ON BOREL MAPPINGS AND σ-IDEALS GENERATED BY CLOSED SETS

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Abstract. We obtain some results about Borel maps with meager fibers on Polish spaces. The results are related to a recent dichotomy by Sabok and Zapletal, concerning Borel maps and σ-ideals generated by closed sets. In particular, we give a “classical” proof of this dichotomy.

We shall also show that for certain natural σ-ideals I generated by closed sets in compact metrizable spaces X, every Borel map on a Borel set in X not in I, either has a fiber not in I or else it is injective on a Borel set not in I. This is the case for the σ-ideal generated by finite-dimensional closed sets in the Hilbert cube, which provides an answer to a question asked by M. Elekes.

1. Introduction

This work is related to a paper by M. Sabok and J. Zapletal [14]. Sabok and Zapletal obtained in this paper a very interesting dichotomy concerning Borel maps and σ-ideals generated by closed sets in Polish (i.e., completely metrizable separable) spaces, cf. Corollary 3.3. Their proof consists of two basic elements: an “open mapping theorem”, justified by some intricate forcing-related arguments, and a variation of a reasoning of Solecki [15]. We shall give another approach, using some ideas from Zakrzewski [19] and [20], to get a factorization theorem for Borel mappings a bit stronger than the open mapping theorem, which provides in effect a proof of the Sabok-Zapletal dichotomy based on a classical descriptive set theory, as presented in Kechris [3].

The factorization theorem is closely related to a dichotomy, presented in Section 4, concerning the structure of Borel mappings on Polish spaces.

In Section 5 we shall show that certain natural σ-ideals I generated by closed sets in compact metrizable spaces X have the following *1-1 or constant property* property, distinguished by Sabok and Zapletal [14]: for any Borel map $f : B \to Y$ on a Borel set $B \subseteq X$ not in I, Y being

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Polish, either f is injective on a Borel subset not in I or there is a fiber of f not in I.

The σ-ideals we consider include the one generated by closed finite-dimensional sets in the Hilbert cube, which provides in effect an answer to a question of M. Elekes raised during his seminar talk at the University of Warsaw.

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2. Terminology and some background

2.1. **Terminology and notation.** All our spaces are separable metrizable. Our terminology concerning descriptive set theory follows Kechris [3].

Given a Polish space E, we denote by $BOR(E)$ and $MGR(E)$ the collections of Borel and meager sets in E, respectively.

By a *σ-ideal* I on E we understand a collection of subsets of E, closed under countable unions and such that for any $A \in I$, all subsets of A are in I. We say that a σ-ideal I is *generated by closed sets* if there is a family $\mathcal{F} \subseteq I$ consisting of sets closed in E such that each element of I can be covered by countably many elements of $\mathcal{F}$.

We shall identify $\mathbb{N}^{\mathbb{N}}$ – the countable product of natural numbers, with the space of irrationals and $2^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}}$ – the subspace consisting of zero-one sequences, is the Cantor set.

2.2. **Boolean algebras $BOR(E)/J$ and Sikorski's theorem.** A key element in our proofs of the main results of this paper is the following approach, developed by Zakrzewski in [19] and [20].

Let E and F be Polish spaces and assume that F has no isolated points. Let $f : E \to F$ be a Borel map whose all fibers $f^{-1}(y)$ are meager in E and let us consider the σ-ideal

(1) $\quad J = \{B \in BOR(F) : \ f^{-1}(B) \in MGR(E)\}$.

A minor modification of reasonings in [20] shows that the quotient Boolean algebras $BOR(F)/J$ and $BOR(F)/MGR(F)$ are isomorphic and Sikorski's theorem (see [3, Theorem 15.10]) asserts that this isomorphism is determined by a bijection of F onto itself which preserves Borel sets in both directions (cf. [19], [20]). In effect, we get a Borel isomorphism

(2) $\quad \varphi : F \to F, \ \forall A \in BOR(F) \ (A \in MGR(F) \Leftrightarrow \varphi(A) \in J)$.

Let us notice that for any Borel set G comeager in E, the set $C = \varphi^{-1}(f(G))$ is comeager in F.

Indeed, for any Borel set A in F disjoint from C, $f^{-1}(\varphi(A))$ is disjoint from G, hence meager in E. Therefore, $\varphi(A) \in J$ and by (2), $A \in$

$MGR(F)$. Now, C being analytic, it is open modulo meager sets, and therefore $F \setminus C \in MGR(F)$.

2.3. **Hyperspaces, function spaces and a parametrization lemma.** Given a compact space X, we shall denote by $\mathscr{K}(X)$ the space of all compact subsets of X, equipped with the Vietoris topology (see [3]). The hyperspace $\mathscr{K}(X)$ is compact and metrizable.

