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ACUTE TRIANGULATIONS OF POLYHEDRA AND $\mathbb{R}^N$

ERYK KOPCZYŃSKI, IGOR PAK*, PIOTR PRZYTYCKI†

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We study the problem of *acute triangulations* of convex polyhedra and the space $\mathbb{R}^n$. Here an acute triangulation is a triangulation into simplices whose dihedral angles are acute. We prove that acute triangulations of the n-cube do not exist for $n \geq 4$. Further, we prove that acute triangulations of the space $\mathbb{R}^n$ do not exist for $n \geq 5$. In the opposite direction, in $\mathbb{R}^3$, we present a construction of an acute triangulation of the cube, the regular octahedron and a non-trivial acute triangulation of the regular tetrahedron. We also prove nonexistence of an acute triangulation of $\mathbb{R}^4$ if all dihedral angles are bounded away from $\pi/2$.

1. Introduction

The subject of acute triangulations is an important area of Discrete and Computational Geometry, with a number of connections to other areas and some real world applications. Until recently, most results dealt with the 2-dimensional case, where the problem has been largely resolved. In the last few years, several papers [15,22,39] broke the dimension barrier in both positive and negative direction (see below). In this paper we continue this exploration, nearly completely (negatively) resolving the problem in dimension 4 and higher, and making further advancement in dimension 3.

The problem of finding acute triangulations has a long history in classical geometry, and is elegantly surveyed in [9], which argues that it goes back to

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Figure 1.

Aristotle. In recent decades, it was further motivated by the *finite element method* which requires “good” meshes (triangulations of surfaces) for the numerical algorithms to run. Although the requirements for meshes largely depend on the algorithm, the sharp angle conditions seem to be a common feature, and especially important in this context. We refer to [37] for the introduction to the subject, and to [34] for the state of art.

Another motivation comes from the recreational literature, where the subject of dissections has been popular in general (see [25]), and of acute triangulations in particular [11,28]. In this context, the problem of acute triangulations of a square, cube, and hypercubes seem to be of special interest [14].

An *acute triangulation* is a dissection into *acute* simplices (i.e. with acute dihedral angles) which form a simplicial complex, so e.g. in the plane, a vertex of one simplex cannot lie in the interior of an edge of another. (See Figure 1, where on the left we have a dissection of the square which forms a simplicial complex, and on the right we have a dissection which does not.)

In one of the first papers on the subject, Burago and Zalgaller proved in [10] that any non-convex polygon (possibly, with holes) has an acute triangulation. Unfortunately, their argument was largely forgotten as it was not explicit and did not give a bound on the number of triangles required. In a long series of papers [6,3,7,27,33,40] first polynomial, and then linear bounds were obtained for non-obtuse, and, eventually, for acute triangulations of polygons. We refer to [41] for the historical outline, a short survey, and further references.

In higher dimensions, several results have been recently obtained. First, Eppstein, Sullivan and Üngör [15] showed that the space $\mathbb{R}^3$ can be triangulated into acute tetrahedra, by adopting a classical tiling construction due to Sommerville. Then, Křížek [22] showed that no vertex in $\mathbb{R}^n$ for $n \geq 5$ can be surrounded by a finite number of acute simplices [1] and conjectured that the space $\mathbb{R}^4$ also cannot be triangulated into acute tetrahedra. Finally, and most recently, VanderZee, Hirani, Zharnitsky and Guoy [39] used an advanced numerical simulation technique to find an acute triangulation for the

[1] There is a crucial error in this proof. We refer to Subsection 6.3 for the details.

(usual) cube in $\mathbb{R}^3$. Their construction is independent of ours and uses fewer tetrahedra.

In this paper we prove several results in higher dimensions.

Theorem A (Theorems 2.7 and 2.8(i)). *There exists an acute triangulation of the* 3*-cube, the regular octahedron, and a non-trivial acute triangulation of the regular tetrahedron.*

Roughly, we first triangulate the cube into a regular 3-simplex and four standard 3-simplices (and the octahedron into eight standard ones). We then subdivide each of these 3-simplices into 543 pieces, to obtain combinatorially what we call the *special subdivision* (based on the 600-cell, see Section 2). This approach was used previously by Przytycki and Świątkowski in [32] to construct the so called *flag-no-square* subdivisions in dimension 3 (see Definition 2.3). Let us mention here that this "curvature" condition was surveyed in the appendix of [32], and that it was used originally to construct Gromov hyperbolic groups with prescribed boundaries. Let us repeat that the case of the cube in $\mathbb{R}^3$ was independently resolved in [39].

In the opposite direction, we prove the following result:

Theorem B (Corollary 4.3). *There is no periodic acute triangulation of the space* $\mathbb{R}^4$*. In particular, there is no acute triangulation of the* 4*-cube.*

The first assertion of Theorem B implies the second one, as 4-cubes tile the space (see Section 4). A short combinatorial proof of Theorem B is based on the generalized Dehn–Sommerville equations. This method also gives new results on flag-no-square triangulations (see Section 4). Moreover, it allows to complete the acute triangulations picture with the following.

Theorem C (Corollary 4.5, [22, Theorem 6.2]). *There is no triangulation of a polyhedron in* $\mathbb{R}^n$*, for* $n \geq 5$*, which contains an interior vertex such that all dihedral angles adjacent to it are acute.*

In particular, there is no acute triangulation of $\mathbb{R}^n$ and the n-cube for $n \geq 5$.

Finally, we prove the following most general result:

Theorem D (Corollary 5.2). *For every* $\varepsilon > 0$*, there is no triangulation of the space* $\mathbb{R}^4$ *into simplices with dihedral angles less than* $\frac{\pi}{2} - \varepsilon$*.*

The proof of Theorem D relies on the generalized Dehn–Sommerville equations and on the relations between isoperimetric inequalities and parabolicity of infinite graphs.

The paper is structured as follows. In Section 2 we study acute triangulations in $\mathbb{R}^3$ and prove Theorem A. In a short Section 3 we recall the generalized Dehn–Sommerville equations. Then, in Section 4, we study their consequences for *rich* triangulations (combinatorial consequence of both acute and flag-no-square, see Definition 2.2), and prove Theorem B and Theorem C. We then switch our attention to Theorem D in Section 5.

Convention. In the entire article we adopt a convention that simplicial complexes and triangulations of (homology) manifolds are not allowed to have edges connecting a vertex to itself, and they are also not allowed to have multiple simplices spanned on the same set of vertices.

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2. Acute triangulations of the 3-cube and the octahedron

In this section we describe acute triangulations of the 3-cube and the octahedron. The starting point is the following observation. Recall that the *link* of a simplex σ in a simplicial complex X is the subcomplex of X formed by all simplices disjoint from σ spanning together with σ a larger simplex.

Observation 2.1. *The link of an interior edge of an acute triangulation of a polyhedron in* $\mathbb{R}^3$ *is a simplicial loop of length at least* 5.

In view of this observation, let us make the following definition.

Definition 2.2. A triangulation of an n-dimensional homology manifold is *rich* if the links of all interior $(n-2)$-simplices are loops of length at least 5.

Note that being rich is a purely combinatorial (i.e. non-metric) property. Observation 2.1 states that an acute triangulation of the 3-cube (or a regular octahedron) must be rich. We compare this definition with the following notion:

Definition 2.3. A simplicial complex (or a triangulation) is called *flag-no-square*, if it is *flag* (i.e. each set of vertices pairwise connected by edges spans a simplex) and each simplicial loop of length four *has a diagonal* (i.e. a pair of opposite vertices of the loop spans an edge).

Remark 2.4. Every flag-no-square triangulation of a homology manifold is rich.

Przytycki–Świątkowski [32, Corollary 2.14] proved that every 3-dimensional polyhedral complex admits a flag-no-square subdivision. We recall this construction, since we also use it to subdivide the 3-cube and the octahedron.

