and factor through all the covers of  $T$  in  $\overline{M}$ .

Let  $\overline{M}' \rightarrow \overline{M}$  be the pullback of the cover  $\widehat{M} \rightarrow M$  via the semicover  $\overline{M} \rightarrow M$ . Then on all of its boundary tori the semicover  $\overline{M}' \rightarrow M$  restricts to the corresponding tori in  $\widehat{M}$ . Gluing  $\overline{M}'$  with appropriately many copies of the components of  $\widehat{M}$  extends  $\overline{M}'$  to a cover  $M'$  of  $M$ .  $\square$

#### 4. SURFACE-INJECTIVE SEMICOVERS

In this section we construct a family of semicovers of  $M$  to which a given surface  $S \subset M$  lifts. We keep the assumptions from Section 2.

We will use the following case of a theorem of Martínez-Pedroza:

**Theorem 4.1** ([MP09, Thm 1.1]). *Let  $S_0 \subset M^h$  be an incompressible geometrically finite surface properly embedded in a hyperbolic manifold  $M^h$ . Let  $\partial S_0 = C_1 \sqcup \dots \sqcup C_k$  and suppose these circles are contained in boundary tori  $T_1, \dots, T_k$  of  $M^h$  (some  $T_i$  may coincide). Then for almost all cyclic covers  $T'_i$  of  $T_i$  to which  $C_i$  lift, the graph of spaces obtained by amalgamating  $S_0$  with  $T'_i$  along  $C_i$  maps  $\pi_1$ -injectively into  $M^h$  and the image of its  $\pi_1$  in  $\pi_1 M^h$  is relatively quasiconvex.*

The separability of double cosets of relatively quasiconvex subgroups of  $\pi_1$  of a hyperbolic 3-manifold with boundary was established in [Wis11, Thm 16.23]. Consequently, we have:

**Corollary 4.2.** *For almost all cyclic covers  $T'_i$  described in Theorem 4.1, the group  $\pi_1(S_0 \sqcup_{\{C_i\}} \{T'_i\})$  is separable in  $\pi_1 M^h$ .*

**Corollary 4.3.** *The subgroup  $\pi_1 S_0$  as well as the double cosets  $\pi_1 S_0 \pi_1 T_i$  are separable in  $\pi_1 M^h$ .*

**Definition 4.4.** Let  $S \subset M$  be an incompressible surface. A semicover  $\overline{M} \rightarrow M$  to which  $S$  lifts is  *$S$ -injective* if for each hyperbolic or graph manifold block  $\overline{B}$  of  $\overline{M}$  the intersection  $S \cap \overline{B}$  is connected. We allow  $S$  itself to be disconnected.

**Lemma 4.5** ([PW11, Constr 4.13]). *Let  $S \subset M^g$  be a possibly disconnected straight incompressible surface in a simple graph manifold. Suppose  $n$  is an integer divisible by all of the degrees of (possibly disconnected) covers  $S \cap E \rightarrow F$ , where  $E \subset M^g$  is a Seifert-fibred component with base surface  $F$ , and  $S \cap E$  is horizontal. Then there is a finite cover  $\overline{M}^g$  of  $M^g$  to which  $S$  lifts such that for each torus  $\overline{T} \subset \partial \overline{M}^g$  intersecting  $S$ :*

- $S \cap \overline{T}$  is connected,

- $\bar{T}$  maps to a torus  $T \subset \partial M^g$  with degree  $\frac{n}{|S \cap T|}$ .

Here  $|S \cap T|$  denotes the number of components in the intersection of the surface  $S$  with the torus  $T$ .

**Proposition 4.6.** *Let  $S \subset M$  be an incompressible surface. Let  $S_0$  be a component of intersection of  $S$  with a hyperbolic or graph manifold block  $M_0$  of  $M$ . Let  $T_i$  be the (possibly repeating) tori of  $\partial M_0$  intersected by  $S_0$ . Let  $g \in \pi_1 M_0 - \pi_1 S_0$  (resp.  $g_i \in \pi_1 M_0 - \pi_1 S_0 \pi_1 T_i$  for each  $i$ ). Then there is a finite  $S$ -injective semicover  $\bar{M}$  with  $g \notin \pi_1 \bar{M}_0$  (resp.  $g_i \notin \pi_1 \bar{M}_0 \pi_1 T_i$ ), where  $\bar{M}_0$  is the block of  $\bar{M}$  containing the lift of  $S_0$ .*

To make sense of the double cosets  $\pi_1 S_0 \pi_1 T_i$  inside  $\pi_1 M_0$ , pick basepoints  $x_i$  of  $M_0$  in  $C_i$  and interpret  $\pi_1 S_0, \pi_1 T_i$  as subgroups of  $\pi_1 M_0$  determined by loops based at  $x_i$  staying in  $S_0, T_i$ , respectively.