If the compact space X is uncountable, $\mathscr{H}(2^{\mathbb{N}}, X)$ is the space of homeomorphic embeddings of the Cantor set into X, endowed with the topology of uniform convergence (which does not depend on a specific metric in X). The Polish space $\mathscr{H}(2^{\mathbb{N}}, X)$ is a convenient tool in some parametrization problems, cf. [10], [7] and [8].

The subject of this paper is related to a vast literature on parametrization of measurable multifunctions, cf. [6], [17], [16, Theorem 5.2.8]. However, we did not find in the literature direct references to the results we need.

In particular, the following observation will be useful in Section 4.

Lemma 2.1. *Let $W \subseteq \mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}}$ be a G_δ-set such that each vertical section $W(t)$ is dense in $2^{\mathbb{N}}$.*

Then there is a comeager copy of the irrationals Z in $\mathbb{N}^{\mathbb{N}}$ and a homeomorphic embedding $\sigma : Z \times \mathbb{N}^{\mathbb{N}} \to W$ such that $\sigma(\{z\} \times \mathbb{N}^{\mathbb{N}})$ is a dense subset of $\{z\} \times 2^{\mathbb{N}}$, for each $z \in Z$.

Proof. We shall use the following result due to van Mill [9]: there are closed nowhere dense sets $K_1 \subseteq K_2 \subseteq \ldots$ in $2^{\mathbb{N}}$ such that for any sequence $C_1 \subseteq C_2 \subseteq \ldots$ of closed nowhere dense sets in $2^{\mathbb{N}}$, there is a homeomorphism $h : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ and a sequence $n(1) < n(2) < \ldots$ such that $C_i \subseteq h(K_{n(i)})$, $i = 1, 2, \ldots$. More specifically, any sequence $K_1, K_2, \ldots$ in the definition of capsets in [9, Section 2] has the required properties, by [9, Theorem 1.6].

Let $(\mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}}) \setminus W = \bigcup_i F_i$, where $F_1 \subseteq F_2 \subseteq \ldots$ are sets closed in $\mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}}$. Let us check that the set

$$\mathscr{L} = \{(t, h) \in \mathbb{N}^{\mathbb{N}} \times \mathscr{H}(2^{\mathbb{N}}, 2^{\mathbb{N}}) : \exists \tau \in \mathbb{N}^{\mathbb{N}}\ F_i(t) \subseteq h(K_{\tau(i)}),\ i = 1, 2, \ldots\}$$

is Borel.

Indeed, for a fixed open base $U_1, U_2, \ldots$ in $2^{\mathbb{N}}$, each set

$$\begin{aligned} \mathscr{L}_{i,m} &= \{(t, h) : F_i(t) \subseteq h(K_m)\} \\ &= \textstyle\bigcap_j \{(t, h) :\ F_i(t) \cap U_j = \emptyset \quad \text{or} \quad h(K_m) \cap U_j \neq \emptyset\} \end{aligned}$$

is G_δ, and $\mathscr{L} = \bigcap_i \bigcup_m \mathscr{L}_{i,m}$.

The result of van Mill guarantees that each vertical section $\mathscr{L}(t)$ of the set $\mathscr{L}$ is nonempty, and therefore, by the Yankov-von Neumann selection theorem, see [3], there is a continuous map $\lambda : Z \to \mathscr{H}(2^{\mathbb{N}}, 2^{\mathbb{N}})$ defined on a comeager G_δ-set Z in $\mathbb{N}^{\mathbb{N}}$ such that $\lambda(z) \in \mathscr{L}(z)$, for $z \in Z$.

The map $\psi(z,s) = (z, \lambda(z)(s))$ is a homeomorphic embedding of $Z \times 2^{\mathbb{N}}$ into itself, $P = 2^{\mathbb{N}} \setminus \bigcup_i K_i$ is a copy of the irrationals, and for each $z \in Z$, $\lambda(z)(P)$ is a dense subset of $2^{\mathbb{N}}$ contained in $W(z)$.

It follows that $\sigma = \psi | Z \times P$ has the required properties.

□

3. A factorization of Borel maps and the Sabok-Zapletal dichotomy

The following factorization theorem yields easily an open mapping theorem of Sabok and Zapletal [14, Lemma 7.2].

Theorem 3.1. *Let E, F be Polish spaces and let $f : E \to F$ be a continuous function with meager fibers.*

Then there exists a G_δ-set M comeager in E, an open continuous surjection $h : M \to \mathbb{N}^{\mathbb{N}}$ and a continuous injection $g : \mathbb{N}^{\mathbb{N}} \to F$ such that $f \mid M = g \circ h$.

Proof. Following closely Zakrzewski [19] and [20], we consider the σ-ideal

(1) $J = \{B \in BOR(F) : \ f^{-1}(B) \in MGR(E)\}$.

Note that the function f being continuous and the fibers of f being meager, the space F has no isolated points, hence, as explained in Section 2.2, we get a Borel isomorphism

(2) $\varphi : F \to F$, $\forall A \in BOR(F)\ (A \in MGR(F) \Leftrightarrow \varphi(A) \in J)$.