Consider the 600-cell, the convex regular 4-polytope with Schläfli symbol $\{3;3;5\}$ (see, e.g., [12]). Denote by X_{600} the boundary complex of the 600-cell, a 3-dimensional simplicial polyhedron homeomorphic to the 3-dimensional sphere. It consists of 600 3-simplices [2] and has 120 vertices. Its vertex links are icosahedra and its edge links are pentagons. We first focus on the combinatorial simplicial structure of X_{600}. Denote by X_{543} the subcomplex of X_{600} which we obtain by removing from X_{600} the interiors of all simplices intersecting a fixed 3-simplex. (The number 543 in the subscript refers to the number of 3-simplices in X_{543}.)

Lemma 2.5 ([32, Lemmas 2.5 and 2.7]).

(1) X_{543} is topologically a 3-ball. It is flag-no-square.
(2) Its boundary is a 2-sphere isomorphic (as a simplicial complex) to the simplicial complex which we obtain from the boundary of a 3-simplex by subdividing each face as in Figure 2.

We recall the following definition:

Definition 2.6 ([32, Definitions 2.2 and 2.8]). Given a simplicial complex of dimension at most 3, its *special subdivision* is the simplicial complex obtained by:

(i) subdividing each edge into two (by adding an extra vertex in the interior of the edge),
(ii) subdividing each 2-simplex as in Figure 2,
(iii) subdividing each 3-simplex so that it becomes isomorphic to X_{543}.

We are ready to describe the combinatorial structure of our triangulation of the 3-cube and the octahedron. Assume that the cube lies in $\mathbb{R}^3$ with the vertices at points $(\pm 1, \pm 1, \pm 1)$. Consider the triangulation W of the

[2] To streamline and simplify the presentation, we refer to triangles as 2-simplices, to tetrahedra as 3-simplices, etc.

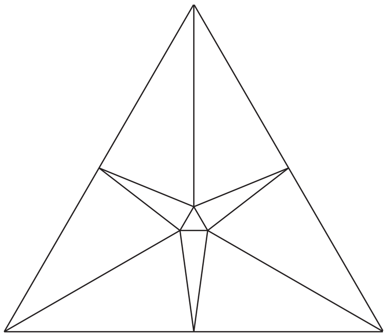

Figure 2.

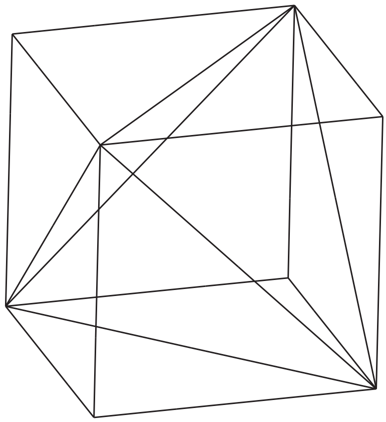

Figure 3.

cube into five 3-simplices so that one of them (denote it by T_0) has vertices $(1,1,1), (-1,-1,1), (-1,1,-1)$ and $(1,-1,-1)$, while the remaining four 3-simplices (denote them $T_1, \ldots, T_4$) are the components of the complement to T_0 in the cube (see Figure 3). Note that $T_1, \ldots, T_4$ are congruent (equal up to a rigid motion); we call such 3-simplices *standard* (see e.g. [31]). [3] Let W^* be the special subdivision of W defined as above.

Similarly, let Y be the triangulation of the octahedron into eight standard 3-simplices obtained as cones from the center over the faces. Let Y^* be the special subdivision of Y.

By [32, Proposition 2.13], subdivisions W^* and Y^* are both flag-no-square. Thus, they are rich and have a potential of giving an acute realiza-

[3] This tetrahedron is also called the *cube-corner*.

tion. This is true indeed, and the main result of this section is the following theorem:

Theorem 2.7 (part of Theorem A).

(1) There is an acute triangulation of the 3*-cube, which is combinatorially equivalent to* W^**.*

(2) There is an acute triangulation of the octahedron, which is combinatorially equivalent to Y^**.*

In fact, we provide acute triangulations of all 3-simplices of W and Y, combinatorially equivalent to X_{543}, and matching on common part of the boundary. In other words, we prove the following intermediate result:

Theorem 2.8 (part of Theorem A). *There is a (non-trivial) acute triangulation, combinatorially equivalent to* X_{543}*, of*

(i) the regular 3*-simplex,*
(ii) the standard 3*-simplex.*

Below we describe the construction for the 3-cube. At some points we use a computer program. We provide the exact position of all vertices of both triangulations from Theorem 2.8 in the appendix. There are three steps of the construction. First, we construct an acute triangulation of T_0. Then we "flatten" it to obtain an acute triangulation of T_1. Then we construct another acute triangulation of T_0 so that it matches the one of T_1 on the common part of the boundary.

Step 1. Note that the vertices of the 600-cell, whose boundary complex we called X_{600}, lie on a sphere in $\mathbb{R}^4$. Moreover all 3-simplices in this realization of X_{600} are regular, hence acute. Let now $\widetilde{X}_{543}$ be the realization of X_{543} in $\mathbb{R}^3$, whose vertices are obtained by stereographic projection of the $\mathbb{R}^4$ realization. We choose the center of the projection to be the center of the (spherical) 3-simplex in X_{600} disjoint from X_{543}. It turns out that this mapping does not disturb the angles significantly.

We move the vertices of $\partial\widetilde{X}_{543}$ radially so that they arrange on the boundary of a regular 3-simplex which we identify with T_0. If we scale the size of ∂T_0 correctly, this triangulation of T_0 is already acute, i.e. it satisfies Theorem 2.8(i). However, it is not the one listed in the appendix, we will modify it later in Step 3 (see also Figure 4).

Step 2. The subdivision of the standard 3-simplex, say T_1, is more difficult. Our computer program uses the following algorithm to find the position of the vertices. We "flatten" the acute triangulation of T_0 obtained in Step 1 in order to obtain an acute triangulation of T_1. We gradually move one of the boundary vertices (marked A on Figure 5) towards the center, keeping the three vertices marked C in the points where the circles inscribed into

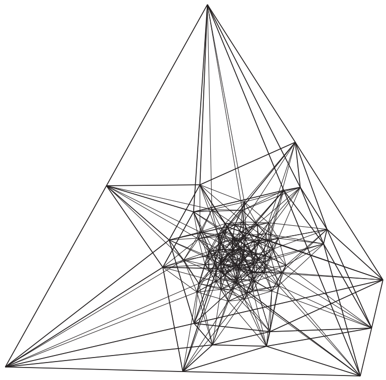

Figure 4.

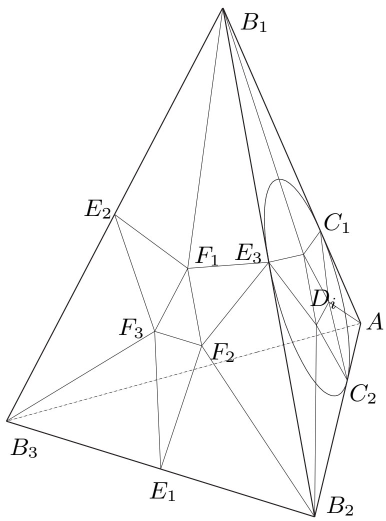

Figure 5.

triangles AB_iB_j meet the edges AB_i. We also keep the nine points marked D on their faces, and scale and translate together all the interior vertices.

Whenever some angle stops being acute during this operation, we suspend the flattening process to correct the angles. This is done by slightly moving the responsible vertices so that the angle becomes smaller. Vertices are moved only in a way that does not disturb the combinatorial structure, i.e. points A, B_i, C_i, E_i are not moved at all; movement of D_i and F_i is restricted to their faces; all the interior vertices except the two outermost

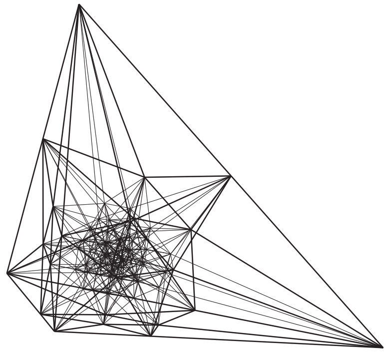

Figure 6.

layers (of 12 and 16 vertices, respectively) are moved together so that the structure is not disrupted.

