UNIVERSALLY MEAGER SETS, II

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Abstract. We present new characterizations of universally meager sets, shown in [12] to be a category analog of universally null sets. In particular, we address the question of how this class is related to another class of universally meager sets, recently introduced by Todorcevic [11].

1. Introduction

In this note we continue the study of universally meager sets undertaken in [12]. Suppose that A is a subset of a perfect (i.e. with no isolated points) Polish (i.e. separable, completely metrizable) topological space X. We say that A is *universally meager* ($A \in \mathbf{UM}$, see [12], [1] and [2]), if for every Borel isomorphism between X and the Cantor space 2^ω (or equivalently, a perfect Polish space Y – see [12]) the image of A is meager in 2^ω (or in Y, respectively) (this class of sets was earlier introduced and studied by Grzegorek [3], [4], [5] under the name of *absolutely of the first category* sets). We say that A is *perfectly meager* ($A \in \mathbf{PM}$), if for all perfect subsets P of X, the set $A \cap P$ is meager relative to the topology of P. Both $\mathbf{UM}$ and $\mathbf{PM}$ may be seen as category analogs of the class of *universally null sets*, i.e., such sets $A \subseteq X$ that for every Borel isomorphism between X and 2^ω the image of A is null in 2^ω, though by the results of [12] it is perhaps more accurate to view universally meager sets in this role. Clearly, $\mathbf{UM} \subseteq \mathbf{PM}$ and under the Continuum Hypothesis or Martin's Axiom the inclusion is proper. On the other hand various examples of uncountable perfectly meager sets that can be constructed in ZFC (see [7]) turn out to be universally meager and in fact Bartoszyński [1] proved that it is consistent with ZFC that $\mathbf{UM} = \mathbf{PM}$.

Recently, Todorcevic [11] defined another notion of universally meager sets in a much broader setup. Recall that a topological space Y is a Baire space if every nonempty open subset of Y is non-meager in

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Y (see [6, 8.B]). Recall also that a function f defined on a topological space Y is *nowhere constant* if it not constant on any nonempty open subset of Y. Let us say that a subset A of a topological space X is *universally meager in the sense of Todorcevic* if for every Baire space Y and continuous nowhere constant map $f : Y \to X$ the preimage $f^{-1}[A]$ is meager in Y. Todorcevic makes the remark that this is a smallness property very much reminiscent of the notion of a perfectly meager set. Actually we will establish the following connection with universally meager sets (note, however, that under a large cardinal assumption no uncountable set of reals is universally meager in the sense of Todorcevic – see [11]):

Theorem 1.1. *For a subset A of a perfect Polish space X, the following are equivalent:*

(1) $A \in \mathbf{UM}$.
(2) For every second countable Baire space Y and continuous nowhere constant map $f : Y \to X$ the preimage $f^{-1}[A]$ is meager in Y.
(3) For every Polish space Y and continuous nowhere constant map $f : Y \to X$ the preimage $f^{-1}[A]$ is meager in Y.

Another smallness property has recently been studied by Nowik and Reardon [9]. Following [9], for a family $\mathcal{A}$ (respectively: $\mathcal{B}$) of subsets of 2^ω (respectively: of X) let

$U_{\mathbf{B}}[\mathcal{A}] = \{B \subseteq X : f^{-1}[B] \in \mathcal{A}$ for each Borel function $f : 2^\omega \to X\}$, and

$H[\mathcal{B}] = \{A \subseteq X : \forall B \subseteq A\ B \in \mathcal{B}\}$.

Thus $H[U_{\mathbf{B}}[\mathrm{BP}]]$ is the collection of all sets $A \subseteq X$ with the property that for every subset B of A and Borel map $f : 2^\omega \to X$ the preimage $f^{-1}[B]$ has the Baire property (BP) in 2^ω. A slight modification of the proof of 1.1 gives the following answer to a question from a preliminary version of [9].

Theorem 1.2. *For a subset A of a perfect Polish space X, the following are equivalent:*

(1) $A \in \mathbf{UM}$.
(2) For every second countable Baire space Y, subset B of A and Borel map $f : Y \to X$ the preimage $f^{-1}[B]$ has the Baire property in Y.
(3) For every Polish space Y, subset B of A and Borel map $f : Y \to X$ the preimage $f^{-1}[B]$ has the Baire property in Y.
(4) $A \in H[U_{\mathbf{B}}[\mathrm{BP}]]$.