Since $\varphi^{-1} \circ f : E \to F$ is Borel, there is a G_δ-set G in E such that

(3) G is comeager in E and $\varphi^{-1} \circ f \mid G$ is continuous.

By (1) and the observation ending Section 2.2, the analytic set

(4) $C = \varphi^{-1}(f(G))$ is comeager in F.

Let G^* be a compact metrizable extension of G, let $\mathscr{K}(G^*)$ be the space of compact subsets of G^* equipped with the Vietoris topology (cf. Section 2.3), and let $\Phi : C \to \mathscr{K}(G^*)$ be defined by

(5) $\Phi(t) = \mathrm{cl}_{G^*}(f^{-1}(\varphi(t)) \cap G)$,

where cl_{G^*} is the closure in the space G^*.

Since Φ is measurable with respect to the σ-algebra generated by analytic sets in F, both maps Φ and φ are continuous on a comeager set in C, and in effect, there is a copy D of the irrationals in C such that

(6) D is comeager in C, $\varphi \mid D$ and $\Phi \mid D$ are continuous.

By (1), (3), (4) and (6), the G_δ-set

(7) $M = f^{-1}(\varphi(D)) \cap G$ is comeager in E,

and the surjection

(8) $h = \varphi^{-1} \circ f \mid M : M \to D$ is continuous.

By (8), (7) and (5), for $t \in D$, $h^{-1}(t) = f^{-1}(\varphi(t)) \cap M = f^{-1}(\varphi(t)) \cap G$ is dense in $\Phi(t)$, and since for each W open in G^*, $h(W \cap M) = \{t \in D :\ h^{-1}(t) \cap W \neq \emptyset\} = \{t \in D :\ \Phi(t) \cap W \neq \emptyset\}$ is open, the map h is open.

To complete the proof it is enough, by (6) and (8), to declare $g = \varphi \mid D$.

□

The factorization theorem above leads to the following version of the Sabok-Zapletal dichotomy [14, Theorem 1.8], cf. Section 2.1.

Theorem 3.2. *Let X be a Polish space. Let I be a σ-ideal on X generated by closed sets and having the following property:*

(0) *for every G_δ set $U \subseteq X$ not in I and every continuous open function $h : U \to \mathbb{N}^{\mathbb{N}}$ there is a closed nowhere dense set $L \subseteq h(U)$ such that $h^{-1}(L) \notin I$.*

Then for every continuous function $f : X \to F$ with all fibers in I, F a Polish space, and each analytic set $A \subseteq X$ not in I, there is a G_δ set $G \subseteq X$ not in I such that $G \subseteq A$ and f is injective on G.

Proof. The proof is a slight variation of a construction from Case 2 of Solecki's proof of [15, Theorem 1].

To begin with, we use Solecki's theorem [15, Theorem 1] to find a G_δ subset H of A not in I. Removing from H all relatively open sets intersecting H in an element of I, one gets a nonempty G_δ set $E \subseteq H$ such that each nonempty, relatively open in E set W is not in I. This implies that all subsets of E that are in I are meager in E. In particular, since all fibers of f are in I, the function $f|E : E \to Y$ satisfies the assumption of Theorem 3.1. It follows that there is a G_δ-set M comeager in E (hence nonempty relatively open subsets of M are not in I), a continuous open surjection $h : M \to \mathbb{N}^{\mathbb{N}}$ and a continuous injection $g : \mathbb{N}^{\mathbb{N}} \to F$ such that $f|M = g \circ h$.

Fix a complete metric on M. The required G_δ set G will be of the form $G = \bigcap_{n\in\mathbb{N}} \bigcup\{U_\tau : \mathrm{lh}\tau = n\}$ for a regular Souslin scheme $\{U_\tau : \tau \in \mathbb{N}^{<\mathbb{N}}\}$ satisfying the following properties (cf. [15]):

(1) U_τ is a nonempty relatively open subset of M,
(2) $\operatorname{diam} U_\tau \leq 1/(\mathrm{lh}\tau + 1)$,
(3) $\tau \subseteq \rho$ and $\tau \neq \rho$ imply $\mathrm{cl}_M(U_\rho) \subseteq U_\tau$,
(4) $h(U_{\tau * n}) \cap h(U_{\tau * m}) = \emptyset$ if $n \neq m$,
(5) $\lim_n \operatorname{diam} U_{\tau * n} = 0$,
(6) $\mathrm{cl}_M(\bigcup_{n\in\mathbb{N}} U_{\tau * n}) \setminus \bigcup_{n\in\mathbb{N}} \mathrm{cl}_M(U_{\tau * n}) \notin I$,
(7) $\mathrm{cl}_M(\bigcup_{n\in\mathbb{N}} U_{\tau * n}) \setminus \bigcup_{n\in\mathbb{N}} \mathrm{cl}_M(U_{\tau * n}) \subseteq U_\tau$.

