Set Theory homework #1

due October 9, 2019

In the first two exercises prove that the given relation $\preceq$ is a well-ordering of $\mathbb{N}\times\mathbb{N}$. Is $(\mathbb{N}\times\mathbb{N},\preceq)$ isomorphic to $(\mathbb{N},\leqslant)$? If not, describe its limit elements ($\leqslant_l$ denotes the lexicographic well-ordering of $\mathbb{N}\times\mathbb{N}$).

Exercise 1.

$$\langle n_1,m_1\rangle \preceq \langle n_2,m_2\rangle \text{ if and only if } \Big(\max(n_1,m_1) < \max(n_2,m_2) \vee \big(\max(n_1,m_1) = \max(n_2,m_2) \;\wedge\; \langle n_1,m_1\rangle \leqslant_l \langle n_2,m_2\rangle\big)\Big).$$

Exercise 2.

$$\langle n_1,m_1\rangle \preceq \langle n_2,m_2\rangle \text{ if and only if } \Big(\min(n_1,m_1) < \min(n_2,m_2) \vee \big(\min(n_1,m_1) = \min(n_2,m_2) \;\wedge\; \langle n_1,m_1\rangle \leqslant_l \langle n_2,m_2\rangle\big)\Big).$$

Exercise 3. Let

$$X = \Big\{x \in \mathbb{N}^{\mathbb{N}} :\ \exists m \in \mathbb{N}\forall n \in \mathbb{N}\ (n > m \Rightarrow x(n) = 0)\Big\}.$$

Prove that the antilexicographic linear ordering of X (given $x \neq y$ in X we say that x is less then y in this ordering, if $x(k) < y(k)$ where $k = \max\{n \in \mathbb{N} :\ x(n) \neq y(n)\}$) is a well-ordering of X.

Exercise 4. Prove that no uncountable family of subsets of $\mathbb{N}$ is well-ordered by inclusion.

Exercise 5. Prove that no uncountable subset of the reals is well-ordered by the natural ordering of the reals.