UNIVERSALLY MEAGER SETS, II

PIOTR ZAKRZEWSKI

Abstract. We present new characterizations of universally meager sets, shown in [11] to be a category analog of universally null sets. In particular, we address the question of how this class is related to another class of universally meager sets, recently introduced by Todorcevic [10].

1. Introduction

In this note we continue the study of universally meager sets undertaken in [11]. Suppose that $A$ is a subset of a perfect (i.e. with no isolated points) Polish (i.e. separable, completely metrizable) topological space $X$. We say that $A$ is universally meager ($A \in \text{UM}$, see [11], [1] and [2]), if for every Borel isomorphism between $X$ and the Cantor space $2^\omega$ (or equivalently, a perfect Polish space $Y$ – see [11]) the image of $A$ is meager in $2^\omega$ (or in $Y$, respectively) (this class of sets was earlier introduced and studied by Grzegorek [3], [4], [5] under the name of absolutely of the first category sets). We say that $A$ is perfectly meager ($A \in \text{PM}$), if for all perfect subsets $P$ of $X$, the set $A \cap P$ is meager relative to the topology of $P$. Both UM and PM may be seen as category analogs of the class of universally null sets, i.e., such sets $A \subseteq X$ that for every Borel isomorphism between $X$ and $2^\omega$ the image of $A$ is null in $2^\omega$, though by the results of [11] it is perhaps more accurate to view universally meager sets in this role. Clearly, $\text{UM} \subseteq \text{PM}$ and under the Continuum Hypothesis or Martin's Axiom the inclusion is proper. On the other hand various examples of uncountable perfectly meager sets that can be constructed in ZFC (see [7]) turn out to be universally meager and in fact Bartoszyński [1] proved that it is consistent with ZFC that $\text{UM} = \text{PM}$.

Recently, Todorcevic [10] defined another notion of universally meager sets in a much broader setup. Recall that a topological space $Y$ is a Baire space if every nonempty open subset of $Y$ is non-meager in $Y$ (see [6, 8.B]). Recall also that a function $f$ defined on a topological space $Y$ is nowhere constant if it not constant on any nonempty open subset of $Y$. Let us say that a subset $A$ of a topological space
X is universally meager in the sense of Todorcevic if for every Baire space $Y$ and continuous nowhere constant map $f : Y \to X$ the preimage $f^{-1}[A]$ is meager in $Y$. Todorcevic makes the remark that this is a smallness property very much reminiscent of the notion of a perfectly meager set. Actually we will establish the following connection with universally meager sets (note, however, that under a large cardinal assumption no uncountable set of reals is universally meager in the sense of Todorcevic – see [10]):

**Theorem 1.1.** For a subset $A$ of a perfect Polish space $X$, the following are equivalent:

1. $A \in \text{UM}$.
2. For every second countable space $Y$ and continuous nowhere constant map $f : Y \to X$ the preimage $f^{-1}[A]$ is meager in $Y$.
3. For every Polish space $Y$ and continuous nowhere constant map $f : Y \to X$ the preimage $f^{-1}[A]$ is meager in $Y$.

Another smallness property has recently been studied by Nowik and Reardon [8]. Let $H_B[U|BP]$ be the collection of all sets $A \subseteq X$ with the property that for every subset $B$ of $A$ and Borel map $f : 2^\omega \to X$ the preimage $f^{-1}[B]$ has the Baire property (BP) in $2^\omega$. A slight modification of the proof of 1.1 gives the following answer to a question from [8].

**Theorem 1.2.** For a subset $A$ of a perfect Polish space $X$, the following are equivalent:

1. $A \in \text{UM}$.
2. For every second countable Baire space $Y$, subset $B$ of $A$ and Borel map $f : Y \to X$ the preimage $f^{-1}[B]$ has the Baire property in $Y$.
3. For every Polish space $Y$, subset $B$ of $A$ and Borel map $f : Y \to X$ the preimage $f^{-1}[B]$ has the Baire property in $Y$.