Once the construction of the system $\{U_\tau : \tau \in \mathbb{N}^{<\mathbb{N}}\}$ is completed, the proof that $G \notin I$ is exactly the same as in [15] (cf. also the proof of Theorem 5.1). Clearly, $G \subseteq A$ and condition (4) guarantees that the restriction $h|G$ and hence also the restriction $f|G$ are 1-1.

Let $U_\emptyset$ be any nonempty open subset of M with diam $U_\emptyset \leq 1$ and assume that U_τ is already constructed. Note that U_τ is a G_δ subset of X, not in I, as a nonempty open subset of M and the function $h|U_\tau : U_\tau \to \mathbb{N}^\mathbb{N}$ is open. In particular, $W = h(U_\tau)$ is a nonempty open subset of $\mathbb{N}^\mathbb{N}$. By assumption (0), there is a closed nowhere dense set $L \subseteq W$ such that $K = h^{-1}(L) \cap U_\tau$ is not in I. Since h is continuous and open, K is closed and nowhere dense in U_τ.

One can find (cf. [15]) a countable discrete set $D = \{x_n : n \in \mathbb{N}\} \subseteq U_\tau$ together with pairwise disjoint open balls $B_n \subseteq U_\tau$ centered at x_n, respectively, so that $\mathrm{cl}_M(D) = K \cup D$, $D \cap K = \emptyset$ and $h(B_n) \cap h(B_m) = \emptyset$ if $n \neq m$.

Indeed, let $\{a_n : n \in \mathbb{N}\}$ be a dense set in K. We pick the points x_n and the balls B_n inductively, additionally requiring that the distance from a_n to x_n is less than $1/(n+1)$ and $L \cup \bigcup\{\mathrm{cl}_{\mathbb{N}^\mathbb{N}}(h(B_i)) : i < n\} \neq W$ for each $n \in \mathbb{N}$. At step n let $R = h^{-1}(\bigcup\{\mathrm{cl}_{\mathbb{N}^\mathbb{N}}(h(B_i)) : i < n\}) \cap U_\tau$. Then R is a closed subset of U_τ disjoint from K. Since K is closed and nowhere dense in U_τ, there is a point $x_n \in U_\tau \setminus (K \cup R)$ close enough to a_n. Let $y_n = h(x_n)$. Since $y_n \in W \setminus (L \cup \bigcup\{\mathrm{cl}_{\mathbb{N}^\mathbb{N}}(h(B_i)) : i < n\})$, we can find an open neighborhood V of y_n such that $\mathrm{cl}_{\mathbb{N}^\mathbb{N}}(V) \subseteq W \setminus (L \cup \bigcup\{\mathrm{cl}_{\mathbb{N}^\mathbb{N}}(B_i) : i < n\})$. Finally, let B_n be an open ball centered at x_n such that $h(B_n) \subseteq V$ and let $U_{\tau * n} = B_n$. Choosing the diameters of B_n's sufficiently small we can arrange that conditions (2), (3) and (5) are satisfied and $\mathrm{cl}_M(\bigcup_{n\in\mathbb{N}} U_{\tau*n}) \setminus \bigcup_{n\in\mathbb{N}} \mathrm{cl}_M(U_{\tau*n}) = K$.

□

Terminology in the following corollary is taken from [14], where this important fact was established.

Corollary 3.3 (The Sabok-Zapletal dichotomy). *Let X be a Polish space. Let I be a σ-ideal on X generated by closed sets and such that the forcing P_I does not add Cohen reals. Then for every Borel set $B \subseteq X$ not in I and every Borel function f from B into a Polish space F with all fibers in I there is a G_δ-set $G \subseteq B$ not in I such that $f|G$ is 1-1.*

Proof. First, like in the proof of Theorem 3.2, we get a nonempty G_δ-set $E \subseteq B$ such that all subsets of E that belong to I are meager in E. Since f is continuous on a comeager G_δ-set in the space E, we get in effect a G_δ-set $H \subseteq B$ not in I such that $f|H : H \to Y$ is continuous. Now the conclusion is an immediate consequence of Theorem 3.2. □

We end this section with yet another consequence of Theorem 3.1.

Remark 3.4. *Let $f : E \to F$ be as in Theorem 3.1, and let M be a comeager G_δ-set described in the assertion of this theorem. Then there is a continuous map $u : M \to M$, constant on each fiber of f, which takes open sets in M to relatively open sets in $u(M)$.*

Proof. Indeed, the map $h : M \to \mathbb{N}^{\mathbb{N}}$ being open and $\mathbb{N}^{\mathbb{N}}$ being zero-dimensional, there is a continuous selection $s : \mathbb{N}^{\mathbb{N}} \to M$, i.e., $(h \circ s)(t) = t$ for each $t \in M$. We claim that $u = s \circ h$ has the required properties.