As a matter of fact, the conditions formulated in Theorems 1.1 and 1.2 are easily seen to be sufficient for $A \in \mathbf{UM}$ and the more difficult part is to prove that they are also necessary. On the other hand, a considerable weakening of these conditions still suffices. Recall that the Baire space ω^ω is the unique, up to homeomorphism, nonempty

Polish zero-dimensional space for which all compact subsets have empty interior (see [6, 7.7]). Our main result is:

Theorem 1.3. *For a subset A of a perfect Polish space X, the following are equivalent:*

(1) $A \in \mathbf{UM}$.
(2) For every continuous bijection $f : \omega^\omega \to X$ the preimage $f^{-1}[A]$ is meager in ω^ω.
(3) For every subset B of A and continuous bijection $f : \omega^\omega \to X$ the preimage $f^{-1}[B]$ has the Baire property in ω^ω.

In the rest of the paper we always assume that A is a subset of a perfect Polish space X. We will sometimes use the notation $\langle X, \tau \rangle$ to indicate which topology is being considered at the moment. The relative topology of a subspace $Z \subseteq X$ will be denoted $\tau | Z$. The σ-algebra of all Borel subsets of a topological space Y will be denoted by $\mathbf{B}(Y)$ (or $\mathbf{B}(\tau)$, where τ is the topology of Y, if there is a need to be more specific). The collection of all meager subsets of Y will be denoted by $\mathcal{M}(Y)$ (or $\mathcal{M}(Y, \tau)$, if needed).

2. Characterizations of UM

The proofs of Theorems 1.1 and 1.2 are based on the following lemma.

Lemma 2.1. *Suppose that $f : Y \to X$ is Borel map defined on a second countable Baire space Y. If $A \in \mathbf{UM}$ and the fibers $f^{-1}[\{x\}]$ of all $x \in A$ are meager in Y, then the preimage of A is meager in Y.*

Proof. Assume that $A \in \mathbf{UM}$, let $Z = f^{-1}[A]$ and suppose that $Z \notin \mathcal{M}(Y)$. The argument is a refinement of the proof of [12, Theorem 2.1, (iii) $\Rightarrow$ (i)].

Define a σ-ideal $\mathcal{I}$ in $\mathbf{B}(A)$ by letting

$$B \in \mathcal{I} \iff f^{-1}[B] \in \mathcal{M}(Y), \text{ for } B \in \mathbf{B}(A).$$

Note that $\mathcal{I}$ is proper ($A \notin \mathcal{I}$) and the quotient Boolean algebra $\mathbf{B}(A)/\mathcal{I}$ is σ-complete. We will reach a contradiction with [12, Theorem 2.1 (vi)] as soon as we prove that the algebra $\mathbf{B}(A)/\mathcal{I}$ is actually complete, atomless and has a countable dense subset.

Note that $f^{-1}[\{x\}] \in \mathcal{M}(Y)$ for all $x \in A$, so I contains all singletons of A. This easily implies that the algebra $\mathbf{B}(A)/\mathcal{I}$ has no atoms (for details see the proof of [12, Theorem 2.1, (iii) $\Rightarrow$ (i)]).

Let $\mathcal{J} = \mathcal{M}(Y) \cap \mathbf{B}(Z)$; $\mathcal{J}$ is a proper σ-ideal in $\mathbf{B}(Z)$. Fix a countable base $\mathcal{B}$ for the topology of Y and let

$$\mathcal{D} = \{Z \cap V : \ V \in \mathcal{B} \text{ and } Z \cap V \notin \mathcal{M}(Y).$$

Then the family $\mathcal{D}$ is countable and we claim that it represents a dense subset of the quotient algebra $\mathbf{B}(Z)/\mathcal{J}$. Indeed, if $C \in \mathbf{B}(Z) \setminus \mathcal{M}(Y)$, then there is an open subset U of Y such that C is equal, modulo a meager subset of Y, to $Z \cap U$. Then, since $\mathcal{B}$ is countable, there is $V \in \mathcal{B}$ such that $V \subseteq U$ and $C \cap V \notin \mathcal{M}(Y)$.

Finally, the function f induces a σ-complete embedding of the algebra $\mathbf{B}(A)/\mathcal{I}$ into the algebra $\mathbf{B}(Z)/\mathcal{J}$. It follows that the algebra $\mathbf{B}(A)/\mathcal{I}$ satisfies the countable chain condition and hence is complete. Moreover, it is (isomorphic to) a complete subalgebra of $\mathbf{B}(Z)/\mathcal{J}$ which implies that it also has a countable dense subset.