When all the angles are corrected, we resume the flattening, until we obtain the standard 3-simplex T_1. This completes the description of the triangulation in Theorem 2.8, part (ii) (see also Figure 6 and the appendix).

Step 3. The position of the vertices F_i on the equilateral face of T_1 is now different from their position on the face of T_0, because we had to move F_i during the correcting process in Step 2. So in the triangulation of T_0 constructed in Step 1 we move all 12 vertices corresponding to F_i to the position matching with the standard T_i. It turns out that it is then enough to scale the interior structure to obtain an acute triangulation (see the appendix).

Now we attach acute triangulations of all T_i, constructed in Steps 2 and 3, to obtain a triangulation of the 3-cube satisfying conditions of Theorem 2.7, part (1) (see Figure 7 and Remark 2.9).

Finally, we obtain a triangulation of the regular octahedron satisfying conditions of Theorem 2.7, part (2), by attaching eight copies of the standard 3-simplex triangulated as in Step 2. For further discussion on the construction we refer to Subsection 6.1.

Remark 2.9. The animation of our triangulation of the 3-cube is available at

`http://www.mimuw.edu.pl/~erykk/papers/acute.html`.

For the details and the exact values of all the parameters which have been guessed, see the implementation of above algorithm, available with the animation.

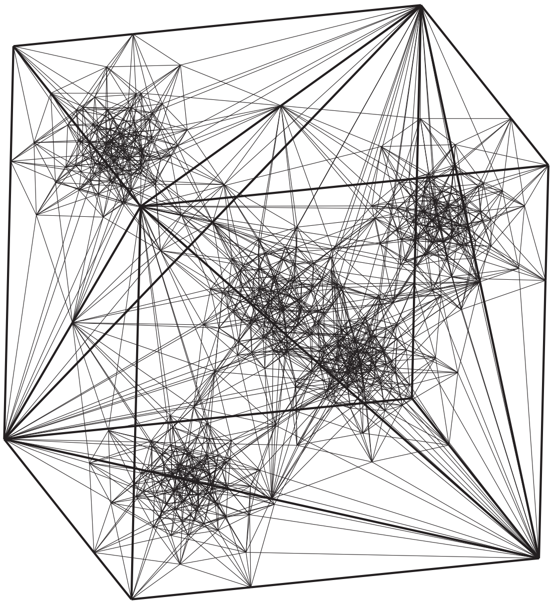

Figure 7.

3. Dehn–Sommerville equations in dimension 4

In this short section we present some known results in geometric combinatorics.

Denote by $f_i(M)$, $f_i(\partial M)$ (we later abbreviate this to f_i, f_i^∂), the number of i-dimensional simplices of a triangulation of a compact m-dimensional homology manifold M and its boundary ∂M. Recall the following Dehn–Sommerville type equations (see Subsection 6.8 for the history of this generalization).

Theorem 3.1 ([20, Theorem 1.1] and [30]). *Let M be a compact m-dimensional triangulated homology manifold with boundary. For $k=0,\dots,m$ we have*

$$f_k(M) - f_k(\partial M) = \sum_{i=k}^{m} (-1)^{i+m} \binom{i+1}{k+1} f_i(M).$$

If $m=4$, then for $k=1,2$ we obtain the following.

Corollary 3.2. *If M is 4-dimensional and we abbreviate $f_i = f_i(M), f_i^\partial = f_i(\partial M)$, then*
(i) $2f_1 - f_1^\partial = 3f_2 - 6f_3 + 10f_4$,
(ii) $-f_2^\partial = -4f_3 + 10f_4$.

These equalities will be used repeatedly in the next two sections.

4. Rich triangulations of 4-manifolds

In this section we prove the following combinatorial result on rich triangulations (see Definition 2.2) of 4-dimensional homology manifolds. This addresses Przytycki–Świątkowski [32, Questions 5.8(3)]. We keep the notation f_i, f_i^∂ from Section 3.

Theorem 4.1. *Every rich triangulation of a compact* 4*-dimensional homology manifold* M *with Euler characteristic* χ *satisfies*

$$2f_0 \leq 2\chi + f_1^\partial.$$

In particular, if M *is closed, then* $f_0 \leq \chi$.

Before we present the proof of the theorem, let us give the following four corollaries in the case when the homology manifold M is closed.

Corollary 4.2. *Any* 4*-dimensional closed homology manifold* M *has only finitely many rich triangulations. In particular* M *has only finitely many flag-no-square triangulations.*

Corollary 4.3 (Theorem B). *There is no periodic (i.e. invariant under a cocompact group of translations) acute triangulation of* $\mathbb{R}^4$*. In particular, there is no acute triangulation of the* 4*-cube.*

Proof. A periodic triangulation τ of $\mathbb{R}^4$ descends to a triangulation τ' of a 4-torus. Since the Euler characteristic of a 4-torus equals 0, by Theorem 4.1, triangulation τ' is not rich. Hence, by Observation 2.1, τ is not acute. This proves the first part of the corollary. For the second part, observe that an acute triangulation of the 4-cube could be promoted, by reflecting, to a periodic acute triangulation of $\mathbb{R}^4$. ∎

Let us show also that the theorem gives a much simplified proof of the following known result:

Corollary 4.4 ([19, Section 2.2]). *There are no rich (in particular no flag-no-square) triangulations of closed homology* n*-manifolds, for* $n \geq 5$.

Proof. The link L of any codimension-5 simplex of a triangulation σ of a closed homology n-manifold (for $n \geq 5$) is a 4-dimensional homology sphere, which implies that its Euler characteristic χ equals 2. Since L is 4-dimensional, it must have at least 6 vertices. Hence, by Theorem 4.1, L is not rich. Thus, triangulation σ is also not rich, which proves the result. ∎

In the same way we obtain the following theorem (cf. Subsection 6.3):

Corollary 4.5 (Theorem C, [22, Theorem 6.2]). *There is no triangulation of a polyhedron in $\mathbb{R}^n$, for $n \geq 5$, which contains an interior vertex such that all dihedral angles adjacent to it are acute.*

Proof. Let v be an interior vertex and let ρ be a codimension-5 simplex (a vertex for $\mathbb{R}^5$, an edge for $\mathbb{R}^6$ etc.) containing v. The link L of ρ is a 4-dimensional homology sphere, hence its Euler characteristic equals 2. Since L is 4-dimensional, it must have at least 6 vertices. Hence, by Theorem 4.1, L is not rich. Thus the link of one of the codimension-2 simplices containing ρ is a cycle of length shorter than 5. Hence one of the dihedral angles adjacent to v is not acute. ∎

Finally we provide the following.

Proof of Theorem 4.1. Let τ be a rich triangulation of M. We compute the number N of flags $(\rho_2 \subset \rho_4)$ of a 2-simplex ρ_2 contained in a 4-simplex ρ_4. On one hand, it equals $10f_4$, since each 4-simplex has ten 2-dimensional faces. On the other hand, by richness, each interior 2-simplex (there are $f_2 - f_2^\partial$ of those) is contained in at least five 4-simplices. Thus, we have:

$$N = 10f_4 \geq 5(f_2 - f_2^\partial). \tag{1}$$

By the definition of the Euler characteristic, we have:

$$\chi - f_0 = -f_1 + f_2 - f_3 + f_4. \tag{2}$$

Applying consecutively formula (2), then Corollary 3.2 parts (i) and (ii), and formula (1), we obtain:

$$\begin{aligned} 2(\chi - f_0) + f_1^\partial &= -2f_1 + 2(f_2 - f_3 + f_4) + f_1^\partial = \\ &= -(f_1^\partial + 3f_2 - 6f_3 + 10f_4) + 2(f_2 - f_3 + f_4) + f_1^\partial = \\ &= -f_2 + 4f_3 - 8f_4 = -f_2 + (f_2^\partial + 10f_4) - 8f_4 = \\ &= 2f_4 - (f_2 - f_2^\partial) \geq 0, \end{aligned}$$

as desired. ∎

5. Acute triangulations of $\mathbb{R}^4$

In this section (see also Section 6.2) we address the problem whether there is an acute triangulation of $\mathbb{R}^4$. We know already that every such acute triangulation of $\mathbb{R}^4$ cannot be periodic (Corollary 4.3). Here we present the following stronger result.

