APPLICATIONS OF THE TARSKI–KANTOROVITCH FIXED-POINT PRINCIPLE IN THE THEORY OF ITERATED FUNCTION SYSTEMS

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Abstract. We show how some results of the theory of iterated function systems can be derived from the Tarski–Kantorovitch fixed–point principle for maps on partially ordered sets. In particular, this principle yields, without using the Hausdorff metric, the Hutchinson–Barnsley theorem with the only restriction that a metric space considered has the Heine–Borel property. As a by–product, we also obtain some new characterizations of continuity of maps on countably compact and sequential spaces.

1. Introduction

Let X be a set and $f_1, \dots, f_n$ be selfmaps of X. The theory of iterated function systems (abbr., IFS) deals with the following *Hutchinson–Barnsley operator*:

$$F(A) := \bigcup_{i=1}^{n} f_i(A) \qquad \text{for } A \subseteq X. \tag{1}$$

The fundamental result of the Hutchinson–Barnsley theory (cf. [2], [7]) says that if (X, d) is a complete metric space and all the maps f_i are Banach's contractions, then F is the Banach contraction on the family $K(X)$ of all nonempty compact subsets of X, endowed with the Hausdorff metric. Consequently, F has then a unique fixed point A_0 in $K(X)$, which is called a fractal in the sense of Barnsley. Moreover, for any set A in $K(X)$, the sequence $(F^n(A))_{n=1}^{\infty}$ of iterations of F converges to A_0 with respect to the Hausdorff metric. For an arbitrary IFS a set A_0 such that $A_0 = F(A_0)$ is called *invariant with respect to the IFS* $\{f_i : \ i = 1, \dots, n\}$ (cf. Lasota–Myjak [10]). If $n = 1$, then such an A_0 is said to be a *modulus set* for the map f_1 (cf. Kuczma [9, p. 13]).

In this paper we study possibilities of applying the Tarski–Kantorovitch fixed–point principle (cf. Dugundji–Granas [3, Theorem 4.2, p. 15]) in the theory of IFS (in the sequel we will use the abbreviation "the T–K principle"). So we will employ the partial ordering technique to obtain results on fixed points of the Hutchinson–Barnsley operator. The idea of treating fractals as Tarski's fixed points appeared earlier in papers of Soto–Andrade

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& Varela [13] and Hayashi [6], however, they considered other version of Tarski's theorem than that studying in this paper. Other consequences of the T–K principle were investigated, e.g., in articles of Baranga [1] (the Banach contraction principle is derived here from the Kleene theorem, an equivalent version of the T–K principle) and Jachymski [8]. See also "Notes and comments" in the Dugundji–Granas monograph [3, p. 169], and references therein.

Our paper is organized as follows. In Section 2 the T–K principle is formulated and a lemma on continuity with respect to a partial ordering is proved.

Section 3 is devoted to a study of the T–K principle for the family 2^X of all subsets of X, endowed with the set–theoretical inclusion $\supseteq$ as a partial ordering. Theorem 2 gives sufficient conditions for the existence of the greatest invariant set with respect to the IFS considered in this purely set–theoretical case.

Section 4 deals with the family $C(X)$ of all nonempty closed subsets of a Hausdorff topological space X, endowed with the inclusion $\supseteq$. In this case the countable chain condition of the T–K principle forces the countable compactness of X (cf. Proposition 4). Our Theorem 3 on an invariant set generalizes an earlier result of Leader [12], established for the case $n = 1$. As a by–product, we obtain a new characterization of continuity of maps on countably compact and sequential spaces (cf. Proposition 5 and Theorem 8). We also study the T–K principle for the following operator F, introduced by Lasota and Myjak [10].

$$F(A) := \mathrm{cl}\left(\bigcup_{i=1}^{n} f_i(A)\right) \qquad \text{for } A \subseteq X, \tag{2}$$

where cl denotes the closure operator. Again, as a by–product, we obtain here another new characterization of continuity (cf. Proposition 6 and Theorem 9).

Section 5 deals with the family $K(X)$ of all nonempty compact subsets of a topological space X, endowed with the inclusion $\supseteq$. This time the condition "$b \le F(b)$" of the T–K principle forces, in some sense, the compactness of a space, in which we work. Nevertheless, using an idea of Williams [14], we show that, in such a case, the T–K principle yields the Hutchinson–Barnsley theorem for a class of the *Heine–Borel metric spaces*, that is, spaces in which every closed and bounded set is compact (cf. Williamson–Janos [15]). We emphasize here that instead of showing that the Hutchinson–Barnsley operator F is contractive with respect to the Hausdorff metric, it suffices to prove the existence of a compact subset A of X such that $F(A) \subseteq A$, which is quite elementary (cf. the proof of Corollary 2). Also it is worth noticing here that many results of the theory of IFS were obtained in the class of the Heine–Borel metric spaces (cf. Lasota–Myjak [10] and Lasota–Yorke [11]).

In Section 6 we assembled some topological results, which, in our opinion, were interesting themselves, and which had been obtained as a by–product of our study of $\supseteq$-continuity of the Hutchinson–Barnsley operator.

Given sets X and Y, and a map $f:\ X \mapsto Y$, the sets $f^{-1}(\{y\})$ $(y \in Y)$ are called *fibres* of f (cf. Engelking [5, p. 14]).

As in [5], we assume that a compact or countably compact space is Hausdorff by the definition.

2. The Tarski–Kantorovitch fixed–point principle

Recall that a relation $\leq$ in a set P is a *partial ordering*, if $\leq$ is reflexive, weakly antisymmetric and transitive. A linearly ordered subset of P is called a *chain*. A selfmap F of P is said to be $\leq$-*continuous* if for each countable chain C having a supremum, $F(C)$ has a supremum and $\sup F(C) = F(\sup C)$. Then F is increasing with respect to $\leq$.