□

Proof of theorem 1.1. First assume that $A \in \mathbf{UM}$ and let $f : Y \to X$ be a continuous nowhere constant map defined on a second countable Baire space Y. Note that since Y is a Baire space and f is continuous the condition that f is nowhere constant simply means that its fibres at all points of X are meager in Y. Then by lemma 2.1 the preimage $f^{-1}[A]$ is meager in Y. This proves that (1) ⇒ (2).

Next assume that $A \notin \mathbf{UM}$. Then by [12, Theorem 2.1 (i)] there is a Borel one-to-one function $g : Z \to A$ defined on a set $Z \notin \mathcal{M}(X)$; shrinking Z, if necessary, we may assume that Z is dense in itself and g is continuous on Z. By Kuratowski's theorem (see [6, 3.8]) extend g to a continuous function $f : Y \to X$ defined on a G_δ subset of X such that $Z \subseteq Y \subseteq \overline{Z}$. Then Y is a Polish space in the relative topology, f is nowhere constant on Y and the preimage $f^{-1}[A]$ is not meager in Y (we even have $f^{-1}[A] \notin \mathcal{M}(X)$ since $Z \subseteq f^{-1}[A]$). This proves that (3) ⇒ (1) and the remaining implication ((2) ⇒ (3)) is obvious.

□

Proof of theorem 1.2. First assume that $A \in \mathbf{UM}$ and let $f : Y \to X$ be a Borel map defined on a second countable Baire space Y; it is enough to prove that the set $f^{-1}[A]$ has the Baire property in Y. Let $A_1 = \{x \in X : \ f^{-1}[x] \notin \mathcal{M}(Y)\}$ and $A_2 = A \setminus A_1$. Notice that A_1 is countable, so $f^{-1}[A_1]$ is Borel. On the other hand, the fibres of f at all points of A_2 are meager in Y, so by lemma 2.1 we have $f^{-1}[A_2] \in \mathcal{M}(Y)$. I follows that the set $f^{-1}[A] = f^{-1}[A_1] \cup f^{-1}[A_2]$ has the BP in Y. This proves that (1) ⇒ (2).

Implications (2) ⇒ (3) and (3) ⇒ (4) are obvious.

So finally assume that $A \in H[U_{\mathbf{B}}[\mathrm{BP}]]$ and let $f : 2^\omega \to X$ be a Borel isomorphism between the Cantor space and X. By (4), the preimage $f^{-1}[B]$ has the BP in 2^ω for every $B \subseteq A$. But this, f being a bijection, means that $f^{-1}[A]$ has the BP hereditarily, which in turn is equivalent to the fact that $f^{-1}[A] \in \mathcal{M}(2^\omega)$.

□

As another corollary of 2.1 we have the following "Recław style" (see [10]) characterization of $\mathbf{UM}$.

Proposition 2.2. *For a subset A of a perfect Polish space X, the following are equivalent:*

(1) $A \in \mathbf{UM}$.
(2) For every Polish space Y *and a Borel set* $B \subseteq X \times Y$ *with every section* B^y *countable, if for each* $x \in A$ *the section* B_x *is meager in* Y*, then the union* $\bigcup_{x \in A} B_x$ *is meager in* Y.

Proof. Use the fact that by the Lusin-Novikov theorem (see [6, 18.10]) B can be written as the countable union of graphs of Borel functions $f_n : \mathrm{proj}_Y(B) \to X$.

□

The proof of Theorem 1.3 is based on the following lemma.

Lemma 2.3. *Let* τ *be the topology of* X *(making it a perfect Polish space) and assume that* $\overline{\tau} \supseteq \tau$ *is a Polish topology on* X.

If $C \subseteq X$ *is countable, then there exists a countable disjoint collection* $\mathcal{P}$ *of subsets of* X *such that:*

(1) each $Q \in \mathcal{P}$ *is a perfect subset of the space* $\langle X, \tau \rangle$,
(2) each $Q \in \mathcal{P}$ *is a closed nowhere dense subset of the space* $\langle X, \overline{\tau} \rangle$,
(3) $C \subseteq \bigcup \mathcal{P}$.