We say that a triangulation of $\mathbb{R}^p$ has *bounded geometry* if there is a global upper bound on the ratio of edge lengths in every p-simplex.

Theorem 5.1. *There is no acute triangulation of* $\mathbb{R}^4$ *with bounded geometry.*

This result can be restated in the following (equivalent) form:

Corollary 5.2 (Theorem D). *There is no acute triangulation of* $\mathbb{R}^4$ *with dihedral angles bounded away from* $\frac{\pi}{2}$.

Proof. If the dihedral angles are bounded away from $\frac{\pi}{2}$, then the angles of 2-simplices are bounded away from $\frac{\pi}{2}$ (see e.g. [22]). Hence the angles of 2-simplices are also bounded away from 0. By the sine law, this gives a bound on the ratio of lengths of edges in each 2-simplex, which results in a bound of the ratio of lengths of edges in each 4-simplex. ∎

Before we prove Theorem 5.1, we need a few preliminary results.

Lemma 5.3. *Let* τ *be a triangulation of* $\mathbb{R}^p$ *with bounded geometry. Then the* 1*-skeleton of* τ *has bounded vertex degree.*

Proof. All p-simplices of τ are affinely quasi-conformal to the regular p-simplex with a universal constant. Hence the spherical volume contributed by any p-simplex in the link of any vertex of τ is bounded from below (here we exceptionally treat the link as a piecewise-spherical cell complex). On the other hand, the total volume of the link is the volume of the unit $(p-1)$-sphere. This bounds the number of p-simplices, and in particular the number of lower dimensional simplices, sharing each vertex. ∎

Lemma 5.4. *Let* M *be a compact connected* 4*-dimensional triangulated homology manifold. Assume that* M *admits a PL embedding into* $\mathbb{R}^4$ *(then in particular it has non-empty boundary). Then the Euler characteristic of* M *is at most* $1+\mathrm{rk}H_2(\partial M)$, *where* H_2 *denotes the second singular homology with* $\mathbb{Z}$ *coefficients.*

Proof. First observe that the natural map $H_2(M)\to H_2(M,\partial M)$ is trivial. Indeed, this mapping factors through

$$H_2(M) \to H_2(S^4) \to H_2(S^4, S^4 \setminus M) = H_2(M, \partial M),$$

where S^4 is the one point compactification of $\mathbb{R}^4$ with $H_2(S^4)=0$. Hence the natural map $H_2(\partial M) \to H_2(M)$ is onto and $\mathrm{rk}H_2(M) \le \mathrm{rk}H_2(\partial M)$. Thus, the Euler characteristic χ of M satisfies

$$\chi \le \mathrm{rk}H_0(M) + \mathrm{rk}H_2(M) = 1 + \mathrm{rk}H_2(M) \le 1 + \mathrm{rk}H_2(\partial M),$$

as desired. ∎

Theorem 4.1 and Lemma 5.4 now imply the following result:

Corollary 5.5. *Let M be a compact connected 4-dimensional homology manifold with a rich triangulation. Assume that M admits a PL embedding into $\mathbb{R}^4$. If f_i, f_i^∂ are defined as in Section 3, then we have*

$$2f_0 \le 2(1 + f_2^\partial) + f_1^\partial.$$

We turn our attention now to the study of isoperimetric functions on infinite graphs.

Definition 5.6. Let $G = (V, E)$ be a simple connected (locally finite) infinite graph, and let $\Omega \subset V$ be a finite subset of vertices. Denote by $\partial\Omega$ the *vertex-boundary* of Ω, defined as the subset of $V \setminus \Omega$ consisting of vertices adjacent to vertices in Ω.

We say that $I\colon \mathbb{Z}_{\ge 0} \to \mathbb{Z}_{\ge 0}$ is an *isoperimetric function* for G, if the inequality $I(|\Omega|) \le |\partial\Omega|$ holds for every finite $\Omega \subset V$.

Proposition 5.7. *The 1-skeleton of any acute triangulation of $\mathbb{R}^4$ with bounded geometry has linear isoperimetric function.*

Proof. Let $G = (V, E)$ be the 1-skeleton of an acute triangulation of $\mathbb{R}^4$ with bounded geometry. Consider any finite $\Omega \subset V$. We want to obtain a linear isoperimetric function for G. We assume that the subgraph spanned by Ω in V is connected. The general case follows from this one: we add up the inequalities coming from components of the subgraph spanned by Ω. In view of Lemma 5.3 each vertex in $\partial\Omega$ will be then counted only a uniformly bounded number of times.

To outline the idea of the proof, assume first that the subcomplex of $\mathbb{R}^4$ which is the closure of the union of all simplices meeting Ω is a (4-dimensional) homology manifold. Denote this subcomplex by M. Then Ω is contained in M and the vertices in ∂M lie in $\partial\Omega$. By Lemma 5.3, f_1^∂ and f_2^∂ are bounded above by Cf_0^∂, for some fixed C. Hence, by Corollary 5.5, we have

$$|\Omega| \le f_0 \le 1 + f_2^\partial + \frac{1}{2}f_1^\partial \le 1 + \frac{3}{2}Cf_0^\partial \le \left(1 + \frac{3}{2}C\right) f_0^\partial \le \left(1 + \frac{3}{2}C\right) |\partial\Omega|,$$

as desired.

In general, as pointed out to us by Jon McCammond and the referee, the closure of the union of all simplices meeting Ω might not be a homology manifold. The strategy then is, roughly speaking, to subdivide the original triangulation in order to find a tubular PL neighborhood M of Ω, which is a homology 4-manifold with rich triangulation. To this end, we need the following construction:

Definition 5.8. Let Y be a subcomplex of a simplicial complex X. Let $N_X(Y)$ be the simplicial complex containing Y defined in the following way. Its set of vertices is the union of the set of vertices of Y and of the set of simplices of X which are not contained in Y, but contain a vertex from Y. Now we describe the simplices in $N_X(Y)$. Assume that a simplex σ is contained in Y and we have simplices $\sigma \subset \tau_1 \subset \ldots \subset \tau_k$ in X with $\tau_1 \not\subseteq Y$. Then in $N_X(Y)$ we span a simplex on the union of the set of vertices from Y lying in σ and on the set $\{\tau_i\}$.

We assume now that X is a homology n-manifold without boundary. Suppose that Y is a *full* subcomplex of X (i.e. if all the vertices of a simplex σ of X belong to Y, then also σ belongs to Y). Then the simplicial complex $M=N_X(Y)$ is also a homology n-manifold. The image of the natural embedding of $M=N_X(Y)$ into X, which restricts to the identity on Y and maps each vertex corresponding to a simplex of X to its barycenter in X, can be regarded as a PL tubular neighborhood of Y in X.

The vertices in the interior of M are exactly the vertices of Y. The link in M of an $(n-2)$-dimensional simplex in Y is obtained by subdividing its link in X. Hence if X is rich, then M is also rich.

We now return to the proof of Proposition 5.7. Let X be the $\mathbb{R}^4$ equipped with the acute triangulation with bounded geometry and let $Y \subset X$ be the full subcomplex spanned by Ω. Put $M=N_X(Y)$. Since Y is connected, M is connected as well. Since Ω is the set of vertices of Y, we have $|\Omega| \le f_0$. On the other hand, every vertex in the boundary of M corresponds to a simplex in X containing a vertex of $\partial\Omega$. Hence by Lemma 5.3 we have $f_0^\partial \le D|\partial\Omega|$, for some fixed D.