Theorem 1 (Tarski–Kantorovitch)**.** *Let $(P, \leq)$ be a partially ordered set, in which every countable chain has a supremum. Let F be a $\leq$-continuous selfmap of P such that there exists a $b \in P$ with $b \leq F(b)$. Then F has a fixed point; moreover, $\sup\{F^n(b) : \ n \in \mathbb{N}\}$ is the least fixed point of F in the set $\{p \in P : \ p \geq b\}$.*

Remark 1. It can be easily verified that the assumption "every countable chain has a supremum" is equivalent to "every increasing sequence (p_n) (that is, $p_n \leq p_{n+1}$ for $n \in \mathbb{N}$) has a supremum". Similarly, in the definition of $\leq$-continuity, we may substitute increasing sequences for countable chains. Such a reformulated Theorem 1 is identical with the Kleene fixed–point theorem (cf., e.g., Baranga [1]).

Lemma 1. *Let $(P, \leq)$ be a partially ordered set, in which every countable chain has a supremum and such that for any $p, q \in P$ there exists an infimum* $\inf \{x, y\}$. *Assume that for any increasing sequences $(p_n)_{n=1}^{\infty}$ and $(q_n)_{n=1}^{\infty}$,*

$$\inf \left\{ \sup_{n\in\mathbb{N}} p_n, \ \sup_{n\in\mathbb{N}} q_n \right\} = \sup_{n\in\mathbb{N}} \inf \{p_n, q_n\}. \tag{3}$$

Let $F_1, \ldots, F_n$ be $\leq$-continuous selfmaps of P and define a map F by

$$F(p) := \inf \{F_1(p), \ldots, F_n(p)\} \qquad \text{for } p \in P.$$

Then F is $\leq$-continuous.

Proof. For the sake of simplicity, assume that $n = 2$; then an easy induction shows that our argument can be extended to the case of an arbitrary $n \in \mathbb{N}$. By Remark 1, it suffices to prove that given an increasing sequence (p_n), $F(p) = \sup_{n\in\mathbb{N}} F(p_n)$, where $p := \sup_{n\in\mathbb{N}} p_n$. Since F_1 and F_2 are increasing, so is F. Thus the sequence $(F(p_n))$ is increasing and by hypothesis, it has a supremum. Then, by (3) and $\leq$-continuity of F_1 and F_2,

$$\begin{aligned}\sup_{n\in\mathbb{N}} F(p_n) &= \sup_{n\in\mathbb{N}} \inf \{F_1(p_n), F_2(p_n)\} = \inf \{\sup_{n\in\mathbb{N}} F_1(p_n), \sup_{n\in\mathbb{N}} F_2(p_n)\} \\ &= \inf \{F_1(\sup_{n\in\mathbb{N}} p_n), F_2(\sup_{n\in\mathbb{N}} p_n)\} = F(\sup_{n\in\mathbb{N}} p_n),\end{aligned}$$

which proves the $\leq$-continuity of F. □

The following example shows that there exists a partially ordered set $(P, \leq)$, in which every countable chain has a supremum and for any $p, q \in P$ there exists $\inf \{p, q\}$, but condition (3) does not hold. In fact, the set $(P, \leq)$ defined below is a *complete lattice*, that is, every subset of P has a supremum and an infimum.

Example 1. Let $C(\mathbb{R})$ be the family of all nonempty closed subsets of the real line and $P := C(\mathbb{R}) \cup \{\emptyset\}$. Endow P with the inclusion $\subseteq$. If $\{A_t : t \in T\} \subseteq P$, then $\inf_{t\in T} A_t = \bigcap_{t\in T} A_t$ and $\sup_{t\in T} A_t = \operatorname{cl}\left(\bigcup_{t\in T} A_t\right)$. Define

$$A_n := [0, 1 - \frac{1}{n}], \quad B_n := [1 + \frac{1}{n}, 2] \qquad \text{for } n \in \mathbb{N}.$$

Then (A_n) and (B_n) are increasing and

$$\inf \{\sup_{n\in\mathbb{N}} A_n, \sup_{n\in\mathbb{N}} B_n\} = \operatorname{cl}\left(\bigcup_{n\in\mathbb{N}} A_n\right) \cap \operatorname{cl}\left(\bigcup_{n\in\mathbb{N}} B_n\right) = \{1\},$$

whereas $\sup_{n\in\mathbb{N}} \inf \{A_n, B_n\} = \operatorname{cl}\left(\bigcup_{n\in\mathbb{N}} (A_n \cap B_n)\right) = \emptyset$, so (3) does not hold.

3. The Hutchinson–Barnsley operator on $(2^X, \supseteq)$

Throughout this section X is an abstract set, 2^X denotes the family of all subsets of X, and $f, f_1, \dots, f_n$ are selfmaps of X. We consider the partially ordered set $(2^X, \supseteq)$. So for $A, B \subseteq X$, $A \leq B$ means that B is a subset of A. A sequence $(A_n)_{n=1}^\infty$ is $\supseteq$-increasing if it is decreasing in the usual sense; moreover, $\sup_{n\in\mathbb{N}} A_n$ in $(2^X, \supseteq)$ coincides with the intersection $\bigcap_{n\in\mathbb{N}} A_n$.

Proposition 1. *Let $F(A) := f(A)$ for $A \subseteq X$ so that $F : 2^X \mapsto 2^X$. The following conditions are equivalent:*

(i) F is $\supseteq$-continuous;

(ii) given a decreasing sequence $(A_n)_{n=1}^\infty$ of subsets of X,

$$f\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \bigcap_{n\in\mathbb{N}} f(A_n);$$

(iii) all fibres of f are finite.