Proof. Let $C = \{c_n : n \in \mathbb{N}\}$. We are going to define inductively a sequence $\langle Q_n : n \in \mathbb{N} \rangle$ of subsets of X so that the family $\mathcal{P} = \{Q_n : n \in \mathbb{N}\}$ satisfies the requirements of the lemma. At step n let $X_n = X \setminus \bigcup_{i<n} Q_i$ (hence $X_0 = X$). If $c_n \notin X_n$, then let $Q_n = Q_{n-1}$. Otherwise, working in the space $\langle X, \tau \rangle$ fix a sequence $\langle U_k : k \in \mathbb{N} \rangle$ of pairwise disjoint open subsets of X_n converging to c_n (in the sense that if $x_k \in U_k$ for each k, then $\lim_{k \to \infty} x_k = c_n$). Next, working in the space $\langle X, \overline{\tau} \rangle$, in each U_k find a nowhere dense homeomorphic copy C_k of the Cantor set (this is possible since $\mathbf{B}(\overline{\tau}) = \mathbf{B}(\tau)$, so U_k is an uncountable Borel subset of $\langle X, \overline{\tau} \rangle$). Finally, let $Q_n = \bigcup_{k \in \mathbb{N}} C_k \cup \{c_n\}$. It is easy to see that this works.

□

Note that continuous bijections from the Baire space ω^ω onto a given perfect Polish space $\langle X, \tau \rangle$ correspond to topologies $\tau' \supseteq \tau$ on X such that the space $\langle X, \tau' \rangle$ is homeomorphic to ω^ω. Thus the following result is actually an equivalent formulation of Theorem 1.3 restated in the form in which we are now going to prove it (compare [12, Theorem 2.1(iv)]).

Theorem 2.4. *For a subset* A *of a perfect Polish space* $\langle X, \tau \rangle$*, the following are equivalent:*

(1) $A \in \mathbf{UM}$.
(2) A *is meager in every Polish topology* $\tau' \supseteq \tau$ *such that the space* $\langle X, \tau' \rangle$ *is homeomorphic to* ω^ω.
(3) Every subset B *of* A *has the Baire property in every Polish topology* $\tau' \supseteq \tau$ *such that the space* $\langle X, \tau' \rangle$ *is homeomorphic to* ω^ω.

Proof. Only implication (2) $\Rightarrow$ (1) requires a proof. So assume that $A \notin \mathbf{UM}$. By [12, Theorem 2.1(iv)], there is a topology τ_1 on X such that $\langle X, \tau_1 \rangle$ is a perfect Polish space, $\mathbf{B}(\tau_1) = \mathbf{B}(\tau)$ and $A \notin \mathcal{M}(X, \tau_1)$.

Since the identity function from $\langle X, \tau_1 \rangle$ to $\langle X, \tau \rangle$ is Borel, by Kuratowski's theorem ([6, 8.38]) there is a dense G_δ subset G of $\langle X, \tau_1 \rangle$ such that $\tau_1|G \supseteq \tau|G$. By further shrinking G, if necessary, we may also assume that the space $\langle G, \tau_1|G \rangle$ is homeomorphic to ω^ω. Let $\tau_G = \tau_1|G$. Then $A \cap G \notin \mathcal{M}(X, \tau_1)$ hence also $A \cap G \notin \mathcal{M}(G, \tau_G)$ and without loss of generality in the rest of the proof we assume that $A \subseteq G$.

Next let $\tau_2 \supseteq \tau$ be a Polish topology on X such that $\mathbf{B}(\tau_2) = \mathbf{B}(\tau)$ and G is clopen in τ_2 (see [6, 13.1]). Then $X \setminus G$ is also clopen in τ_2 so the topology $\tau_2|(X \setminus G)$ is Polish.

Let $X \setminus G = P \cup C$ be the Cantor-Bendixon decomposition of the space $\langle X \setminus G, \tau_2|(X \setminus G) \rangle$ (see [6, 6.4]). Since the space $\langle P, \tau_2|P \rangle$ is perfect, there is a topology $\tau_P \supseteq \tau_2|P$ on P such that the space $\langle P, \tau_P \rangle$ is homeomorphic to ω^ω (see [6, 7.15]).