As a matter of fact, the conditions formulated in Theorems 1.1 and 1.2 are easily seen to be sufficient for $A \in \text{UM}$ and the more difficult part is to prove that they are also necessary. On the other hand, a considerable weakening of these conditions still suffices. Recall that the Baire space $\omega^\omega$ is the unique, up to homeomorphism, nonempty Polish zero-dimensional space for which all compact subsets have empty interior (see [6, 7, 7]). Our main result is:

**Theorem 1.3.** For a subset $A$ of a perfect Polish space $X$, the following are equivalent:

1. $A \in \text{UM}$.
2. For every continuous bijection $f : \omega^\omega \to X$ the preimage $f^{-1}[A]$ is meager in $\omega^\omega$.
(3) For every subset $B$ of $A$ and continuous bijection $f : \omega^\omega \to X$ the preimage $f^{-1}[B]$ has the Baire property in $\omega^\omega$.

In the rest of the paper we always assume that $A$ is a subset of a perfect Polish space $X$. We will sometimes use the notation $\langle X, \tau \rangle$ to indicate which topology is being considered at the moment. The relative topology of a subspace $Z \subseteq X$ will be denoted $\tau|Z$. The $\sigma$-algebra of all Borel subsets of a topological space $Y$ will be denoted by $\mathcal{B}(Y)$ (or $\mathcal{B}(\tau)$, where $\tau$ is the topology of $Y$, if there is a need to be more specific). The collection of all meager subsets of $Y$ will be denoted by $\mathcal{M}(Y)$ (or $\mathcal{M}(Y, \tau)$, if needed).

2. Characterizations of UM

The proofs of Theorems 1.1 and 1.2 are based on the following lemma.

Lemma 2.1. Suppose that $f : Y \to X$ is Borel map defined on a second countable Baire space $Y$. If $A \in \text{UM}$ and the fibers $f^{-1}([x])$ of all $x \in A$ are meager in $Y$, then the preimage of $A$ is meager in $Y$.

Proof. This is a refinement of the proof of [11, Theorem 2.1, (iii) $\Rightarrow$ (i)].

Assume that $A \in \text{UM}$ and let $Z = f^{-1}[A]$. We are going to prove that $Z \in \mathcal{M}(Y)$.

Let us first convince ourselves that with no loss of generality we may assume that $Y$ is dense in itself. To see this, let $P = \overline{Z}$. If $P \in \mathcal{M}(Y)$, then we are done. Otherwise, let $U = \text{Int}(P)$ and note that $U$ is a nonempty open set hence a Baire subspace of $Y$. Since $Z \setminus U \subseteq P \setminus U \in \mathcal{M}(Y)$, it suffices to prove that $Z \cap U \in \mathcal{M}(Y)$. However, since $U$ is open, a subset of $U$ is meager in $Y$ if and only if it is meager in $U$. In particular, $f^{-1}([x]) \cap U \in \mathcal{M}(U)$ for all $x \in A$ and it is enough to prove that $Z \cap U \in \mathcal{M}(U)$. Finally, note that $U$ has no isolated points. Indeed,

$z \in f^{-1}([f(z)]) \in M(Y) \text{ for all } z \in Z,$

which shows that points of $Z$ are not isolated and it follows that the same is true for points of $U = \text{Int}(Z)$. This means that by replacing $Y$ with $U$ and $f$ with $f|U$ we reduce the problem to the “dense-in-itself” case.

Suppose now that $Z \notin \mathcal{M}(Y)$. Define a $\sigma$-ideal $\mathcal{I}$ in $\mathcal{B}(A)$ by letting

$B \in \mathcal{I} \iff f^{-1}[B] \in \mathcal{M}(Y), \text{ for } B \in \mathcal{B}(A).$

