Abstract. We show that the Hilbert space is coarsely embeddable into any $\ell_p$ for $1 \leq p \leq \infty$. It follows that coarse embeddability into $\ell_2$ and into $\ell_p$ are equivalent for $1 \leq p < 2$.

Coarse embeddings were defined by M. Gromov [Gr, 7.E2] to express the idea of inclusion in the large scale geometry of groups. G. Yu showed later that the case when a finitely generated group with a word length metric is being embedded into the Hilbert space is of great importance in solving the Novikov Conjecture [Yu], while recent work of G. Kasparov and G. Yu [KY] treats the case when the Hilbert space is replaced with just a uniformly convex Banach space. Due to these remarkable theorems coarse embeddings gain a great deal of attention, but still embeddability into the Hilbert, and more generally Banach spaces, is not entirely understood with many question remaining open.

In this context the class of $\ell_p$-spaces seems to be particularly interesting. Their embeddability into the Hilbert space is known - $\ell_p$ admits such an embedding when $0 < p \leq 2$ but do not if $p > 2$ due to a recent result of W. Johnson and N. Randrianarivony [JR]. In this note we study the opposite situation, i.e. we show that the separable Hilbert space embeds into $\ell_p$ for any $1 \leq p \leq \infty$. As a consequence we obtain a new characterization of embeddability into $\ell_2$, namely that the properties of embeddability into $\ell_p$ for $1 \leq p \leq 2$ are all equivalent.

In [GK, Section 6] the authors advertised a conjecture stated by A.N. Dranishnikov [Dr, Conjecture 4.4]: a discrete metric space has Property A if and only if it admits a coarse embedding into the space $\ell_1$. The results presented in this note show, that this is the same as asking whether Property A is equivalent to embeddability into the Hilbert space, and although it is a folk conjecture that such statement is not true, no example distinguishing between the two is known.
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$L_p$-spaces and the Mazur Map

In everything what follows we consider only separable $L_p(\mu)$-spaces and we will specialize to the most interesting case of the spaces $\ell_p$, the case for $L_p(\mu)$ for other, including non-separable, measures follows easily and is left to the reader. We use the standard notation $\ell_p = \ell_p(\mathbb{N})$ and by $S(X)$ we denote the unit sphere in the Banach space $X$.

The Mazur map $M_{p,q} : S(\ell_p) \to S(\ell_q)$ is defined by the formula

$$M_{p,q}(x) = \left\{ |x_i|^p \, \text{sign} \, x_i \right\}_{i=1}^\infty$$

where $x = \{x_i\}_{i=1}^\infty \in \ell_p$. It is a uniform homeomorphism between unit spheres of $\ell_p$-spaces. More precisely, for some $C$ depending only on $\frac{p}{q}$ it satisfies the inequalities:

$$\frac{p}{q} \|x - y\|_p \leq \|M_{p,q}(x) - M_{p,q}(y)\|_q \leq C \|x - y\|^{p/q}_p$$

for all $x, y \in S(\ell_p)$ and $p < q$, and the opposite inequalities if $p > q$ (note that $M_{p,q} = M_{q,p}^{-1}$). For the proof of these estimates and details on the Mazur map and its applications we refer the reader to [BL, Chapter 9.1].

If $\{X_n\}_{n \in \mathbb{N}}$ is a sequence of Banach spaces, we denote by $\left(\sum X_n\right)_p$ the direct sum of $X_n$ with the $p$-norm, i.e.

$$\left(\sum_{n=1}^\infty X_n\right)_p = \left\{ x = \{x_n\}_{n \in \mathbb{N}} : x_n \in X_n, \sum_{n=1}^\infty \|x_n\|^p < \infty \right\},$$

$$\|x\|_p = \left(\sum_{n=1}^\infty \|x_n\|^p\right)^{\frac{1}{p}}.$$

Clearly, $\ell_p$ is isometric to $\left(\sum\ell_p\right)_p$.

We will also need the following classification of separable $L_p$-spaces.

Theorem 1 (see e.g. [Wo, III.A]). A separable space $L_p(\mu)$ is isometric to one of the following spaces: $\ell_p^n$ for $n = 1, 2, 3, \ldots$, $L_p[0, 1]$, $\ell_p$, $(L_p[0, 1] \oplus \ell_p^n)_p$ for $n = 1, 2, 3, \ldots$, $(L_p[0, 1] \oplus \ell_p)_p$. 
A CONDITION FOR COARSE EMBEDDABILITY

We recall the definition of a coarse embedding.