Moreover, again by Lemma 5.3, f_1^∂ and f_2^∂ are bounded above by Cf_0^∂, for some fixed C. Finally, by Corollary 5.5, we have

$$|\Omega| \le f_0 \le 1 + f_2^\partial + \frac{1}{2}f_1^\partial \le 1 + \frac{3}{2}Cf_0^\partial \le \left(1+\frac{3}{2}C\right) f_0^\partial \le \left(1+\frac{3}{2}C\right) D|\partial\Omega|,$$

as desired. ∎

Summarizing, we showed that acute triangulations with bounded geometry have a linear isoperimetric function. In the remaining part of this section, we show that this leads to a contradiction. The argument that follows was suggested to us by Marc Bourdon. Following Benjamini–Curien [4, Section 2.2], we recall the following definition:

Definition 5.9. Let $G=(V,E)$ be a locally finite connected graph and let $\Gamma(v)$ be the set of all semi-infinite self avoiding simplicial paths (i.e. edge paths) in G starting from $v \in V$. For any $m\colon V \to \mathbb{R}_+$ (so called *metric*), the *length* of a path γ in G is defined by

$$\mathrm{Length}_m(\gamma) = \sum_{v \in \gamma} m(v).$$

If $m \in L^p(V)$, we denote by $||m||_p$ the usual L^p norm. The graph G is *p-parabolic* if the *p-extremal length* of $\Gamma(v)$,

$$\sup_{m \in L^p(V)} \inf_{\gamma \in \Gamma(v)} \frac{\mathrm{Length}_m(\gamma)^p}{||m||_p^p}$$

is infinite. This definition does not depend on the choice of $v \in V$.

Lemma 5.10. *Let G be the 1-skeleton of a triangulation of $\mathbb{R}^p$ with bounded geometry, where $p \geq 2$. Then G is p-parabolic.*

This lemma can be obtained from the Bonk and Kleiner result [8, Corollary 8.8]. To make the proof complete and self-contained, we include a concise proof.

Proof. Let $\ell, L\colon V \to \mathbb{R}_+$ be the length functions of the shortest and the longest edge adjacent to a vertex. Since our triangulation has bounded geometry, by Lemma 5.3 there is a constant $C > 0$, such that $L(v) \leq C\ell(v)$ for all $v \in V$.

We fix a basepoint vertex $v \in V$. Let $m\colon V \to \mathbb{R}_{\geq 0}$ be a function defined by

$$m(w) = \frac{\ell(w)}{||w - v||} \qquad \text{for all } w \in V,\, w \neq v,$$

and let $m(v) = 0$. For every $R \geq 0$, define $m_R\colon V \to \mathbb{R}_{\geq 0}$ by $m_R(w) = m(w)$, for all $w \in V \cap B_{e^R}(v)$, and $m_R(w) = 0$ for all $w \notin B_{e^R}(v)$. Here and throughout this section, $B_t(v)$ denotes the closed ball in $\mathbb{R}^n$ of radius t, around the vertex v. We claim that

$$\inf_{\gamma \in \Gamma(v)} \frac{\mathrm{Length}_{m_R}(\gamma)^p}{||m_R||_p^p} \to \infty \qquad \text{as } R \to \infty, \tag{3}$$

and therefore, the p-extremal length of $\Gamma(v)$ is infinite.

Step 1. Let $\gamma \in \Gamma(v)$. We begin with bounding $\mathrm{Length}_{m_R}(\gamma)$ from below. Let ω be the 1-form on $\mathbb{R}^p$ which is zero on $B_{L(v)}(v)$ and equals $\frac{dr}{r}$ outside $B_{L(v)}(v)$, where r is the radial coordinate w.r.t. the basepoint v. Let γ_R be the maximal initial part of γ consisting of vertices in $B_{e^R}(v)$ and edges starting at vertices in $B_{e^R}(v)$. Since γ_R starts at v and eventually leaves

$B_{e^R}(v)$, we have $\int_{\gamma_R}\omega \geq R - \ln L(v)$. On the other hand, for an edge f of γ_R starting at $w \neq v$ we have

$$\int_f \omega \leq \frac{L(w)}{||w-v||}.$$

Altogether, we obtain:

$$\begin{aligned}\text{Length}_{m_R}(\gamma) &= \sum_{w\in\gamma} m_R(w) \geq \sum_{w\in\gamma_R} m_R(w) = \sum_{w\in\gamma_R} m(w)\\ &\geq \frac{1}{C}\sum_{w\in\gamma_R\setminus v}\frac{L(w)}{||w-v||} \geq \frac{1}{C}\sum_{f\in\gamma_R}\int_f\omega = \frac{1}{C}\int_{\gamma_R}\omega \geq \frac{1}{C}(R-\ln L(v)).\end{aligned}$$

Step 2. We now bound $||m_R||_p$ from above. With each vertex $w \in V$ we associate the ball $B(w)$ of radius $\frac{l(w)}{2}$ centered at w. All these balls have disjoint interiors.

Let σ be the p-form on $\mathbb{R}^p$ which is zero on $B(v)$ and is equal to $\frac{1}{r^p}$vol outside $B(v)$, where vol denotes the Euclidean volume form.

Let $w \neq v$. We estimate $\sigma(B(w))$. Since the radius of $B(w)$ is at most $\frac{||w-v||}{2}$, we have $B(w) \subset B_{\frac{3||w-v||}{2}}(v)$. Hence:

$$\begin{aligned}\sigma(B(w)) &= \int_{B(w)}\frac{\text{vol}}{r^p} \geq \left(\frac{2}{3||w-v||}\right)^p \text{vol}(B(w)) =\\ &= \left(\frac{2}{3||w-v||}\right)^p V_p\left(\frac{l(w)}{2}\right)^p = cm^p(w),\end{aligned}$$

for a universal constant $c > 0$, where V_p is the volume of the unit p-ball. Hence

$$||m_R||_p^p = \sum_{w\in B_{e^R}(v)} m^p(w) \leq \frac{1}{c}\sum_{w\in B_{e^R}(v)}\sigma(B(w)) \leq \frac{1}{c}\sigma(B_{\frac{3}{2}e^R}(v)).$$

The latter is bounded above by $\frac{1}{c}A_{p-1}(\ln\frac{3}{2} + R - \ln\frac{l(v)}{2})$, where A_{p-1} is the volume of the unit $(p-1)$-sphere, and that is a linear function in R.

Combining the steps. In Step 1 we have bounded the numerator of (3) below by a polynomial of degree p in R. In Step 2 we have bounded the denominator of (3) above by a function linear in R. Hence the p-extremal length of $\Gamma(v)$ is infinite, as desired. ∎

To finish the proof, we need the following known result:

Proposition 5.11 ([4, Proposition 4.1(1)]). *Let $G=(V,E)$ be an infinite locally finite connected graph. If G is p-parabolic and I is a positive isoperimetric function, then we have*

$$\sum_{k=1}^{\infty} \frac{1}{I(k)^{\frac{p}{p-1}}} = \infty.$$

Proof of Theorem 5.1. Assume that there is an acute triangulation τ of $\mathbb{R}^4$ with bounded geometry. Let G be the 1-skeleton of τ. By Lemma 5.10 we have that the graph G is 4-parabolic. By Proposition 5.11, we have that $k \to Ck$ is not an isoperimetric function for G. This contradicts Proposition 5.7. ∎

6. Final remarks and open problems

6.1.

It is unclear how far the results of Section 2 extend to other polytopes in $\mathbb{R}^3$. For example, the regular icosahedron has an easy acute triangulation using cones from the center over every facet. Similarly, the regular dodecahedron, one can easily subdivide it into 120 congruent tetrahedra all meeting at the center. It turns out that the special subdivision of this triangulation has an acute realization in this case as well. For the subdivision of one of the congruent tetrahedra see

`http://www.mimuw.edu.pl/~erykk/papers/acute.html`.

Putting this together, we obtain the following result:

Theorem 6.1. *All Platonic solids have non-trivial acute triangulations.*

Now, it is possible that *every* convex polytope in $\mathbb{R}^3$ has an acute triangulation. We conjecture this to be the case. Unfortunately, we are very far from proving this, given that this paper and [39] have the first examples of non-trivial acute triangulations of convex polytopes (cf. [9]). Perhaps, it is even possible that every 3-dimensional abstract polyhedral manifold has an acute triangulation, in the style of [10]. For example, we conjecture that the boundary of every convex polytope in $\mathbb{R}^4$ has an acute triangulation. Of course, in the spirit of [3,27,33], the problem might prove much easier for non-obtuse triangulation.