Note that $\mathcal{I}$ is indeed a $\sigma$-ideal in $\mathcal{B}(A)$, since $A \notin \mathcal{I}$ and the fact that $f^{-1}([x]) \in \mathcal{M}(Y)$ for all $x \in A$ implies that $\mathcal{I}$ contains all singletons. Let $\mathcal{J} = \mathcal{M}(Y) \cap \mathcal{B}(Z)$; $\mathcal{J}$ is a $\sigma$-ideal in $\mathcal{B}(Z)$ (note again that $Z \notin \mathcal{J}$ and it follows from $(\ast)$ that $\mathcal{J}$ contains all singletons). Now the function $f$ induces a complete embedding of the Boolean algebra $\mathcal{B}(A)/\mathcal{I}$ into the algebra $\mathcal{B}(Z)/\mathcal{J}$. We claim that $\mathcal{B}(Z)/\mathcal{J} \cong \mathbb{C}$, the unique, up to an isomorphism, complete, atomless Boolean algebra with a countable
dense subset. The point is that the assumption that $Y$ is a second countable Baire space without isolated points is enough to conclude (see [12, Lemma 2.2]), that:

- $\mathbf{B}(Y)/(\mathcal{M}(Y) \cap \mathbf{B}(Y)) \cong \mathbf{C},$
- $\mathbf{B}(Z)/\mathcal{I} \cong \mathbf{B}(Y)/(\mathcal{M}(Y) \cap \mathbf{B}(Y)).$

It follows that the algebra $\mathbf{B}(A)/\mathcal{I}$ is also isomorphic to $\mathbf{C}$ being (isomorphic to) its complete subalgebra with no atoms. This, however, contradicts [11, Theorem 2.1 (vi)].

\qed

Proof of theorem 1.1. First assume that $A \in \text{UM}$ and let $f : Y \to X$ be a continuous nowhere constant map defined on a second countable Baire space $Y$. Note that since $Y$ is a Baire space and $f$ is continuous the condition that $f$ is nowhere constant simply means that its fibres at all points of $X$ are meager in $Y$. Then by lemma 2.1 the preimage $f^{-1}[A]$ is meager in $Y$.

Next assume that $A \not\in \text{UM}$. Then by [11, Theorem 2.1 (i)] there is a Borel one-to-one function $f : Z \to A$ defined on a set $Z \not\in \mathcal{M}(X)$; by dropping countably many points, if necessary, we may assume that $Z$ is dense in itself. By Kuratowski's theorem (see [6, 3, 8]) extend $f$ to a continuous function $g : Y \to X$ defined on a $G_\delta$ subset of $X$ such that $A \subseteq Y \subseteq \overline{G}$. Then $Y$ is a Polish space in the relative topology, $g$ is nowhere constant on $Y$ and the preimage $f^{-1}[A]$ is not meager in $Y$ (we even have $f^{-1}[A] \not\in \mathcal{M}(X)$ since $Z \subseteq f^{-1}[A]$).

\qed

Proof of theorem 1.2. First assume that $A \in \text{UM}$ and let $f : Y \to X$ be a Borel map defined on a second countable space $Y$; it is enough to prove that the set $f^{-1}[A]$ has the Baire property in $Y$. Let $A_1 = \{x \in X : f^{-1}[x] \not\in \mathcal{M}(Y)\}$ and $A_2 = A \setminus A_1$. Notice that $A_1$ is countable, so $f^{-1}[A_1]$ is Borel. On the other hand, the fibres of $f$ at all points of $A_2$ are meager in $Y$, so by lemma 2.1 we have $f^{-1}[A_2] \in \mathcal{M}(Y)$. I follows that the set $f^{-1}[A] = f^{-1}[A_1] \cup f^{-1}[A_2]$ has the BP in $Y$. This proves that (1) $\Rightarrow$ (2).

Implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are obvious.

So finally assume that $A \in H_{\mathbf{B}}[U[\text{BP}]]$ and let $g : 2^\omega \to X$ be a Borel isomorphism between the Cantor space and $X$. By (4), the preimage $f^{-1}[B]$ has the BP in $2^\omega$ for every $B \subseteq A$. But this, $g$ being a bijection, means that $f^{-1}[A]$ has the BP hereditarily, which in turn is equivalent to the fact that $f^{-1}[A] \in \mathcal{M}(2^\omega)$.

\qed

As another corollary of 2.1 we have the following “Reclaw style” (see [9]) characterization of $\text{UM}$.