In particular, (iii) holds if f is injective.

Proof. The equivalence (i)$\Longleftrightarrow$(ii) follows from Remark 1. To prove (ii)$\Longrightarrow$(iii) suppose, on the contrary, that (iii) does not hold. Then there exist a $y \in X$ and a sequence $(x_n)_{n=1}^\infty$ such that $y = f(x_n)$ and $x_n \neq x_m$ if $n \neq m$. Set $A_n := \{x_k : k \geq n\}$ for $n \in \mathbb{N}$. Clearly, $(A_n)_{n=1}^\infty$ is decreasing and $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$. Simultaneously, $f(A_n) = \{y\}$ so that

$$\bigcap_{n\in\mathbb{N}} f(A_n) = \{y\} \neq \emptyset = f\left(\bigcap_{n\in\mathbb{N}} A_n\right),$$

which violates (ii).

To prove (iii)$\Longrightarrow$(ii) assume that a sequence $(A_n)_{n=1}^\infty$ is decreasing. It suffices to show that $\bigcap_{n\in\mathbb{N}} f(A_n) \subseteq f(\bigcap_{n\in\mathbb{N}} A_n)$. Let $y \in \bigcap_{n\in\mathbb{N}} f(A_n)$. Then there is a sequence $(x_n)_{n=1}^\infty$ such that $x_n \in A_n$ and $y = f(x_n)$, that is, the set $\{x_n : n \in \mathbb{N}\}$ is a subset of the fibre $f^{-1}(\{y\})$. Condition (iii) implies that there is an $x \in X$ and a subsequence $(x_{k_n})_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $x_{k_n} = x$. Hence $x \in \bigcap_{n\in\mathbb{N}} A_{k_n}$. Since $(A_n)_{n=1}^\infty$ is decreasing, $\bigcap_{n\in\mathbb{N}} A_{k_n} = \bigcap_{n\in\mathbb{N}} A_n$ so $x \in \bigcap_{n\in\mathbb{N}} A_n$. Moreover, $y = f(x)$ and thus $y \in f(\bigcap_{n\in\mathbb{N}} A_n)$. □

As an application of Proposition 1, Theorem 1 and Lemma 1, we obtain the following result on invariant sets of IFS in the set–theoretical case.

Theorem 2. *Let F be defined by (1). If for $i = 1, \dots, n$ all fibres of the maps f_i are finite, then for each set $A \subseteq X$ such that $F(A) \subseteq A$, the set $\bigcap_{n\in\mathbb{N}} F^n(A)$ is invariant with respect to the* IFS $\{f_1, \dots, f_n\}$. *In particular, the set $\bigcap_{n\in\mathbb{N}} F^n(X)$ is the greatest invariant set with respect to this* IFS. *Hence, the system $\{f_1, \dots, f_n\}$ has a nonempty invariant set if and only if the set $\bigcap_{n\in\mathbb{N}} F^n(X)$ is nonempty.*

Proof. We will apply Theorem 1 for the partially ordered set $(2^X, \supseteq)$ and the operator F. Clearly, $(2^X, \supseteq)$ is a complete lattice. We verify condition (3). Let $(A_n)_{n=1}^{\infty}$ and $(B_n)_{n=1}^{\infty}$ be decreasing sequences of subsets of X. Then (3) is equivalent to the equality

$$\bigcap_{n\in\mathbb{N}} A_n \cup \bigcap_{n\in\mathbb{N}} B_n = \bigcap_{n\in\mathbb{N}} (A_n \cup B_n),$$

which really holds. Let $F_i(A) := f_i(A)$ for $A \subseteq X$ and $i = 1, \dots, n$. By Proposition 1, all the maps F_i are $\supseteq$-continuous. Thus all the assumptions of Theorem 1 are satisfied.

To show that $\bigcap_{n\in\mathbb{N}} F^n(X)$ is the greatest invariant set, observe that if $A_0 = F(A_0)$, then $A_0 = F^n(A_0)$ so that $A_0 = \bigcap_{n\in\mathbb{N}} F^n(A_0)$. Since F is increasing, so are all its iterates F^n and hence, $F^n(A_0) \subseteq F^n(X)$, which implies that $A_0 \subseteq \bigcap_{n\in\mathbb{N}} F^n(X)$. The last statement of Theorem 2 is obvious. □

Let us notice that if X is a finite set, then condition (iii) of Proposition 1 is automatically satisfied so, by Theorem 2, for each map $f : X \mapsto X$ the set $\bigcap_{n\in\mathbb{N}} f^n(X)$ is a modulus set for f. It turns out that this property characterizes finite sets only, according to the following

Proposition 2. *The following conditions are equivalent:*

(*i*) *X is a finite set;*
(*ii*) *for each map $f : X \mapsto X$, the set $\bigcap_{n\in\mathbb{N}} f^n(X)$ is a modulus set for f.*

Proof. The implication (i)$\Longrightarrow$(ii) follows from Theorem 2. To prove (ii)$\Longrightarrow$(i) suppose, on the contrary, that X is infinite. Let X_0 be a countable subset of X. Without loss of generality we may assume that

$$X_0 = \{a, b\} \cup \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} \{a_{nk}\}$$

where elements a, b and a_{nk} are distinct. Set

$$\begin{aligned} &f(x) := b \quad \text{for } x \in (X \setminus X_0) \cup \{a, b\};\\ &f(a_{n1} := a \quad \text{for } n \in \mathbb{N};\\ &f(a_{nk}) := a_{n,k-1} \quad \text{for } n \geq 2 \text{ and } 2 \leq k \leq n. \end{aligned}$$