To see this, let U be open in M, $a \in U$ and $b = u(a)$. Let V be a neighbourhood of b with $h(V)$ contained in $h(U)$ (recall that h is an open map). Then for any $x \in u(M) \cap V$ there is $y \in U$ with $h(y) = h(x)$ and therefore $u(y) = u(x) = x$. This shows that $u(M) \cap V$ is a neighbourhood of b in $u(M)$ contained in $u(U)$.

□

4. A dichotomy concerning Borel maps

The following result is closely related to Theorem 3.1.

Theorem 4.1. *Let E, F be Polish spaces and let $f : E \to F$ be a Borel map with meager fibers. Then one of the following two mutually exclusive possibilities holds true:*

(A) *there is a non-meager G_δ set in E on which f is injective;*
(B) *there is a homeomorphic embedding $u : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to E$ onto a comeager set in E and a continuous injection $v : \mathbb{N}^{\mathbb{N}} \to F$ such that $f(u(t, s)) = v(t)$ for every $t,\ s \in \mathbb{N}^{\mathbb{N}}$.*

Proof. As in Theorem 3.2, a key element of our proof will be an approach from [19] and [20] (cf. Section 2.2).

With no loss of generality we can assume that the space F has no isolated points (otherwise we replace F with the set P of condensation points of F and then replace E with its dense G_δ-subset contained in $f^{-1}(P)$).

As in the proof of Theorem 3.1, let J be the σ-ideal

(1) $\quad J = \{B \in BOR(F) :\ f^{-1}(B) \in MGR(E)\}$,

$\varphi : F \to F$ a Borel isomorphism such that

(2) $\quad \forall A \in BOR(F)\ (A \in MGR(F) \Leftrightarrow \varphi(A) \in J)$,

cf. Section 2.2, and let G be a comeager copy of the irrationals in E such that both maps

(3) $\quad f|G : G \to F \quad$ and $\quad \varphi^{-1} \circ f|G : G \to F$ are continuous.

We shall consider

(4) $H = \{x \in G : x$ is a point of condensation of $f^{-1}(f(x)) \cap G\}$.

The set H is the projection of the set (cf. Section 2.3)

$\{(x, K) \in E \times \mathscr{K}(E) :\ K$ is a Cantor set in $G, x \in K$ and $|f(K)| = 1\}$,

Borel in the product of E and the hyperspace $\mathscr{K}(E)$, and hence

(5) H is analytic and $f^{-1}(f(x)) \cap H$ is dense in itself for $x \in H$.

Let us notice also that

(6) $f|(G \setminus H)$ is countable-to-one.

If $G \setminus H$ is non-meager, being coanalytic it contains a non-meager Borel set, and in effect, by (6) and a theorem of Lusin (see [3]), f is injective on a non-meager G_δ-set in E, contained in $G \setminus H$. Thus we have arrived at case (A) of the assertion.

Let us assume now that

(7) H is comeager in E.

Then, by (1), (2) and (5), (7), the set

(8) $C = \varphi^{-1}(f(H))$ is analytic and comeager in F,

cf. a remark after (2) in Section 2.2.

Let G^* be a zero-dimensional compactification of G and let $\Phi : C \to \mathscr{K}(G^*)$, cf. Section 2.3, be defined by, cf. the proof of Theorem 3.1, (5),

(9) $\Phi(t) = \mathrm{cl}_{G^*}(f^{-1}(\varphi(t)) \cap H)$.

The map Φ being measurable with respect to the σ-algebra generated by analytic sets in C, there is a copy D of the irrationals such that, cf. (2) and (9),

(10) $D \subseteq C$ is comeager in F and $\varphi|D$, $\Phi|D$ are continuous.

Let us check that, cf. (9), the set, cf. Section 2.3,

(11) $\mathcal{A} = \{(t,h) \in D \times \mathscr{H}(2^{\mathbb{N}}, G^*) : \ h(2^{\mathbb{N}}) = \Phi(t)\}$

is Borel.

Indeed, let $V_1,\ V_2, \ldots$ be an open base in G^* and let us consider the following open sets in the product $D \times \mathscr{H}(2^{\mathbb{N}}, G^*)$:

$$\mathscr{B}_i = \{(t,h) : h(2^{\mathbb{N}}) \cap V_i \neq \emptyset\} \text{ and } \mathscr{C}_i = \{(t,h) : \Phi(t) \cap V_i \neq \emptyset\}.$$

Then

$$\mathcal{A} = (D \times \mathscr{H}(2^{\mathbb{N}}, G^*)) \setminus \bigcup_i [(\mathscr{B}_i \setminus \mathscr{C}_i) \cup (\mathscr{C}_i \setminus \mathscr{B}_i)].$$

Moreover, by (5) and (9), each $\Phi(t)$ is a Cantor set, and therefore, for each vertical section of $\mathcal{A}$, cf. (11),

(12) $\mathcal{A}(t) \neq \emptyset,\ t \in D$.