Proposition 2.2. For a subset $A$ of a perfect Polish space $X$, the following are equivalent:

- $A \in \text{UM}$
- $X \setminus A$ is perfectly meager in $X$.
- There is a Borel $g : X \to 2^\omega$ such that $g^{-1}[A]$ has the BP in $2^\omega$. 

\qed
(1) $A \in \text{UM}$.

(2) For every Polish space $Y$ and a Borel set $B \subseteq X \times Y$ with every section $B^x$ countable, if for each $x \in A$ the section $B_x$ is meager in $Y$, then the union $\bigcup_{x \in A} B_x$ is meager in $Y$.

Proof. Use the fact that by the Lusin-Novikov theorem (see [6, 18.10]) $B$ can be written as the countable union of graphs of Borel functions $f_n : \text{proj}_Y(B) \to X$. 

The proof of Theorem 1.3 is based on the following lemma.

**Lemma 2.3.** Let $\tau$ be the topology of $X$ (making it a perfect Polish space) and assume that $\overline{\nu} \supseteq \tau$ is a Polish topology on $X$.

If $C \subseteq X$ is countable, then there exists a countable disjoint collection $\mathcal{P}$ of subsets of $X$ such that:

1. each $Q \in \mathcal{P}$ is a perfect subset of the space $\langle X, \tau \rangle$,
2. each $Q \in \mathcal{P}$ is a $F_{\sigma}$ nowhere dense subset of the space $\langle X, \overline{\nu} \rangle$,
3. $C \subseteq \bigcup \mathcal{P}$.

Proof. Let $C = \{c_n : n \in \mathbb{N}\}$. We are going to define inductively a sequence $\langle Q_n : n \in \mathbb{N} \rangle$ of subsets of $X$ so that the family $\mathcal{P} = \{Q_n : n \in \mathbb{N}\}$ satisfies the requirements of the lemma. At step $n$ let $X_n = X \setminus \bigcup_{i < n} Q_i$ (hence $X_0 = X$). If $c_n \notin X_n$, then let $Q_n = Q_{n-1}$. Otherwise, working in the space $\langle X, \tau \rangle$ fix a sequence $\langle U_k : k \in \mathbb{N} \rangle$ of pairwise disjoint open subsets of $X_n$ converging to $c_n$ (in the sense that if $x_k \in U_k$ for each $k$, then $\lim_{k \to \infty} x_k = c_n$). Next, working in the space $\langle X, \overline{\nu} \rangle$, in each $U_k$ find a nowhere dense homeomorphic copy $C_k$ of the Cantor set (this is possible since $B(\overline{\nu}) = B(\tau)$, so $U_k$ is an uncountable Borel subset of $\langle X, \overline{\nu} \rangle$). Finally, let $Q_n = \bigcup_{k \in \mathbb{N}} C_k$. It is easy to see that this works.

Note that continuous bijections from the Baire space $\omega^\omega$ onto a given perfect Polish space $\langle X, \tau \rangle$ correspond to topologies $\tau' \supseteq \tau$ on $X$ such that the space $\langle X, \tau' \rangle$ is homeomorphic to $\omega^\omega$. Thus the following result is actually an equivalent formulation of Theorem 1.3 restated in the form in which we are now going to prove it (compare [11, Theorem 2.1 (iv)]).

**Theorem 2.4.** For a subset $A$ of a perfect Polish space $\langle X, \tau \rangle$, the following are equivalent:

1. $A \in \text{UM}$.
2. $A$ is meager in every Polish topology $\tau' \supseteq \tau$ such that the space $\langle X, \tau' \rangle$ is homeomorphic to $\omega^\omega$.
3. Every subset $B$ of $A$ has the Baire property in every Polish topology $\tau' \supseteq \tau$ such that the space $\langle X, \tau' \rangle$ is homeomorphic to $\omega^\omega$.
Proof. Only implication (2) \( \Rightarrow \) (1) requires a proof. So assume that \( A \not\subset \text{UM} \). By [11, Theorem 2.1(iv)], there is a topology \( \tau_1 \) on \( X \) such that \( \langle X, \tau_1 \rangle \) is a perfect Polish space, \( B(\tau_1) = B(\tau) \) and \( A \not\subset \mathcal{M}(X, \tau_1) \).