Then $b = f^n(b)$ and $a = f^n(a_{nn})$ so $\{a, b\} \subseteq \bigcap_{n\in\mathbb{N}} f^n(X)$. On the other hand, it is easily seen that $\bigcap_{n\in\mathbb{N}} f^n(X) \subseteq \{a, b\}$. Therefore, we get

$$f\left(\bigcap_{n\in\mathbb{N}} f^n(X)\right) = f(\{a, b\}) = \{b\} \neq \{a, b\} = \bigcap_{n\in\mathbb{N}} f^n(X),$$

which violates (ii). □

We emphasize that condition (iii) of Proposition 1 is not necessary for the set $\bigcap_{n\in\mathbb{N}} f^n(X)$ to be a modulus set for f. This fact can be deduced from Proposition 3 and Example 2 given below.

Proposition 3. *Let (X,d) be a bounded metric space and $f: X \mapsto X$ be a Banach contraction with a contractive constant $h \in (0,1)$. Then for each set $A \subseteq X$ (not necessarily $f(A) \subseteq A$), $\bigcap_{n\in\mathbb{N}} f^n(A)$ is a modulus set for f.*

Proof. Let $A \subseteq X$. Clearly, if the set $\bigcap_{n\in\mathbb{N}} f^n(A)$ is empty, then it is a modulus set for f. If this set is nonempty, then the diameter, $\delta\left(\bigcap_{n\in\mathbb{N}} f^n(A)\right)$, can be estimated as follows:

$$\delta\left(\bigcap_{n\in\mathbb{N}} f^n(A)\right) \le \delta\left(f^n(A)\right) \le \delta\left(f^n(X)\right) \le h^n\delta(X) \to 0 \quad \text{as } n \to \infty,$$

which implies that $\bigcap_{n\in\mathbb{N}} f^n(A) = \{a\}$ for some $a \in X$. Hence, to prove that $\bigcap_{n\in\mathbb{N}} f^n(A)$ is a modulus set for f, it suffices to show that a is a fixed point of f. Since $a \in f^n(A)$ for $n \in \mathbb{N}$, there is a sequence $(a_n)_{n=1}^{\infty}$ such that $a = f^n(a_n)$. Then

$$d\big(a, f(a)\big) = d\big(f^n(a_n), f^{n+1}(a_n)\big) \le h^n d\big(a_n, f(a_n)\big) \le h^n\delta(X) \to 0,$$

which implies that $a = f(a)$. □

Example 2. Let $X := [-1,1]$, $\alpha \in (0,1/3)$, $f(0) := 0$ and $f(x) := \alpha x^2 \sin(1/x)$ for $x \in X\setminus\{0\}$. Endow X with the euclidean metric. Since $|f'(x)| \le 3\alpha < 1$, f is a Banach contraction, so the assumptions of Proposition 3 are satisfied. On the other hand, Theorem 2 is not applicable here, since the fibre $f^{-1}(\{0\})$ is infinite.

Remark 2. The proof of Theorem 1 (cf. Dugundji–Granas [3, p. 15]) can suggest to introduce the following definition: a selfmap F of a partially ordered set $(P,\le)$ is said to be *iteratively $\le$-continuous* if F is increasing and F preserves a supremum of each increasing sequence $(p_n)_{n=1}^{\infty}$ such that $p_n = F^n(p)$ for some $p \in P$ (compare it with Remark 1). Then Theorem 1 holds for such a class of maps. Moreover, this class is essentially wider than the class of $\le$-continuous maps: the map $F: 2^X \mapsto 2^X$ generated by the map f from Example 2 is iteratively $\le$-continuous by Proposition 3 and is not $\le$-continuous by Proposition 1.

4. The Hutchinson–Barnsley operator on $(C(X), \supseteq)$

Throughout this section X is a Hausdorff topological space and $C(X)$ denotes the family of all nonempty closed subsets of X, endowed with the inclusion $\supseteq$. We start with examining the countable chain condition in this case.

Proposition 4. *The following conditions are equivalent:*

(*i*) *every countable chain in $(C(X), \supseteq)$ has a supremum;*
(*ii*) *for every decreasing sequence $(A_n)_{n=1}^{\infty}$ of nonempty closed subsets of X, the intersection $\bigcap_{n\in\mathbb{N}} A_n$ is nonempty;*
(*iii*) *X is countably compact.*

Proof. (i)$\Longleftrightarrow$(ii) follows from Remark 1. For (ii)$\Longleftrightarrow$(iii), see Engelking [5, Theorem 3.10.2]. □

Recall that a space X is *sequential* if every sequentially closed subset A of X (that is, A contains limits of all convergent sequences of its elements) is closed. In particular, every first–countable space is sequential (cf. Engelking [5, Theorem 1.6.14]). Our next result deals with $\supseteq$-continuity of the Hutchinson–Barnsley operator in such spaces. It is interesting that $\supseteq$-continuity is connected with appropriate properties of fibres of f (similarly, as in the set–theoretical space; cf. Proposition 1 and Theorem 6), which, however, leads directly to continuity with respect to topology, according to the following

Proposition 5. *Let X be a countably compact and sequential space, $f : X \mapsto X$ and $F(A) := f(A)$ for $A \subseteq X$. The following conditions are equivalent:*

(i) $F(C(X)) \subseteq C(X)$ and F is continuous on $C(X)$ with respect to the inclusion $\supseteq$;
(ii) f is continuous on X with respect to the topology.