The Yankov-von Neumann theorem (see [3]) provides a copy T of the irrationals in D and a continuous mapping $\chi : T \to \mathscr{H}(2^{\mathbb{N}}, G^*)$ such that, cf. (10), (12),

(13) T is comeager in F and $\chi(t) \in \mathcal{A}(t)$, for $t \in T$.

The map

(14) $k : T \times 2^{\mathbb{N}} \to G^*$, $k(t,s) = \chi(t)(s)$ is continuous.

We have that

(15) $W = k^{-1}(G)$ is G_δ in $T \times 2^{\mathbb{N}}$ and $k|W : W \to G$ is injective,
(16) $f \circ (k|W) = \varphi \circ proj|W$, where $proj(t,s) = t$.

Moreover, for each $t \in T$,

(17) $\chi(t)^{-1}(f^{-1}(\varphi(t)) \cap H) \subseteq W(t)$,

cf. (11), (13), (9), (15), and hence

(18) $W(t)$ is dense in $2^{\mathbb{N}}$, for $t \in T$.

We shall check that

(19) $k(W)$ is comeager in E,

(20) $k|W : W \to k(W)$ is a homeomorphism.

By (17), $k(W) \supseteq H \setminus f^{-1}(\varphi(F \setminus T))$ and (19) follows from (7), (8), (10) and (2).

Since $k|W$ is continuous and injective, cf. (15), to get (20) we have to verify that $k(t_n, s_n) \longrightarrow k(t_0, s_0)$ implies $(t_n, s_n) \longrightarrow (t_0, s_0)$, whenever $(t_n, s_n) \in W$.

By (3) and (16),

$$t_n = \varphi^{-1} \circ f(k(t_n, s_n)) \longrightarrow \varphi^{-1} \circ f(k(t_0, s_0)) = t_0$$

and hence, $\chi(t_n) \longrightarrow \chi(t_0)$, by the continuity of χ. By (14), $\chi(t_n)(s_n) \longrightarrow \chi(t_0)(s_0)$ and, $\chi(t_0)$ being a homeomorphism, $s_n \longrightarrow s_0$, which completes a justification of (20).

Now, to end the proof, it is enough to take $\sigma : Z \times \mathbb{N}^{\mathbb{N}} \to W$ from Lemma 2.1, cf. (18) where we identify T with $\mathbb{N}^{\mathbb{N}}$, and to declare $u = k \circ \sigma$ and $v = \varphi|Z$, cf. (16).

Indeed, $\sigma(Z \times \mathbb{N}^{\mathbb{N}})$ is comeager in W, cf. Lemma 2.1, hence by (20), $u(Z \times \mathbb{N}^{\mathbb{N}}) = k(\sigma(Z \times \mathbb{N}^{\mathbb{N}}))$ is comeager in $k(W)$ and, by (19), $u(Z \times \mathbb{N}^{\mathbb{N}})$ is comeager in E.

□

5. Certain σ-ideals generated by closed sets in compact spaces

Following Sabok and Zapletal [14], we say that a σ-ideal I on a Polish space X has the *1-1 or constant property* if it satisfies the conclusion of the Sabok-Zapletal dichotomy (see 3.3), i.e., if for every Borel set $B \subseteq X$ not in I and every Borel function f from B into a Polish space Y with all fibers in I, there is a G_δ-set $G \subseteq B$ not in I such that $f|G$ is 1-1.

We shall describe a property of σ-ideals I on compact metrizable spaces which guarantees that I has the 1-1 or constant property (Theorem 5.1).

In particular, for such σ-ideals I, the forcing P_I introduced in the Sabok-Zapletal dichotomy (Corollary 3.3) does not add Cohen reals. Indeed, while the implication "if P_I does not add Cohen reals, then I has the 1-1 or constant property" is essentially the content of the Sabok-Zapletal dichotomy (see Theorems 3.2 and 3.3), the converse is much easier to establish. For the sake of the reader's convenience we shall present a short proof of this (belonging to folklore) fact in Remark 5.7 at the end of this section.

Adapting for our needs the standard terminology concerning σ-ideals of closed sets (cf. [5]) we say that a σ-ideal I of subsets of a compact metric space X generated by closed sets

- is *calibrated* if for every $F \in \mathscr{K}(X)$ (see Section 2.3) not in I and countably many compact sets $K_i \in I$, $i \in \mathbb{N}$, there is $K \in \mathscr{K}(X)$ not in I such that $K \subseteq F \setminus \bigcup_{i\in\mathbb{N}} K_i$,
- has the *covering property* if every analytic set (G_δ suffices, by Solecki's theorem ([15, Theorem 1])) not in I contains a compact set not in I.

Clearly, if a σ-ideal I has the covering property then it is calibrated. Numerous examples of σ-ideals with these properties can be found in [5].