**Definition 1.** Let $X, Y$ be metric spaces. A map $f : X \to Y$ is a coarse embedding if there exist non-decreasing functions $\rho_1, \rho_2 : [0, \infty) \to [0, \infty)$ satisfying

1. $\rho_1(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho_2(d_X(x, y))$ for all $x, y \in X$,
2. $\lim_{t \to \infty} \rho_1(t) = +\infty$.

In [DG] M. Dadarlat and E. Guentner characterized spaces coarsely embeddable into the Hilbert $H$ space in terms of existence of maps into the unit sphere $S(H)$. Their result is a reminiscence of a characterization of uniform embeddability (meaning an existence of a uniform homeomorphism onto a subset) into a Hilbert space obtained by Aharoni et al in [AMM].

**Theorem 2 ([DG, Theorem 2.1]).** A metric space $X$ admits a coarse embedding into the Hilbert space $H$ if and only if for every $R > 0$ and $\varepsilon > 0$ there is a map $\phi : X \to S(H)$ and $S > 0$ satisfying

1. $\sup \{ \| \phi(x) - \phi(y) \|_H : x, y \in X, d(x, y) \leq R \} \leq \varepsilon$,
2. $\lim_{S \to \infty} \inf \{ \| \phi(x) - \phi(y) \|_H : x, y \in X, d(x, y) \geq S \} = \sqrt{2}$

We are going to use this idea to prove a similar condition for embeddings into the spaces $\ell_p$. The proof relies on the original proof of Theorem 2.

**Theorem 3.** Let $X$ be a metric space and $1 \leq p < \infty$. If there is a $\delta > 0$ such that for every $R > 0$, $\varepsilon > 0$ there is a map $\phi : X \to S(\ell_p)$ satisfying

1. $\sup \{ \| \phi(x) - \phi(y) \|_p : x, y \in X, d(x, y) \leq R \} \leq \varepsilon$,
2. $\lim_{S \to \infty} \inf \{ \| \phi(x) - \phi(y) \|_p : x, y \in X, d(x, y) \geq S \} \geq \delta$,

then $X$ admits a coarse embedding into $\ell_p$.

**Proof.** By the assumptions for every $n \in \mathbb{N}$ there is a map $\phi_n : X \to S(\ell_p)$ and a number $S_n > 0$ such that $\| \phi_n(x) - \phi_n(y) \|_p \leq \frac{1}{2^n}$ whenever $d(x, y) \leq n$ and $\| \phi_n(x) - \phi_n(y) \|_p \geq \frac{\delta}{2^n}$ whenever $d(x, y) \geq S_n$. Without loss of generality we can choose the sequence of $S_n$’s to be strictly increasing and tending to infinity as $n \to \infty$.

Choose $x_0 \in X$ and define a map $\Phi : X \to \bigoplus_{n=1}^{\infty} \ell_p$ by the formula

$$ \Phi(x) = \bigoplus_{n=1}^{\infty} (\phi_n(x) - \phi_n(x_0)) $$

It is easy to see that

$$ \| \Phi(x) \|_p = \sum_{n=1}^{\infty} \| \phi_n(x) - \phi_n(x_0) \|_p < \infty $$

which shows that $\Phi$ is well-defined.
We will show that \( \Phi \) is a coarse embedding. Take \( k \in \mathbb{N} \) and \( \sqrt[k]{k-1} \leq d(x, y) < \sqrt[k]{k} \). Then
\[
\|\Phi(x) - \Phi(y)\|_p^p = \sum_{n=1}^{k-1} \|\varphi_n(x) - \varphi_n(y)\|_p^p + \sum_{n=k}^{\infty} \|\varphi_n(x) - \varphi_n(y)\|_p^p
\leq 2^p(k-1) + \sum_{n=k}^{\infty} \frac{1}{2^kp} \leq 2^p(k-1) + 1 \leq 2^p d(x, y)^p + 1
\]
The first estimate comes from the fact that unit vectors cannot be more than distance 2 apart.
On the other hand for \( S_{k-1} \leq d(x, y) < S_k \) we have
\[
\|\Phi(x) - \Phi(y)\|_p^p \geq \sum_{n=1}^{k-1} \|\varphi_n(x) - \varphi_n(y)\|_p^p \geq (k-1) \left( \frac{\delta}{2} \right)^p
\]
Thus we can choose \( \rho_1(t) = \sum_{n=1}^{\infty} \delta \sqrt{n} \chi_{[s_{n-1}, s_n]}(t) \), \( \rho_2(t) = 2t + 1 \) and it is clear that \( \Phi \) is a coarse embedding.

G. Yu defined Property A \([Yu]\), which gives a sufficient condition for embeddability of a discrete metric space into a Hilbert space. We recall a characterization of Property A given by J.L. Tu.