Proof. This equivalence follows from Remark 1, the fact that for a decreasing sequence $(A_n)_{n=1}^{\infty}$ of sets in $C(X)$, $\sup_{n\in\mathbb{N}} A_n$ in $(C(X), \supseteq)$ coincides with $\bigcap_{n\in\mathbb{N}} A_n$, and Theorem 8 (see Appendix). □

The following example shows that in Proposition 5 we cannot omit the assumption that X is a sequential space. Also observe that there exist countably compact and sequential spaces, which are not compact as, for example, the space W_0 defined below.

Example 3. Let ω_1 denote the smallest uncountable ordinal number, W_0 be the set of all countable ordinal numbers and $W := W_0 \cup \{\omega_1\}$. It is known that W is a compact space (cf. Engelking [5, Example 3.1.27]) and W_0 is countably compact, but not compact (cf. [5, Example 3.10.16]). Moreover, W_0 is a first–countable space, hence sequential. Let $X := W_0 \times W$. Then X is countably compact as the Cartesian product of a countably compact space and a compact space (cf. [5, Corollary 3.10.14]). Define a map f by

$$f(x_1, x_2) := (0, x_2) \qquad \text{for } (x_1, x_2) \in X.$$

Clearly, f is a continuous selfmap of X so (ii) of Proposition 5 holds. Let $A := \{(x_1, x_1) : \ x_1 \in W_0\}$. Since the space W is Hausdorff, A is a closed subset of X. On the other hand $f(A) = \{0\} \times W_0$ so $\mathrm{cl}(f(A)) = \{0\} \times W$. Hence condition (i) of Proposition 5 does not hold: the operator F is not a selfmap of $C(X)$.

As an immediate consequence of Propositions 4 and 5, we obtain the following

Corollary 1. *Let X be a sequential space, f and F be as in Proposition 5. The following conditions are equivalent:*

(i) $(C(X), \supseteq)$ and F satisfy the assumptions of the T–K principle;
(ii) X is countably compact and f is continuous on X.

In view of Corollary 1 the following theorem is the best result on invariant sets with respect to IFS on a sequential Hausdorff space, which can be deduced from the T–K principle for the family $(C(X), \supseteq)$.

Theorem 3. *Let X be a countably compact and sequential space, and $f_1, \cdots, f_n$ be continuous selfmaps of X. Let F be defined by (1) and $A_0 := \bigcap_{n\in\mathbb{N}} F^n(X)$. Then the set A_0 is nonempty and closed, $A_0 = F(A_0)$, and A_0 is the greatest invariant set with respect to the* IFS $\{f_1, \dots, f_n\}$. *Moreover, if X is metrizable, then the sequence $(F^n(X))_{n=1}^{\infty}$ converges to A_0 with respect to the Hausdorff metric.*

Proof. Denote $F_i(A) := f_i(A)$ for $A \in C(X)$ and $i = 1, \dots, n$. By Corollary 1, $(C(x), \supseteq)$ and F_i satisfy the assumptions of Theorem 1. Clearly, for $A \in C(X)$ the set $F(A)$ is closed as a finite union of closed sets. Moreover, condition (3) is satisfied here (cf. the proof of Theorem 2) so, by Lemma 1, F is $\supseteq$-continuous. Thus, by Theorem 1, the set A_0 is invariant with respect to $\{f_1, \dots, f_n\}$. Since $F(X) \subseteq X$ and F is increasing, the sequence $(F^n(X))_{n=1}^{\infty}$ is decreasing. Therefore, if X is metrizable, then $(F^n(X))_{n=1}^{\infty}$ converges to A_0 with respect to the Hausdorff metric as a decreasing sequence of compact sets (cf. Edgar [4, Proposition 2.4.7]). □

We close this section with a result on $\supseteq$-continuity of the operator F defined by (2). It is rather surprising that the $\supseteq$-continuity of such an F forces that F coincides with the operator defined by (1).

Proposition 6. *Let X be a countably compact and sequential space, $f : X \mapsto X$ and $F(A) := \mathrm{cl}(f(A))$ for $A \in C(X)$. The following conditions are equivalent:*

(*i*) *F is continuous on $C(X)$ with respect to the inclusion $\supseteq$;*
(*ii*) *f is continuous on X with respect to the topology.*

Hence, if F is $\supseteq$-continuous, then $F(A) = f(A)$ for $A \in C(X)$.

Proof. By Remark 1, the $\supseteq$-continuity of F on $C(X)$ means that given a decreasing sequence $(A_n)_{n=1}^{\infty}$ of nonempty closed subsets of X,

$$\mathrm{cl}\left(f\left(\bigcap_{n\in\mathbb{N}} A_n\right)\right) = \bigcap_{n\in\mathbb{N}} \mathrm{cl}\left(f(A_n)\right).$$

By Theorem 9 ((i)$\Longleftrightarrow$(ii)), this condition is equivalent to the topological continuity of f. Then, by Theorem 8 ((i)$\Longleftrightarrow$(ii)), for $A \in C(X)$ the image $f(A)$ is closed so $F(A) = f(A)$. □

5. The Hutchinson–Barnsley operator on $(K(X), \supseteq)$

Throughout this section X is (with one exception) a Hausdorff topological space and $K(X)$ denotes the family of all nonempty compact subsets of X, endowed with the inclusion $\supseteq$. Then every countable chain in $(K(X), \supseteq)$ has a supremum. Let F be defined by (1) for $A \in K(X)$. If we are to apply Theorem 1 then, without loss of generality, we may assume that the space X is compact (in particular, countably compact), because the assumption of Theorem 1 "there is an $X_0 \in K(X)$ such that $X_0 \supseteq F(X_0)$" implies that all the maps $f_i\big|_{X_0}$ (the restriction of f_i to X_0) are selfmaps of the same compact set. Thus we arrive at the case considered in the previous section, however, this time we need not assume that a space X is sequential, since each continuous map f on X is closed so it generates the operator F, which is a selfmap of $K(X)$.

