Set Theory Homework #1

Due February 22

Exercise 1. Prove that every countable well-ordered set is isomorphic to a subset of the rationals with the natural order.

Exercise 2. Prove that no uncountable well-ordered set is isomorphic to a subset of the reals with the natural order.

Exercise 3. Prove that no uncountable well-ordered set is isomorphic to a family of subsets of the set of natural numbers ordered by inclusion.

Exercise 4. Prove that there exists an uncountable family of subsets of the set of natural numbers linearly ordered by inclusion.

Exercise 5. Prove that every well-ordered set of cardinality the continuum is isomorphic to a family of subsets of the reals ordered by inclusion.

Exercise 6. Prove that no uncountable well-ordered set is isomorphic to a family of closed subsets of the reals ordered by inclusion.

Exercise 7. Let $\langle X, \leq \rangle$ be an arbitrary uncountable well-ordered set with the smallest element $0$. Let $f : X \to X$ be a function such that if $0 < x$, then $f(x) < x$. Prove that $f$ is not one-to-one.

Exercise 8. With the help of a well-ordering of the reals prove that there exists a partition $\mathbb{R}^2 = A \cup B$ into disjoint sets $A$ and $B$ such that all sections $A_x = \{ y \in \mathbb{R} : (x, y) \in A \}$ and $B_y = \{ x \in \mathbb{R} : (x, y) \in B \}$ have cardinalities less than continuum.