Finally, numerical results would also be of interest. What is the smallest number of tetrahedra required for a non-trivial acute triangulation of the regular tetrahedron? For example, can one beat our record of 543? How about the cube? Can one bound the smallest maximum dihedral angle? Dreaming of the future, can one always find a linear size acute triangulation of a convex polytope in $\mathbb{R}^3$?

6.2.

Although Křížek conjectured in [22] (see also [9]), that there are no acute triangulations of the space $\mathbb{R}^4$, our results resolve only a special case of this problem. The conjecture remains open in full generality, when the geometry of simplices is unbounded. On the one hand, another (plausible) conjecture in [22,9] states that locally such acute triangulation must have at least 600 simplices around each vertex, making a construction of such triangulation exceedingly difficult. On the other hand, in the plane and the space there are known very general combinatorial tiling constructions which require an unbounded geometry (see e.g. [17,35]). We conjecture that there exists an acute triangulation of $\mathbb{R}^4$, although we think that to construct it, one first has to master acute triangulations in dimension 3 (see Section 6.1), and in the spherical case (see Section 6.4 below).

Similarly, what happens with individual convex polytopes in $\mathbb{R}^4$ is much less clear, and will obviously depend on the polytope. For example, by analogy with the icosahedron, there is an easy acute triangulation of the 600-cell. On the other hand, it is unclear whether the 16-cell (the regular cross-polytope), the 24-cell and the 120-cell have acute triangulations (we conjecture not). One is tempted to conclude the 16-cell does not admit an acute triangulation, since it tiles the space $\mathbb{R}^4$. Unfortunately, this argument is incorrect for the following reason. In order to have consistency on the boundary, two subdivisions of a 16-cell adjacent by the tetrahedra must have the opposite orientations. However, in the tiling, there are three (an odd number) top dimensional cells around each codimension-2 simplex, and thus not every triangulation of the 16-cell gives rise to a triangulation of $\mathbb{R}^4$ (the same problem appears for the 24-cell).

Interestingly, the space tessellation argument does work for some notable polytopes in $\mathbb{R}^4$. For example, it is well known that the 4-cube can be dissected into 24 congruent orthoschemes [4] (see e.g. [12]), in such a way that around each interior codimension-2 simplex there are 4 or 6 (an even number) orthoschemes. This implies that such "isosceles" orthoscheme does not have an acute triangulation, because otherwise one could extend it by reflecting to an acute triangulation of the 4-cube. Similarly, the standard (cube corner) 4-dimensional simplex tiles the space $\mathbb{R}^4$ by reflections (it is a fundamental chamber for $\widetilde{D}_4$ affine Coxeter group action). Hence the standard 4-simplex also does not admit an acute triangulation.

[4] They are also called path-simplices.

In summary, we believe that finding a useful criterion for a polytope in $\mathbb{R}^4$ to have an acute triangulation is a challenging problem, which we expect to be more difficult than the 3-dimensional version.

6.3.

We believe that the proof of our Theorem C given in [22, Theorem 6.2] has a crucial gap and is either incomplete or incorrect as written. [5] On p. 387, in the proof of Theorem 5.1, in the sentence "the sum of all dihedral angles of tetrahedra around a given edge E from ∂P cannot be greater than 2π," the author seems to be referring to 3-faces of convex 4-polytope P, which would make this statement true (and Lemma 3.3 in the paper applicable). However, throughout the paper, the polytope P is in fact in $\mathbb{R}^5$, in which case the above sum is a priori unbounded. It seems, this mistake has not been discovered until now. We should mention, however, that the reduction of higher dimensions to dimension 5 given in [22] (see the proof of Theorem 6.2), is independent of Theorem 5.1 and completely correct.

6.4.

It would be interesting to consider the spherical and hyperbolic analogues of the acute triangulation problem. The spherical analogue might prove particularly insightful as it might allow the use of a dimension reduction in the Euclidean case (in particular in the case of $\mathbb{R}^4$).

6.5.

The Burago–Zalgaller original result in [10] is a technical lemma used towards the 3-dimensional analogue of the classical Nash–Kuiper embedding theorem. This result (by a different technique) was recently extended by Akopyan to higher dimensions [1], despite the absence of acute triangulations.

[5] After this paper was written, Michal Křížek graciously confirmed the error in his paper. He informed us that he first learned about it in 2008 from Jan Brandts, and that he has prepared a correction (to appear). Since neither the error nor the correction has been announced nor are publicly available, we decided not to change our presentation and keep the details.

6.6.

In the plane, one can start with a given triangulation and "improve it" by using 2-flips, by increasing the smallest angle in a triangle. This results in the *Delaunay triangulation* which (among all triangulations on this set of vertices) has the largest possible minimal angle, and has a number of other useful properties (though not necessarily the smallest maximal angle). Thus, by strategically placing new points into the interior of a polygon one can then efficiently construct a "good" triangulation. In higher dimensions this approach breaks down for several reasons, both due to the lack of connectedness of triangulations via flips, and non-monotonicity of the angle functionals. We refer to [13,31] for an introduction and an extended discussion of the problem.

Interestingly, a variation on the Delaunay approach does give useful meshes in $\mathbb{R}^3$, as described in [38]. The paper [39] is a followup on this approach, which uses a more refined idea of incremental changes in a triangulation, by moving the vertices one at a time.

6.7.

There is a large body of work on tessellations of the space by convex polyhedra with bounded geometry. Perhaps the earliest, is the result of Alexandrov that in every triangulation of the plane into bounded triangles the average degree of vertices (when defined) must be at least 6 [2]. Another is a classical result by Niven that convex n-gons of bounded geometry cannot tile the plane for $n \geq 7$ [29] (see also [16]). The idea is always to use the isoperimetric inequalities compared with direct counting estimates, an approach which works in higher dimensions as well (see e.g. [5,23,24]). Our proof of Theorem D is a variation on the same line of argument. We refer to [18] for historical background and further references.

6.8.

The classical Dehn–Sommerville equations are defined for f-vectors of simplicial convex polytopes in $\mathbb{R}^d$ (see e.g. [31, Section 8]). They are extended in a number of directions, notably the beautiful *flag f-vectors* by Bayer and Billera, leading to the *cd-index* (see [36]). The version for manifolds without boundary was first given by Klee in [21]. It seems, the version with the boundary goes back to Macdonald [26] and was rediscovered a number of times. We refer to [30] for the most general version of the equations, various applications, and further references.

6.9.

Let us note that the ad hoc acute triangulation of the cube discovered in [39] has 1370 tetrahedra as opposed to $5 \cdot 543 = 2715$ tetrahedra in our construction. On the other hand, one can argue that our construction is more symmetric. It is invariant under the natural action of S_4 permuting the standard tetrahedra. It also has some "hidden" symmetries arising from the 600-cell (the construction in [39] also has a number of symmetries).

After we made our computations publicly available, Evan VanderZee kindly informed us that he re-checked our coordinates for the triangulation of the cube and computed that dihedral angles range between 26.425 and 89.992. [6] For comparison, the dihedral angles found in [39] range between 35.89 and 84.65, which is significantly better for numerical algorithms, since it has fewer tetrahedra, *smaller* the maximal and *larger* the minimal dihedral angles. On the other hand, after performing simulations with our mesh of 2715 tetrahedra (i.e. when combinatorial structure is fixed while positions of points are allowed to vary), VanderZee obtained an acute triangulation with dihedral angles between 25.310 and 88.902. Hence this triangulation and ours are incomparable as far as the dihedral angles are concerned.