Theorem 4. *Let X be a compact space and $f_1, \dots, f_n$ be continuous selfmaps of X. Let F be defined by* (1) *and $A_0 := \bigcap_{n\in\mathbb{N}} F^n(X)$. Then the set A_0 is nonempty and compact, $A_0 = F(A_0)$, and A_0 is the greatest invariant set with respect to the* IFS $\{f_1, \dots, f_n\}$.

Proof. Let $F_i(A) := f_i(A)$ for $A \in K(X)$ and $i = 1, \dots, n$. The $\supseteq$-continuity of F_i follows from Proposition 7 (see Appendix). By Lemma 1, F is $\supseteq$-continuous, so Theorem 1 is applicable. □

Theorem 5. *Let X be a topological space (not necessarily Hausdorff), $f_1, \cdots, f_n$ be continuous selfmaps of X and F be defined by* (1)*. The following conditions are equivalent:*

(*i*) *there exists a nonempty compact set A_0 such that $F(A_0) = A_0$;*
(*ii*) *there exists a nonempty compact set A such that $F(A) \subseteq A$.*

Proof. Obviously, it suffices to show that (ii) implies (i). This follows immediately from Theorem 4 applied to the compact set A and the restrictions $f_i\big|_A$ of the maps f_i to the set A. □

We will demonstrate the utility of Theorem 5 in the theory of IFS. As was mentioned in Section 1, if all the maps f_i are Banach contractions on a complete metric space X, then it can be shown that the operator F is a Banach contraction on $K(X)$ endowed with the Hausdorff metric and, consequently, there is a set $A_0 \in K(X)$ such that $A_0 = F(A_0)$. With a help of Theorem 5 we can give another proof of this fact without using Hausdorff metric. Instead, the contractive condition for f_i enables to show the existence of a nonempty compact set A such that $F(A) \subseteq A$. The only restriction is that we will work in the class of the Heine–Borel metric spaces (cf. Section 1). Nevertheless, this class is large enough for applications since, obviously, the euclidean space $\mathbb{R}^n$ is Heine–Borel. The closed ball around a point $x \in X$ with a radius r is denoted by $B(x, r)$.

Corollary 2. *Let X be a Heine–Borel metric space, $f_1, \dots, f_n$ be Banach's contractions on X with contractive constants $h_1, \dots, h_n$ in $(0, 1)$, and F be defined by* (1)*. Then there exists a nonempty compact set A_0 such that $F(A_0) = A_0$.*

Proof. We use an idea of Williams [14] (also cf. Hayashi [6]). Since a Heine–Borel metric space is complete, each map f_i has a unique fixed point x_i by the Banach contraction principle. Let $A := B(x_1, r)$, a radius r will be specified later. Denote $h := \max\{h_i : \ i = 1, \dots n\}$ and $M := \max\{d(x_i, x_1) : i = 1, \dots, n\}$. If $x \in A$, then by the triangle inequality and the contractive condition

$$\begin{aligned} d(f_i x, x_1) &\le d(f_i x, f_i x_i) + d(x_i, x_1) \le h d(x, x_i) + M \\ &\le h(d(x, x_1) + d(x_1, x_i)) + M \le hr + (1 + h)M. \end{aligned} \tag{4}$$

Now if we set $r := [(1 + h)/(1 - h)]M$, then $hr + (1 + h)M = r$ so, by (4), $f_i(x) \in A$. Since A does not depend on an integer i, we may infer that $F(A) \subseteq A$. Clearly, by the Heine–Borel property, A is compact and the existence of the set A_0 follows from Theorem 5. □

Remark 3. It follows from the above proof and Theorem 3 applied to the IFS $\{f_i\big|_A\colon\ i = 1, \dots, n\}$ that the sequence $(F^n(B(x_1, r)))_{n=1}^{\infty}$ with r defined above is convergent with respect to the Hausdorff metric. We may set

$$A_0 := \bigcap_{n\in\mathbb{N}} F^n(B(x_1, r)),$$

which is the limit of this sequence. Actually, this set is a unique invariant set with respect to $\{f_1, \dots, f_n\}$, but the uniqueness of it follows from the Hutchinson–Barnsley theorem and is not obtainable via the T–K principle.

6. Appendix: continuity of maps on countably compact and sequential spaces

In the proof of Theorem 4 we used the following

Proposition 7. *Let X be a countably compact space, Y be a set and $f : X \mapsto Y$. If all fibres of f are closed, then given a decreasing sequence $(A_n)_{n=1}^{\infty}$ of closed subsets of X,*

$$f\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \bigcap_{n\in\mathbb{N}} f(A_n).$$

Proof. Let $(A_n)_{n=1}^{\infty}$ be a decreasing sequence of closed subsets of X. It suffices to show that $\bigcap_{n\in\mathbb{N}} f(A_n) \subseteq f\left(\bigcap_{n\in\mathbb{N}} A_n\right)$. Let $y \in \bigcap_{n\in\mathbb{N}} f(A_n)$. Then there is a sequence $(a_n)_{n=1}^{\infty}$ such that $y = f(a_n)$ and $a_n \in A_n$. Thus the sets B_n defined by

$$B_n := A_n \cap f^{-1}(\{y\})$$

are nonempty, closed and $B_{n+1} \subseteq B_n$. By the countable compactness of X, there exists an $x \in \bigcap_{n\in\mathbb{N}} B_n$. Then $y = f(x)$ and $x \in \bigcap_{n\in\mathbb{N}} A_n$, which means that $y \in f\left(\bigcap_{n\in\mathbb{N}} A_n\right)$. □

The next result is a partial converse to Proposition 7.