*Proof.* In the case where we assume  $g \notin \pi_1 S_0$ , we use that  $\pi_1 S_0$  is separable in  $\pi_1 M_0$ . If  $M_0$  is hyperbolic, this follows from Corollary 4.3. If  $M_0$  is a graph manifold, we use separability of embedded surfaces in graph manifolds [PW11, Thm 1.1]. Hence there is a finite cover  $\bar{M}_0^* \rightarrow M_0$  to which  $S_0$  lifts with  $g \notin \pi_1 \bar{M}_0^*$ .

In the case where we assume  $g_i \notin \pi_1 S_0 \pi_1 T_i$  for all  $i$ , we use that each double coset  $\pi_1 S_0 \pi_1 T_i$  is separable in  $\pi_1 M_0$ . This follows from Corollary 4.3 and [PW11, Thm 1.2]. Hence there exists a cover  $\bar{M}_0^* \rightarrow M_0$  to which  $S_0$  lifts with  $g_i \notin \pi_1 \bar{M}_0^* \pi_1 T_i$ . Let  $n_i$  be the degree of the restriction of  $\bar{M}_0^* \rightarrow M_0$  to the torus intersecting (the lift of)  $S_0$  along (the lift of)  $C_i$ .

Choose  $n$  so that it is divisible by the numbers in (a)–(c), and also satisfies (d):

- (a) every  $|S \cap T|$ , where  $T$  is a transitional or boundary torus,
- (b) the degrees of (possibly disconnected) covers  $S \cap E \rightarrow F$ , where  $E \subset M$  is a Seifert-fibred component with base surface  $F$ , and  $S \cap E$  is horizontal,
- (c) each  $n_i |S \cap T_i|$  as above.
- (d) We also require  $\frac{n}{|S \cap T|}$  to be the degree of one of the covers  $T' \rightarrow T$  given by Theorem 4.1 for a geometrically finite component of  $S \cap M^h$  in a hyperbolic block  $M^h$  of  $M$ .

We construct the semicover  $\bar{M}$  in the following way. Start with a copy  $\bar{S}$  of  $S$ . Let  $T$  be a transitional or boundary torus of  $M$ . For each component of  $S \cap T$  we attach along the corresponding circle in  $\bar{S}$  the degree  $\frac{n}{|S \cap T|}$  cyclic cover  $\bar{T}$  of  $T$ . The value  $\frac{n}{|S \cap T|}$  is an integer by (a).

For each graph manifold block  $M^g$  of  $M$  consider the finite (possibly disconnected) cover  $\bar{M}^g$  from Lemma 4.5 applied to the surface  $S \cap M^g$ . The boundary components of  $\bar{M}^g$  intersecting  $S$  coincide with the  $\bar{T}$  attached to  $\bar{S}$  above.

Consider now a hyperbolic block  $M^h$  of  $M$  such that  $S \cap M^h$  is a union of fibers. In this case we choose  $\bar{M}^h$  to be the union of  $|S \cap M^h|$  copies of degree  $\frac{n}{|S \cap M^h|}$

cyclic covers of  $M^h$  to which components of  $S \cap M^h$  lift. Again, components of  $\partial\overline{M^h}$  coincide with  $\overline{T}$ , so that we can consistently attach the  $\overline{M^h}$  to  $\overline{S}$ .

Finally, if  $S \cap M^h$  is not a union of fibers, then  $\pi_1$  of each of its components is relatively quasiconvex in  $\pi_1 M^h$ , so by (d) and Corollary 4.2, there is a finite cover  $\overline{M^h}$  extending  $S \cap M^h \cup \{\overline{T}\}$ , and we consistently attach the  $\overline{M^h}$  to  $\overline{S}$ .