We say that a σ-ideal I of subsets of a compact metric space X has *a coanalytic stratified calibration* if it is generated by a family $\mathcal{F} \subseteq \mathscr{K}(X)$ such that $\mathcal{F} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ for certain families $\mathcal{F}_n$ satisfying the following properties for each $n \in \mathbb{N}$:

(A) $\mathcal{F}_n$ is a coanalytic subspace of $\mathscr{K}(X)$,
(B) $\mathcal{F}_n$ is hereditary (i.e., whenever A is a subset of B and B is in $\mathcal{F}_n$, the closure of A is in $\mathcal{F}_n$),
(C) $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$,
(D) for every $m \in \mathbb{N}$, $F \in \mathscr{K}(X)$ not in I and countably many sets $K_i \in \mathcal{F}_n$, $i \in \mathbb{N}$, there is $K \in \mathscr{K}(X) \setminus \mathcal{F}_m$ such that $K \subseteq F \setminus \bigcup_{i\in\mathbb{N}} K_i$.

Note that if I is calibrated and $I \cap \mathscr{K}(X)$ is coanalytic, then letting $\mathcal{F}_n = I \cap \mathscr{K}(X)$ for each $n \in \mathbb{N}$, one sees that I has a coanalytic stratified calibration.

A σ-ideal which has a coanalytic stratified calibration but is *not* calibrated is presented in Remark 5.5.

Theorem 5.1. *If a σ-ideal I of subsets of a compact metric space X has a coanalytic stratified calibration then it has also the 1-1 or constant property.*

Proof. Assume that $\mathcal{F}$ and $\mathcal{F}_n's$ are as in the definition above.

Let $B \subseteq X$ be a Borel set not in I and assume that $f : B \to Y$ is a Borel function from B into a Polish space Y with all fibers in I.

Our first objective is to show that there is a nonempty G_δ-set $M \subseteq B$ such that $f|M : M \to Y$ is continuous and if we let $h = f|M$, then:

(1) every nonempty relatively open subset of M is not in I,
(2) $\exists n \in \mathbb{N} \forall y \in Y\ \mathrm{cl}_X(h^{-1}(y)) \in \mathcal{F}_n$,
(3) $x \mapsto \mathrm{cl}_X(f^{-1}(f(x)))$ is a continuous mapping from M to $\mathscr{K}(X)$.

Applying a theorem of Burgess and Hillard (see [3, Theorem 35.43]) to the (hereditary and coanalytic) family $\mathcal{F}$, we can assume without loss of generality that:

(4) $\forall y \in Y \exists n \in \mathbb{N}\ \mathrm{cl}_X(f^{-1}(y)) \in \mathcal{F}_n$.

Indeed, the Burgess-Hillard theorem applied to the Borel set $A = \{(x, f(x)) :\ x \in X\} \subset X \times Y$ whose horizontal sections $f^{-1}(y)$ are in I, provides Borel sets $A_1, A_2, \ldots$ with horizontal sections in $\mathcal{F}$, which cover A. The projection B_k of $A \cap A_k$ onto X is Borel (A being the graph of a Borel fuction), and since these sets cover B, some B_k is not in I. Then, replacing B by B_k and f by the restriction od f to B_k, we get (4).

Moreover, after further shrinking B, if necessary (see the beginning of the proof of Theorem 3.2), we shall assume that $B \neq \emptyset$ is a G_δ-set and no nonempty relatively open subset of B belongs to I.

Let $\Psi : B \to \mathscr{K}(X)$ be defined by

(5) $\quad \Psi(x) = \mathrm{cl}_X(f^{-1}(f(x)))$.

The function Ψ is measurable with respect to the σ-algebra of X generated by analytic sets so replacing B, if necessary, by its dense G_δ-subset, we can assume that Ψ is continuous on B.

Since for each n, the family $\mathcal{F}_n$ is coanalytic in $\mathscr{K}(X)$, the set $B_n = \Psi^{-1}(\mathcal{F}_n)$ is coanalytic in X. Note that by (4) and (B), $B = \bigcup_{n\in\mathbb{N}} B_n$, so we can fix n such that B_n is non-meager in X. Let $E \subseteq B_n$ be a G_δ-set, non-meager in X (so E is not in I).

Finally, following the proof of Corollary 3.3, we get a G_δ-set $M \subseteq E$ not in I such that $h = f|M : M \to Y$ is continuous and no nonempty relatively open subset of M belongs to I.

Let us fix a complete metric ρ on M and a metric d on X. Subscripts will indicate that the metric notion under consideration is related to the corresponding metric (e.g., $\mathrm{diam}_d(A)$ or $\mathrm{dist}_d(A, B) = \inf\{d(x, y) : x \in A,\ y \in B\}$ for $A,\ B \subseteq X$).

Having defined M and h satisfying (1)–(3), we shall modify the inductive construction used in the proof of Theorem 3.2.