A. The exact position of vertices

In order for the coordinates to be integers, we triangulate the cube whose vertices are at the eight points all of whose coordinates are in the set $\{0, 60000\}$ (instead of $\{\pm 1\}$). Vertices of the Step 3 (Section 2) triangulation of the regular 3-simplex T_0 have the following coordinates (we list four vertices in each row):

0-3	60000,	0,	0;	60000,	60000,	60000;	0,	0,	60000;	0,	60000,	0
0-3	60000,	0,	0;	60000,	60000,	60000;	0,	0,	60000;	0,	60000,	0
4-7	0,	30000,	30000;	60000,	30000,	30000;	30000,	30000,	0;	30000,	60000,	30000
8-11	30000,	30000,	60000;	30000,	0,	30000;	33916,	43042,	16958;	43042,	26084,	43042
12-15	16958,	16958,	26084;	16958,	33916,	43042;	43042,	16958,	33916;	43042,	33916,	16958
16-19	43042,	43042,	26084;	16958,	26084,	16958;	16958,	43042,	33916;	26084,	43042,	43042
20-23	26084,	16958,	16958;	33916,	16958,	43042;	34171,	39326,	34171;	20674,	34171,	25829
24-27	25829,	25829,	39326;	34171,	25829,	20674;	25829,	20674,	34171;	39326,	34171,	34171
28-31	39326,	25829,	25829;	20674,	25829,	34171;	34171,	20674,	25829;	34171,	34171,	39326
32-35	25829,	34171,	20674;	25829,	39326,	25829;	24956,	35044,	35044;	39033,	32132,	27868
36-39	27868,	27868,	20967;	32132,	20967,	32132;	32132,	32132,	20967;	24956,	24956,	24956
40-43	32132,	39033,	27868;	20967,	27868,	27868;	39033,	27868,	32132;	35044,	35044,	24956
44-47	20967,	32132,	32132;	27868,	20967,	27868;	35044,	24956,	35044;	27868,	32132,	39033
48-51	32132,	27868,	39033;	27868,	39033,	32132;	35393,	30000,	35393;	24607,	30000,	24607
52-55	35393,	24607,	30000;	24607,	35393,	30000;	35393,	30000,	24607;	24607,	30000,	35393
56-59	24607,	24607,	30000;	35393,	35393,	30000;	30000,	35393,	24607;	30000,	24607,	35393
60-63	30000,	24607,	24607;	30000,	35393,	35393;	33844,	26156,	26156;	33844,	33844,	33844
64-67	26156,	26156,	33844;	26156,	33844,	26156;	24207,	28632,	31368;	31368,	31368,	35793
68-71	28632,	35793,	28632;	28632,	28632,	35793;	28632,	24207,	31368;	35793,	31368,	31368
72-75	24207,	31368,	28632;	35793,	28632,	28632;	31368,	35793,	31368;	31368,	28632,	24207
76-79	28632,	31368,	24207;	31368,	24207,	28632;	31992,	31992,	25546;	34454,	28008,	31992

[6] It should be noted that optimizing the angles was not our goal. The reason the angles are so large, is because our angle correction procedure described in Section 2 works only with angles larger than or equal to $\pi/2$.

80-83	28008,	25546,	28008;	28008,	31992,	34454;	31992,	34454,	28008;	25546,	28008,	28008
84-87	25546,	31992,	31992;	34454,	31992,	28008;	28008,	34454,	31992;	28008,	28008,	25546
88-91	31992,	28008,	34454;	31992,	25546,	31992;	32775,	32775,	30655;	27225,	30655,	27225
92-95	29345,	27225,	32775;	32775,	29345,	27225;	27225,	32775,	29345;	27225,	29345,	32775
96-99	27225,	27225,	30655;	32775,	30655,	32775;	32775,	27225,	29345;	30655,	27225,	27225
100-103	30655,	32775,	32775;	29345,	32775,	27225;	28494,	31506,	31506;	30865,	29135,	29135
104-107	33159,	30000,	30000;	28494,	28494,	28494;	29135,	29135,	30865;	26841,	30000,	30000
108-111	30000,	30000,	33159;	30000,	26841,	30000;	31506,	31506,	28494;	29135,	30865,	29135
112-115	30000,	30000,	26841;	31506,	28494,	31506;	30865,	30865,	30865;	30000,	33159,	30000

Vertices of the triangulation of the standard 3-simplex T_1 defined in Step 2 of Section 2, have the following coordinates (rotate to obtain the coordinates for the other standard T_i):

0-3	60000,	0,	0;	0,	0,	0;	0,	0,	60000;	0,	60000,	0
4-7	0,	30000,	30000;	17574,	0,	0;	30000,	30000,	0;	0,	17574,	0
8-11	0,	0,	17574;	30000,	0,	30000;	12384,	20726,	0;	10445,	0,	10445
12-15	16958,	16958,	26084;	0,	12384,	20726;	20726,	0,	12384;	20726,	12384,	0
16-19	10445,	10445,	0;	16958,	26084,	16958;	0,	20726,	12384;	0,	10445,	10445
20-23	26084,	16958,	16958;	12384,	0,	20726;	6257,	10104,	6257;	9498,	21496,	13569
24-27	7743,	7743,	19656;	21496,	13569,	9498;	13569,	9498,	21496;	10104,	6257,	6257
28-31	19656,	7743,	7743;	9498,	13569,	21496;	21496,	9498,	13569;	6257,	6257,	10104
32-35	13569,	21496,	9498;	7743,	19656,	7743;	5137,	13344,	13344;	15349,	8879,	5284
36-39	17685,	17685,	11260;	16563,	7777,	16563;	16563,	16563,	7777;	16026,	16026,	16026
40-43	8879,	15349,	5284;	11260,	17685,	17685;	15349,	5284,	8879;	13344,	13344,	5137
44-47	7777,	16563,	16563;	17685,	11260,	17685;	13344,	5137,	13344;	5284,	8879,	15349
48-51	8879,	5284,	15349;	5284,	15349,	8879;	10649,	6407,	10649;	13477,	17719,	13477
52-55	16305,	7821,	12063;	7821,	16305,	12063;	16305,	12063,	7821;	7821,	12063,	16305
56-59	13477,	13477,	17719;	10649,	10649,	6407;	12063,	16305,	7821;	12063,	7821,	16305
60-63	17719,	13477,	13477;	6407,	10649,	10649;	17102,	11055,	11055;	9039,	9039,	9039
64-67	11055,	11055,	17102;	11055,	17102,	11055;	10544,	14025,	16177;	8666,	8666,	12147
68-71	9383,	15016,	9383;	9383,	9383,	15016;	14025,	10544,	16177;	12147,	8666,	8666
72-75	10544,	16177,	14025;	15016,	9383,	9383;	8666,	12147,	8666;	16177,	14025,	10544
76-79	14025,	16177,	10544;	16177,	10544,	14025;	13876,	13876,	8805;	13231,	8160,	11294
80-83	14921,	12984,	14921;	8160,	11294,	13231;	11294,	13231,	8160;	12984,	14921,	14921
84-87	8805,	13876,	13876;	13231,	11294,	8160;	8160,	13231,	11294;	14921,	14921,	12984
88-91	11294,	8160,	13231;	13876,	8805,	13876;	10992,	10992,	9324;	12447,	15145,	12447
92-95	11891,	10223,	14589;	14589,	11891,	10223;	10223,	14589,	11891;	10223,	11891,	14589
96-99	12447,	12447,	15145;	10992,	9324,	10992;	14589,	10223,	11891;	15145,	12447,	12447
100-103	9324,	10992,	10992;	11891,	14589,	10223;	10088,	12458,	12458;	13197,	11836,	11836
104-107	12891,	10406,	10406;	13247,	13247,	13247;	11836,	11836,	13197;	11234,	13719,	13719
108-111	10406,	10406,	12891;	13719,	11234,	13719;	12458,	12458,	10088;	11836,	13197,	11836
112-115	13719,	13719,	11234;	12458,	10088,	12458;	11382,	11382,	11382;	10406,	12891,	10406

In both cases, the edges are spanned on the following pairs of vertices:

4-2, 4-3, 5-0, 5-1, 6-0, 6-3, 7-1, 7-3, 8-1, 8-2, 9-0, 9-2, 10-3, 10-6, 10-7, 11-1, 11-5, 11-8, 12-2, 12-4, 12-9, 13-2, 13-4, 13-8, 14-0, 14-5, 14-9, 14-11, 15-0, 15-5, 15-6, 15-10, 16-1, 16-5, 16-7, 16-10, 16-15, 17-3, 17-4, 17-6, 17-12, 18-3, 18-4, 18-7, 18-13, 19-1, 19-7, 19-8, 19-13, 19-18, 20-0, 20-6, 20-9, 20-12, 20-17, 21-2, 21-8, 21-9, 21-11, 21-14, 22-1, 22-7, 22-16, 22-19, 23-3, 23-4, 23-17, 23-18, 24-2, 24-8, 24-13, 24-21, 25-0, 25-6, 25-15, 25-20, 26-2, 26-9, 26-12, 26-21, 26-24, 27-1, 27-5, 27-11, 27-16, 27-22, 28-0, 28-5, 28-14, 28-15, 28-25, 29-2, 29-4, 29-12, 29-13, 29-24, 29-26, 30-0, 30-9, 30-14, 30-20, 30-25, 30-28, 31-1, 31-8, 31-11, 31-19, 31-22, 31-27, 32-3, 32-6, 32-10, 32-17, 32-23, 33-3, 33-7, 33-10, 33-18, 33-23, 33-32, 34-13, 34-18, 34-19, 35-5, 35-15, 35-16, 35-27, 35-28, 36-6, 36-17, 36-20, 36-25, 36-32, 37-9, 37-14, 37-21, 37-26, 37-30, 38-6, 38-10, 38-15, 38-25, 38-32, 38-36, 39-12, 39-17, 39-20, 39-36, 40-7, 40-10, 40-16, 40-22, 40-33, 41-4, 41-12, 41-17, 41-23, 41-29, 41-39, 42-5, 42-11, 42-14, 42-27, 42-28, 42-35, 43-10, 43-15, 43-16, 43-35, 43-38, 43-40, 44-4, 44-13, 44-18, 44-23, 44-29, 44-34, 44-41, 45-9, 45-12, 45-20, 45-26, 45-30, 45-37, 45-39, 46-11, 46-14, 46-21, 46-37, 46-42, 47-8, 47-13, 47-19, 47-24, 47-31, 47-34, 48-8, 48-11, 48-21, 48-24, 48-31, 48-46, 48-47, 49-7, 49-18, 49-19, 49-22, 49-33, 49-34, 49-40, 50-11, 50-27, 50-31, 50-42, 50-46, 50-48, 51-17, 51-23, 51-32, 51-36, 51-39, 51-41, 52-14, 52-28, 52-30, 52-37, 52-42, 52-46, 53-18, 53-23, 53-33, 53-34, 53-44, 53-49, 54-15, 54-25, 54-28, 54-35, 54-38, 54-43, 55-13, 55-24, 55-29, 55-34, 55-44, 55-47, 56-12, 56-26, 56-29, 56-39, 56-41, 56-45, 57-16, 57-22, 57-27, 57-35, 57-40, 57-43, 58-10, 58-32, 58-33, 58-38, 58-40, 58-43, 59-21, 59-24, 59-26, 59-37, 59-46, 59-48, 60-20, 60-25, 60-30, 60-36, 60-39, 60-45, 61-19, 61-22, 61-31, 61-34, 61-47, 61-49, 62-25, 62-28, 62-30, 62-52, 62-54, 62-60, 63-22, 63-27, 63-31, 63-50, 63-57, 63-61, 64-24, 64-26, 64-29, 64-55, 64-56, 64-59, 65-23, 65-32, 65-33, 65-51, 65-53, 65-58, 66-29, 66-41, 66-44, 66-55, 66-56, 66-64, 67-31, 67-47, 67-48, 67-50, 67-61, 67-63, 68-33, 68-40, 68-49, 68-53, 68-58, 68-65, 69-24, 69-47, 69-48, 69-55, 69-59, 69-64, 69-67, 70-26, 70-37, 70-45, 70-56, 70-59, 70-64, 71-27, 71-35, 71-42, 71-50, 71-57, 71-63, 72-23, 72-41, 72-44, 72-51, 72-53, 72-65, 72-66, 73-28, 73-35, 73-42, 73-52, 73-54, 73-62, 73-71, 74-22, 74-40, 74-49, 74-57, 74-61, 74-63, 74-68, 75-25, 75-36, 75-38, 75-54, 75-60, 75-62, 76-32, 76-36, 76-38, 76-51, 76-58, 76-65, 76-75, 77-30, 77-37, 77-45, 77-52, 77-60, 77-62, 77-70, 78-38, 78-43, 78-54, 78-58, 78-75, 78-76, 79-42, 79-46, 79-50, 79-52, 79-71, 79-73, 80-39, 80-45, 80-56, 80-60, 80-70, 80-77, 81-34, 81-47, 81-55, 81-61, 81-67, 81-69, 82-40, 82-43, 82-57, 82-58, 82-68, 82-74, 82-78, 83-39, 83-41, 83-51, 83-56, 83-66, 83-72, 83-80, 84-34, 84-44, 84-53, 84-55, 84-66, 84-72, 84-81, 85-35, 85-43, 85-54, 85-57, 85-71, 85-73, 85-78, 85-82, 86-34, 86-49, 86-53, 86-61, 86-68, 86-74, 86-81, 86-84, 87-36, 87-39, 87-51, 87-60, 87-75, 87-76, 87-80, 87-83, 88-46, 88-48, 88-50, 88-59, 88-67, 88-69, 88-79, 89-37, 89-46, 89-52, 89-59, 89-70, 89-77, 89-79, 89-88, 90-57, 90-63, 90-71, 90-74, 90-82, 90-85, 91-51, 91-65, 91-72, 91-76, 91-83, 91-87, 92-59, 92-64, 92-69, 92-70, 92-88, 92-89, 93-54, 93-62, 93-73, 93-75, 93-78, 93-85, 94-53, 94-65, 94-68, 94-72, 94-84, 94-86, 94-91, 95-55, 95-64, 95-66, 95-69, 95-81, 95-84, 95-92, 96-56, 96-64, 96-66, 96-70, 96-80, 96-83, 96-92, 96-95, 97-50, 97-63, 97-67, 97-71, 97-79, 97-88, 97-90, 98-52, 98-62, 98-73, 98-77, 98-79, 98-89, 98-93, 99-60, 99-62, 99-75, 99-77, 99-80, 99-87, 99-93, 99-98, 100-61, 100-63, 100-67, 100-74, 100-81, 100-86, 100-90, 100-97, 101-58, 101-65, 101-68, 101-76, 101-78, 101-82, 101-91, 101-94, 102-81, 102-84, 102-86, 102-94, 102-95, 102-100, 103-93, 103-98, 103-99, 104-71, 104-73, 104-79, 104-85, 104-90, 104-93, 104-97, 104-98, 104-103, 105-80, 105-83, 105-87, 105-91, 105-96, 105-99, 105-103, 106-92, 106-95, 106-96, 106-102, 106-103, 106-105, 107-66, 107-72, 107-83, 107-84, 107-91, 107-94, 107-95, 107-96, 107-102, 107-105, 107-106, 108-67, 108-69, 108-81, 108-88, 108-92, 108-95, 108-97, 108-100, 108-102, 108-106, 109-70, 109-77, 109-80, 109-89, 109-92, 109-96, 109-98, 109-99, 109-103, 109-105, 109-106, 110-78, 110-82, 110-85, 110-90, 110-93, 110-101, 110-103, 110-104, 111-91, 111-94, 111-101, 111-102, 111-103, 111-105, 111-106, 111-107, 111-110, 112-75, 112-76, 112-78, 112-87, 112-91, 112-93, 112-99, 112-101, 112-103, 112-105, 112-110, 112-111, 113-79, 113-88, 113-89, 113-92, 113-97, 113-98, 113-103, 113-104, 113-106, 113-108, 113-109, 114-90, 114-97, 114-100, 114-102, 114-103, 114-104, 114-106, 114-108, 114-110, 114-111, 114-113, 115-68, 115-74, 115-82, 115-86, 115-90, 115-94, 115-100, 115-101, 115-102, 115-110, 115-111, 115-114.

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Eryk Kopczyński

Institute of Informatics
Warsaw University
Banacha 2, 02-097 Warsaw
Poland
erykk@mimuw.edu.pl

Igor Pak

Department of Mathematics
UCLA
Los Angeles, CA 90095
USA
pak@math.ucla.edu

Piotr Przytycki

Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8, 00-956 Warsaw
Poland
pprzytyc@mimuw.edu.pl