Let $\overline{\tau}$ be the direct sum of the topologies τ_G on G, τ_P on P and $\tau|C$ on C. Clearly, $\overline{\tau} \supseteq \tau$ and $\overline{\tau}$ is a Polish topology on X. Moreover, the space $\langle G \cup P, \overline{\tau}|(G \cup P) \rangle$ is homeomorphic to ω^ω. Apply Lemma 2.3 to get a collection $\mathcal{P}$ of subsets of X satisfying the conditions of the lemma. Note that since $\bigcup \mathcal{P}$ is a closed nowhere dense subset of $\langle X, \overline{\tau} \rangle$, the space $\langle G \cup P \setminus \bigcup \mathcal{P}, \overline{\tau}|(G \cup P \setminus \bigcup \mathcal{P}) \rangle$ is homeomorphic to ω^ω too. Also, since each $Q \in \mathcal{P}$ is a perfect subset of the space $\langle X, \tau \rangle$, there is a topology $\tau_Q \supseteq \tau|Q$ on Q such that the space $\langle Q, \tau_Q \rangle$ is homeomorphic to ω^ω.

Finally, let τ' be the direct sum of the topology $\overline{\tau}|(G \cup P \setminus \bigcup \mathcal{P})$ and the respective topologies τ_Q for all $Q \in \mathcal{P}$. Clearly, $\tau' \supseteq \tau$ and the space $\langle X, \tau' \rangle$ is homeomorphic to ω^ω. Moreover, $\tau'|(G \setminus \bigcup \mathcal{P}) = \tau_G|(G \setminus \bigcup \mathcal{P})$, $A \notin \mathcal{M}(G, \tau_G)$ and $G \setminus \bigcup \mathcal{P}$ is a dense G_δ subset of $\langle G, \tau_G \rangle$. It follows that $A \setminus \bigcup \mathcal{P} \notin \mathcal{M}(G, \tau_G)$ hence $A \notin \mathcal{M}(X, \tau')$ which is what we wanted to prove.

□

3. Final comments

1. A refinement of the proof of Theorem 1.1 gives yet another characterization of universally meager sets.

Proposition 3.1. *For a subset A of a perfect Polish space X, the following are equivalent:*

(1) *$A \in \mathbf{UM}$.*
(2) *For every non-empty second countable Baire space Y there is no continuous nowhere constant map $f : Y \to A$.*

Proof. Implication (1) $\Rightarrow$ (2) is a special case of Theorem 1.1 ((1) $\Rightarrow$ (2)).

For the other direction, assume that $A \notin \mathbf{UM}$ and, as in the proof of Theorem 1.1, let $g : Z \to A$ be a continuous, one-to-one function defined on a set $Z \notin \mathcal{M}(X)$. After dropping from Z all points $x \in Z$ with the property that there exists an open neighborhood V of x in X with $Z \cap V \in \mathcal{M}(X)$ we are left with a non-empty Baire space Y and a continuous nowhere constant map $f = g|Y : Y \to A$. □

The collection of sets $A \subseteq X$ satisfying condition (2) of Proposition 3.1 for *all* non-empty Baire spaces was studied by Namioka and Pol [8]. Recall that a subset A of a topological space X is universally meager in the sense of Todorcevic if for every Baire space Y and continuous nowhere constant map $f : Y \to X$ the preimage $f^{-1}[A]$ is meager in Y. In fact, an argument similar to the proof of Proposition 3.1 above leads to the following (perhaps well-known) observation.

Proposition 3.2. *For a subset A of a topological space X, the following are equivalent:*

(1) A is universally meager in the sense of Todorcevic.
(2) Each continuous map $f : Y \to A$ defined on a non-empty Baire space Y is constant on some non-empty open subset of Y.

2. It immediately follows from Theorem 1.1 that if a subset A of a perfect Polish space X is universally meager in the sense of Todorcevic ($A \in \mathrm{T}\mathbf{UM}$), then it is universally meager as well. Taking also Theorem 1.2 into account, we have the following picture describing relations between classes of small sets (from the category branch), mentioned in this paper:

$$\mathrm{T}\mathbf{UM} \subseteq \mathbf{UM} = H[U_{\mathbf{B}}[\mathrm{BP}]] \subseteq \mathbf{PM}.$$

Recall that the inclusion $\mathbf{UM} \subseteq \mathbf{PM}$ may be proper or not, depending on a model of set theory (see [1]). Recall also that under a large cardinal assumption $\mathrm{T}\mathbf{UM}$ consists of countable sets only (see [11]; the latter was established earlier by Namioka and Pol [8] under a stronger large cardinal assumption) whereas uncountable sets in $\mathbf{UM}$ always exist. On the other hand, it easily follows from the results of Namioka and Pol [8], that it is consistent that all subsets of $\mathbb{R}$ of cardinality ω_1 are in $\mathrm{T}\mathbf{UM}$. The following question, however, seems to be open: is it consistent with ZFC that $\mathrm{T}\mathbf{UM} = \mathbf{UM}$?

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