Proposition 8. *Let X be a Hausdorff topological space, Y be a set and $f : X \mapsto Y$. If for every decreasing sequence $(A_n)_{n=1}^{\infty}$ of nonempty compact subsets of X, $f\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \bigcap_{n\in\mathbb{N}} f(A_n)$, then all fibres of f are sequentially closed.*

Proof. Suppose, on the contrary, that there is a $y \in X$ such that the fibre $f^{-1}(\{y\})$ is not sequentially closed. Then there exist an $x \in X$ and a sequence $(x_n)_{n=1}^{\infty}$ such that $f(x_n) = y$ and $f(x) \neq y$. Set $A_n := \{x\} \cup \{x_k :\ k \geq n\}$. Then the sets A_n are compact, since X is Hausdorff, and $A_{n+1} \subseteq A_n$. Clearly, $x \in \bigcap_{n\in\mathbb{N}} A_n$. Suppose that $x' \in \bigcap_{n\in\mathbb{N}} A_n$ and $x' \neq x$. Then there is a subsequence $(x_{k_n})_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $x_{k_n} = x'$. Simultaneously, $(x_{k_n})_{n=1}^{\infty}$ converges to x so $x = x'$ (since, in particular, X is a T_1-space), a contradiction. Therefore $\bigcap_{n\in\mathbb{N}} A_n = \{x\}$ so that

$$f\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \{f(x)\} \neq \{f(x), y\} = \bigcap_{n\in\mathbb{N}} f(A_n),$$

which violates the hypothesis. □

As an immediate consequence of Propositions 7 and 8, we get the following

Theorem 6. *Let X be a countably compact and sequential space, Y be a set and $f : X \mapsto Y$. The following conditions are equivalent:*

(*i*) *all fibres of f are closed;*
(*ii*) *given a decreasing sequence $(A_n)_{n=1}^{\infty}$ of nonempty closed subsets of X, $f\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \bigcap_{n\in\mathbb{N}} f(A_n)$;*
(*iii*) *given a decreasing sequence $(A_n)_{n=1}^{\infty}$ of nonempty compact subsets of X, $f\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \bigcap_{n\in\mathbb{N}} f(A_n)$.*

Proof. The implication (i)$\Longrightarrow$(ii) follows from Proposition 7, (ii)$\Longrightarrow$(iii) is obvious and (iii)$\Longrightarrow$(i) follows from Proposition 8. □

Remark 4. Observe that under the assumptions of Theorem 6, the classes $C(X)$ and $K(X)$ need not coincide, so the equivalence (ii)$\Longleftrightarrow$(iii) is not trivial. For example, define X as the set of all countable ordinal numbers; then $X \in C(X) \setminus K(X)$ (cf. Example 3).

In the sequel we will need the following lemma (cf. Engelking [5, Proposition 1.6.15]).

Lemma 2. *Let X be a sequential space, Y be a topological space and $f : X \mapsto Y$. Then f is continuous if and only if f is sequentially continuous, that is, given a sequence $(x_n)_{n=1}^{\infty}$ in X,*

$$f(\lim x_n) \subseteq \lim f(x_n).$$

Proposition 9. *Let X be a topological space, Y be a countably compact and sequential space and $f : X \mapsto Y$. Then f is sequentially continuous if and only if the graph of f is sequentially closed in the Cartesian product $X \times Y$.*

Proof. ($\Longrightarrow$). Let a sequence $(x_n, f(x_n))_{n=1}^{\infty}$ converge to (x, y) in $X \times Y$. Then $x \in \lim x_n$ and $\{y\} = \lim f(x_n)$ since Y is Hausdorff. By hypothesis,

$$f(x) \in f(\lim x_n) \subseteq \lim f(x_n) = \{y\},$$

which means that $f(x) = y$. Thus the graph of f is sequentially closed.

($\Longleftarrow$). Suppose, on the contrary, that f is not sequentially continuous. Then there exist a sequence $(x_n)_{n=1}^{\infty}$ and an $x \in X$ such that $x \in \lim x_n$ and $f(x) \notin \lim f(x_n)$. Without loss of generality, we may assume, by passing to a subsequence if necessary, that there is a neighborhood V of $f(x)$ such that $f(x_n) \notin V$ for all $n \in \mathbb{N}$. Since Y is also sequentially compact (cf. Engelking [5, Theorem 3.10.31]), there is a convergent subsequence $(f(x_{k_n}))_{n=1}^{\infty}$ of $(f(x_n))_{n=1}^{\infty}$. Set $y := \lim f(x_{k_n})$ (this limit is unique since Y is Hausdorff). Since $x \in \lim x_{k_n}$ and the graph of f is sequentially closed, we infer that $y = f(x)$, that is, $(f(x_{k_n}))_{n=1}^{\infty}$ converges to $f(x)$. This yields a contradiction, since $f(x_{k_n}) \notin V$ and $f(x) \in V$. □

The next result is a closed graph theorem for maps on sequential spaces.

Theorem 7. *Let X and Y be sequential spaces and Y be countably compact. For a map $f : X \mapsto Y$ the following conditions are equivalent:*

(*i*) *f is continuous;*
(*ii*) *the graph of f is closed in $X \times Y$;*
(*iii*) *the graph of f is sequentially closed in $X \times Y$;*
(*iv*) *f is sequentially continuous.*

Proof. That (i) implies (ii) follows from Engelking [5, Corollary 2.3.22]. (ii)⇒(iii) is obvious. (iii)⇒(iv) follows from Proposition 9 and finally, (iv)⇒(i) holds by Lemma 2. □

The main result of this section is the following theorem, which gives a characterization of continuity of maps on countably compact and sequential spaces. This result was obtained as a by–product of our study of continuity of the Hutchinson–Barnsley operator with respect to the inclusion $\supseteq$ (cf. Proposition 5).