Since the identity function from \( \langle X, \tau_1 \rangle \) to \( \langle X, \tau \rangle \) is Borel, by Kuratowski’s theorem ([6, 8.38]) there is a dense \( G_\delta \) subset \( G \) of \( \langle X, \tau_1 \rangle \) such that \( \tau_1 | G \supseteq \tau | G \). By further shrinking \( G \), if necessary, we may also assume that the space \( \langle G, \tau_1 | G \rangle \) is homeomorphic to \( \omega^\omega \). Let \( \tau_G = \tau_1 | G \).

Then \( A \cap G \not\subset \mathcal{M}(X, \tau_1) \) hence also \( A \cap G \not\subset \mathcal{M}(G, \tau_G) \) and without loss of generality in the rest of the proof we assume that \( A \subseteq G \).

Next let \( \tau_2 \supseteq \tau \) be a Polish topology on \( X \) such that \( B(\tau_2) = B(\tau) \) and \( G \) is clopen in \( \tau_2 \) (see [6, 13.1]). Then \( X \setminus G \) is also clopen in \( \tau_2 \) so the topology \( \tau_2 \) \( (X \setminus G) \) is Polish.

Let \( X \setminus G = P \cup C \) be the Cantor-Bendixon decomposition of the space \( \langle X \setminus G, \tau_2 \rangle \) \( (X \setminus G) \) (see [6, 6.4]). Since the space \( \langle P, \tau_2 | P \rangle \) is perfect, there is a topology \( \tau_P \supseteq \tau_2 | P \) on \( P \) such that the space \( \langle P, \tau_P \rangle \) is homeomorphic to \( \omega^\omega \) (see [6, 7.15]).

Let \( \mathfrak{P} \) be the direct sum of the topologies \( \tau_G \) on \( G \), \( \tau_P \) on \( P \) and \( \tau_C \) on \( C \). Clearly, \( \mathfrak{P} \supseteq \tau \) and \( \mathfrak{P} \) is a Polish topology on \( X \). Apply Lemma 2.3 to get a collection \( \mathcal{P} \) of subsets of \( X \) satisfying the conditions of the lemma. Note that since \( \bigcup \mathcal{P} \) is a \( F_\sigma \) nowhere dense subset of \( \langle X, \tau \rangle \), the space \( \langle G \cup P \setminus \bigcup \mathcal{P}, \mathfrak{P} | (G \cup P \setminus \bigcup \mathcal{P}) \rangle \) is homeomorphic to \( \omega^\omega \). Also, since each \( Q \in \mathcal{P} \) is a perfect subset of the space \( \langle X, \tau \rangle \), there is a topology \( \tau_Q \supseteq \tau | Q \) on \( Q \) such that the space \( \langle Q, \tau_Q \rangle \) is homeomorphic to \( \omega^\omega \).

Finally, let \( \tau' \) be the direct sum of the topology \( \mathfrak{P} | (G \cup P \setminus \bigcup \mathcal{P}) \) and the respective topologies \( \tau_Q \) for all \( Q \in \mathcal{P} \). Clearly, \( \tau' \supseteq \tau \) and the space \( \langle X, \tau' \rangle \) is homeomorphic to \( \omega^\omega \). Moreover, \( \tau' | (G \cup \bigcup \mathcal{P}) = \tau_G | (G \setminus \bigcup \mathcal{P}), A \not\subset \mathcal{M}(G, \tau_G) \) and \( G \setminus \bigcup \mathcal{P} \) is a dense \( G_\delta \) subset of \( \langle G, \tau_G \rangle \). It follows that \( A \setminus \bigcup \mathcal{P} \not\subset \mathcal{M}(G, \tau_G) \) hence \( A \not\subset \mathcal{M}(X, \tau') \) which is what we wanted to prove.

\[ \square \]

References


Institute of Mathematics, University of Warsaw, ul. Banacha 2, 02-097 Warsaw, Poland

E-mail address: pioracz@impan.pl