**Proposition 1** ([Tu]). A metric space \( X \) has property A if and only if for every \( R > 0 \) and \( \varepsilon > 0 \) there is a map \( \eta : X \to S(\ell_2(X)) \) and \( S > 0 \) such that
\[
\begin{align*}
(1) & \quad \|\eta(x) - \eta(y)\|_2 \leq \varepsilon \text{ when } d(x, y) \leq R; \\
(2) & \quad \text{supp } \eta(x) \subset B(x, S) \text{ for all } x \in X.
\end{align*}
\]

Theorem 2 and the above characterization exhibit the subtle relation between Property A and coarse embeddability.

The following proposition shows that the property of Theorem 3 is not sensitive to changing the index \( p \).

**Proposition 2.** Let \( X \) have the property described in Theorem 3 with respect to some \( 1 \leq p < \infty \). Then \( X \) has the same property with respect to any \( 1 \leq q < \infty \).

**Proof.** For \( R > 0 \) and \( \varepsilon > 0 \) given a map \( f_p : X \to S(\ell_p) \) which satisfies conditions (1) and (2) of Theorem 3 define \( f_q : X \to S(\ell_q) \) by the formula
\[
f_q(x) = M_{p,q}[f_p(x)],
\]
where \( M_{p,q} : S(\ell_p) \to S(\ell_q) \) is the Mazur map.
If $p < q$, by inequalities (1) we have
\[ \frac{p}{q} \| f_p(x) - f_p(y) \|_p \leq \| f_q(x) - f_q(y) \|_q \leq C \| f_p(x) - f_p(y) \|_{p/q}. \]
Consequently
\[ \sup \{ \| f_q(x) - f_q(y) \|_q : x, y \in X, \; d(x, y) \leq R \} \leq C \varepsilon_{p/q}, \]
and
\[ \lim_{S \to \infty} \inf \{ \| f_q(x) - f_q(y) \|_q : x, y \in X, \; d(x, y) \geq S \} \geq \frac{p}{q} \delta. \]
The case $p > q$ is proved similarly. \hfill \Box

In the case of Property A a statement similar to Proposition 2 was studied by Dranishnikov under the name of Property $A_p$ in [Dr].

**Corollary 4.** If $X$ admits a coarse embedding into $\ell_2$ then it admits a coarse embedding into any $\ell_p$ with $1 \leq p \leq \infty$. In particular, the separable Hilbert space embeds into all $\ell_p$.

**Proof.** If $X$ admits a coarse embedding into $\ell_2$ then, by Theorem 2, $X$ has the property from Theorem 3 for $\ell_2$. By Proposition 2 has this property also for $\ell_p$, $1 \leq p < \infty$ and by Theorem 3 admits an embedding into $\ell_p$.

The case $p = \infty$ is clear since $\ell_\infty$ is a universal space for isometric embeddings. \hfill \Box

It follows from the above proof that Theorem 3 cannot be extended to a characterization of coarse embeddability into $\ell_p$ if $p > 2$. Indeed, in that case the procedure described in the above proof would imply that $\ell_p$ for $p > 2$ embeds coarsely into the Hilbert space, which is not the case by a result of Johnson and Randrianarivony [JR].

In [No] it was shown that $L_p(\mu)$ for $1 \leq p \leq 2$ admit a coarse embedding into the Hilbert space and that coarse embeddability into $\ell_2$ is equivalent to coarse embeddability into $L_p[0,1]$ again for $1 \leq p \leq 2$. This allows us to state

**Theorem 5.** Let $X$ be a separable metric space. Then the following conditions are equivalent:

1. $X$ admits a coarse embedding into the Hilbert space;
2. $X$ admits a coarse embedding into $\ell_p$ for some (equivalently all) $1 \leq p < 2$;
3. $X$ admits a coarse embedding into $L_p[0,1]$ for some (equivalently all) $1 \leq p < 2$.

Note that this covers all separable $L_p(\mu)$-spaces with $1 \leq p \leq 2$. This is particularly interesting since the spaces $L_p$ for different $p$’s are not coarsely equivalent. To see this assume they are and take $f : L_p(\mu) \to L_q(\mu)$ to be
the coarse equivalence. Since $L_p$-spaces are geodesic, $f$ is in fact a quasi-isometry and it induces a Lipschitz equivalence on their ultrapowers. By a theorem of Heinrich [He] ultrapowers of $L_p$ spaces are again $L_p$ spaces (possibly on a different measure), and the assertion follows from a classical fact that Lipschitz equivalence on $L_p$-spaces induces a linear isomorphism.

References


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