At this point we have constructed a finite  $S$ -injective semicover  $\overline{M}$ , without yet separating  $g$  (resp.  $g_i$ ). Now we replace the block  $\overline{M_0}$  with its fiber product with  $\overline{M_0^*}$ . (Algebraically  $\pi_1$  of the fiber product is  $\pi_1\overline{M_0} \cap \pi_1\overline{M_0^*} \subset \pi_1\overline{M_0}$ .) This is possible by (c) which guarantees that the fiber product agrees with  $\overline{M_0}$  on its boundary components intersecting  $S_0$ . After this replacement,  $\overline{M}$  satisfies the requirement on  $g$  (resp.  $g_i$ ), by definition of  $\overline{M_0^*}$ .  $\square$

## 5. SEPARABILITY

*Proof of Theorem 1.1.* Choose a basepoint of  $M$  in  $S$  outside all JSJ and boundary tori. Let  $f \in \pi_1 M - \pi_1 S$ . Consider the based cover  $M^S$  of  $M$  with fundamental group  $\pi_1 S$ . Let  $\gamma^S$  be a path in  $M^S$  starting at the basepoint and representing  $f$ . Then  $\gamma^S$  does not terminate on  $S$ . Assume that  $\gamma^S$  is chosen so that it does not *backtrack*, i.e. its image in  $M$  intersects the transitional tori a minimal number of times.

Firstly, consider the case where  $\gamma^S$  terminates in a block  $M_0^S \subset M^S$  that intersects the lift of  $S$ . Denote  $S_0 = S \cap M_0^S$  and let  $M_0 \subset M$  be the block covered by  $M_0^S$ . In the case where  $S_0$  contains the basepoint, let  $g \in \pi_1 M_0$  be an element represented by a path in  $M_0^S$  from the basepoint to the endpoint of  $\gamma^S$ .

By Proposition 4.6 there is a finite  $S$ -injective semicover  $\overline{M}$  of  $M$  with  $g \notin \pi_1\overline{M_0}$ . Thus  $\gamma^S$  projects to a path  $\overline{\gamma}$  in  $\overline{M}$  that ends in  $\overline{M_0}$  outside the lift of  $S_0$ . By Proposition 3.4 the semicover  $\overline{M}$  has a finite cover  $\overline{M'}$  that extends to a finite cover  $M'$  of  $M$ . Since the endpoint of the lift of  $\overline{\gamma}$  to  $M'$ , which lies in  $\overline{M'}$ , does not terminate on the based connected component of the preimage of  $S$ , we have  $f \notin \pi_1 M' \pi_1 S$ , as desired.

Secondly, consider the case where  $\gamma^S$  terminates in a block of  $M^S$  disjoint from the lift of  $S$ . Let  $T^S \subset M^S$  be then the first connected component of the preimage of a transitional torus  $T \subset M$  crossed by  $\gamma^S$  and disjoint from  $S$ . Let  $M_0^S$  be the last block that  $\gamma^S$  travels through before it hits  $T^S$ . Let  $S_0 = S \cap M_0^S$  and let  $M_0 \subset M$  be the block covered by  $M_0^S$ . If  $T$  coincides with one of the tori  $T_i \subset M_0$  crossed by  $S_0$  along  $C_i$ , then let  $x_i \in C_i$  be a basepoint for  $M_0$ . Let  $x'_i$  be a lift of  $x_i$  in  $T^S$ . We keep the notation  $x_i$  for the lift of  $x_i$  to  $S_0 \subset M_0^S$ . Let  $g_i \in \pi_1 M_0$  be an element represented by a path in  $M_0^S$  from  $x_i$  to  $x'_i$ .

Since  $T^S$  is disjoint from  $S_0$ , we have  $g_i \notin \pi_1 S_0 \pi_1 T_i$ . By Proposition 4.6 there is a finite  $S$ -injective semicover  $\overline{M}$  of  $M$  with  $g_i \notin \pi_1\overline{M_0} \pi_1 T_i$  for all  $i$ . In other words,  $\overline{\gamma}$  leaves  $\overline{M_0}$  through a torus disjoint from  $S_0$ .

By Proposition 3.4 the semicover  $\overline{M}$  has a finite cover  $\overline{M}'$  that extends to a finite cover  $M'$  of  $M$ . By separability of the transitional tori groups (since they are maximal abelian) and residual finiteness of the free group (dual to transitional tori), by replacing  $M'$  with a further cover we can assume that the lift of  $\overline{\gamma}$  to  $M'$  does not pass twice through the same transitional torus.

Let  $T' \subset M'$  be the projection of  $T^S$ . Consider the double cover  $M''$  obtained by taking two copies of  $M'$ , cutting along  $T'$ , and regluing. Then the based connected component of the preimage of  $S$  lies in one copy of (the cut)  $M'$  in  $M''$ , while the endpoint of the lift of  $\overline{\gamma}$  lies in the other copy. Hence  $f \notin \pi_1 M'' \pi_1 S$ , as desired.  $\square$

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