Theorem 8. *Let X and Y be countably compact and sequential spaces. For a map $f: X \mapsto Y$ the following conditions are equivalent:*

(*i*) *f is continuous;*
(*ii*) *for every closed subset A of X, the image $f(A)$ is closed, and all fibres of f are closed;*
(*iii*) *for every closed subset A of X, the image $f(A)$ is closed, and given a decreasing sequence $(A_n)_{n=1}^{\infty}$ of nonempty closed subsets of X, $f\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \bigcap_{n\in\mathbb{N}} f(A_n)$;*
(*iv*) *for every compact subset A of X, the image $f(A)$ is compact, and given a decreasing sequence $(A_n)_{n=1}^{\infty}$ of nonempty compact subsets of X, $f\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \bigcap_{n\in\mathbb{N}} f(A_n)$.*

Proof. (i)$\Longrightarrow$(ii). Let A be a closed subset of X. Since X is sequentially compact (cf. Engelking [5, Theorem 3.10.31]), so is A (cf. [5, Theorem 3.10.33]). Hence and by continuity of f, the image $f(A)$ is sequentially compact (cf. [5, Theorem 3.10.32]). In particular, $f(A)$ is sequentially closed, hence closed since Y is sequential. Since, in particular, Y is a T_1-space it is clear that the fibres of f are closed.

(ii)$\Longrightarrow$(iii) follows immediately from Theorem 6.

We give a common proof of the implications (iii)$\Longrightarrow$(i) and (iv)$\Longrightarrow$(i). By Theorem 7, it suffices to show that the graph of f is sequentially closed. Let a sequence $(x_n, f(x_n))_{n=1}^{\infty}$ converge to (x, y) in $X \times Y$. Since both X and Y are Hausdorff, we may infer that $x = \lim x_n$ and $y = \lim f(x_n)$. Set $A_n := \{x\} \cup \{x_k : \ k \geq n\}$ for $n \in \mathbb{N}$. The the sets A_n are compact (hence closed), $A_{n+1} \subseteq A_n$ and $\bigcap_{n\in\mathbb{N}} A_n = \{x\}$. By hypothesis, $\bigcap_{n\in\mathbb{N}} f(A_n) = f\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \{f(x)\}$. Since both (iii) and (iv) imply that the set $f(A_n)$ is closed and $f(x_k) \in f(A_n)$ for $k \geq n$, we may infer that $y = \lim_{k\to\infty} f(x_k) \in f(A_n)$ so that $y \in \bigcap_{n\in\mathbb{N}} f(A_n) = \{f(x)\}$, that is, $y = f(x)$. This proves that the graph of f is sequentially closed.

We have shown that conditions (i), (ii) and (iii) are equivalent, and that (iv) implies (i). To finish the proof it suffices to show that (iii) implies (iv). Since (iii) implies the continuity of f, the first part of (iv) holds. The second part of (iv) follows immediately from (iii). □

Our last theorem gives another characterization of continuity. This result was obtained as a by product of our study of $\supseteq$-continuity of operator F defined by Lasota and Myjak [10] (cf. Proposition 6).

Theorem 9. *Let X and Y be countably compact and sequential spaces. For a map $f: X \mapsto Y$ the following conditions are equivalent:*

(*i*) *f is continuous;*

(*ii*) *given a decreasing sequence* $(A_n)_{n=1}^{\infty}$ *of nonempty closed subsets of* X, $\mathrm{cl}\left(f(\bigcap_{n\in\mathbb{N}} A_n)\right) = \bigcap_{n\in\mathbb{N}} \mathrm{cl}(f(A_n))$;

(*iii*) *given a decreasing sequence* $(A_n)_{n=1}^{\infty}$ *of nonempty compact subsets of* X, $\mathrm{cl}\left(f(\bigcap_{n\in\mathbb{N}} A_n)\right) = \bigcap_{n\in\mathbb{N}} \mathrm{cl}(f(A_n))$.

Proof. (i)$\Longrightarrow$(ii). Let $(A_n)_{n=1}^{\infty}$ be a decreasing sequence of nonempty closed subsets of X. Since the intersection $\bigcap_{n\in\mathbb{N}} A_n$ is closed, we may conclude by Theorem 8 ((i)$\Longrightarrow$(ii)) that all the sets $f(\bigcap_{n\in\mathbb{N}} A_n)$ and $f(A_n)$ $(n \in \mathbb{N})$ are closed. Therefore, (ii) follows immediately from condition (iii) of Theorem 8.

(ii)$\Longrightarrow$(iii) is obvious.

(iii)$\Longrightarrow$(i). By Theorem 7, it suffices to show that the graph of f is sequentially closed. We use the same argument as in the proof of (iv)$\Longrightarrow$(i) in Theorem 8. So let $x = \lim x_n$ and $y = \lim f(x_n)$. Set $A_n := \{x\} \cup \{x_k : k \geq n\}$. By (iii),

$$\bigcap_{n\in\mathbb{N}} \mathrm{cl}(f(A_n)) = \mathrm{cl}\left(f\left(\bigcap_{n\in\mathbb{N}} A_n\right)\right) = \mathrm{cl}\left(\{f(x)\}\right) = \{f(x)\}.$$

Since $y \in \mathrm{cl}(f(A_n))$ for all $n \in \mathbb{N}$, we may infer that $y = f(x)$, which proves that the graph of f is sequentially closed. □

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