More precisely, we shall construct a Souslin scheme $\{U_\tau : \tau \in \mathbb{N}^{<\mathbb{N}}\}$ satisfying the following conditions (cf. the proof of Theorem 3.2):

(1′) U_τ is a nonempty relatively open subset of M,
(2′) $\mathrm{diam}_\rho\, U_\tau \leq 1/(\mathrm{lh}\tau + 1)$,
(3′) $\tau \subseteq \rho$ and $\tau \neq \rho$ imply $\mathrm{cl}_M(U_\rho) \subseteq U_\tau$,
(4′) $h(U_{\tau*n}) \cap h(U_{\tau*m}) = \emptyset$ if $n \neq m$,
(5′) $\lim_n \mathrm{diam}_d\, U_{\tau*n} = 0$,
(6′) $\mathrm{cl}_X(\bigcup_{n\in\mathbb{N}} U_{\tau*n}) \setminus \bigcup_{n\in\mathbb{N}} \mathrm{cl}_X(U_{\tau*n}) \notin \mathcal{F}_{lh\tau}$.
(7′) $\mathrm{cl}_X(\bigcup_{n\in\mathbb{N}} U_{\tau*n}) \setminus \bigcup_{n\in\mathbb{N}} \mathrm{cl}_X(U_{\tau*n}) \subseteq \mathrm{cl}_X(U_\tau)$.

Once the construction of the system $\{U_\tau : \tau \in \mathbb{N}^{<\mathbb{N}}\}$ is completed, we let $G = \bigcap_{n\in\mathbb{N}} \bigcup\{U_\tau : \mathrm{lh}\tau = n\}$. Clearly, the construction guarantees that $h|G = f|G$ is 1-1. Moreover, the proof that $G \notin I$ from [15] requires only a minor modification.

Indeed, if $G \subseteq \bigcup_{i\in\mathbb{N}} F_i$, $F_i \in \mathcal{F}_{n(i)}$, then the Baire category theorem provides an open set W in X such that $W \cap G \neq \emptyset$ and $\mathrm{cl}_X(W \cap G) \subseteq F_n$ for some n. But then, by ($2'$), W contains U_τ for some $\tau \in \mathbb{N}^{<\mathbb{N}}$ with $\mathrm{lh}\tau > n$. Moreover, by ($3'$), ($5'$), ($7'$) and the fact that $U_{\tau * n} \cap G \neq \emptyset$, for every $n \in \mathbb{N}$ we have:

$$\mathrm{cl}_X(\bigcup_{n\in\mathbb{N}} U_{\tau*n}) \setminus \bigcup_{n\in\mathbb{N}} \mathrm{cl}_X(U_{\tau*n}) \subseteq \mathrm{cl}_X(U_\tau \cap G) \subseteq \mathrm{cl}_X(W \cap G).$$

This however, taking into account ($6'$), contradicts the choice of τ.

So let $U_\emptyset$ be an arbitrary relatively open subset of M with $\mathrm{diam}_\rho\, U_\emptyset \leq 1$ and assume that U_τ is already defined. Let $F = \mathrm{cl}_X(U_\tau)$ and $m = \mathrm{lh}(\tau)$. Note that, by (1), $F \notin \mathscr{K}(X) \setminus I$.

Fix $C \subseteq U_\tau$ such that C is countable and dense in U_τ. Then, by (D) and (2), there is $K \in \mathscr{K}(X) \setminus \mathcal{F}_m$ such that $K \subseteq F \setminus \bigcup_{x\in C} \Psi(x)$.

We shall find (cf. the proof of Theorem 3.2) a countable set $D = \{x_n : n \in \mathbb{N}\} \subseteq C$ together with pairwise disjoint open (in M) neighbourhoods $B_n \subseteq U_\tau$ of x_n, respectively, so that:

(6) $\mathrm{cl}_X(D) = K \cup D$ and $D \cap K = \emptyset$,
(7) $h(B_n) \cap h(B_m) = \emptyset$ if $n \neq m$.

To that end let $\{a_n : n \in \mathbb{N}\}$ be a dense set in K.

We shall pick inductively points $x_i \in C$ and open (in M) neighbourhoods B_i, U_i of x_i, demanding that for every i and j:

(8) $B_i \subseteq U_i$ and $\mathrm{cl}_Y(h(B_i)) \subseteq h(U_i)$,
(9) $h(B_i) \cap h(B_j) = \emptyset$ if $i \neq j$,
(10) $d(a_i, x_i) < 1/(i+1)$,
(11) $\mathrm{dist}_d(K, \bigcup \Psi(U_i)) > 0$.

At step n find $m \geq n+1$ such that $1/m < \min_{i<n} \mathrm{dist}_d(K, \bigcup \Psi(U_i))$ (cf (11)). Pick $c \in C$ such that $d(a_n, c) < 1/m$ and let $x_n = c$, so that (10) is satisfied. Then, by the choice of K, $\Psi(c) \cap K = \emptyset$, so $\mathrm{dist}_d(K, \Psi(x_n)) > 0$.

Let $\delta = \mathrm{dist}_d(K, \Psi(x_n))$ and let d_H denote the Hausdorff metric in $\mathscr{K}(X)$ corresponding to d.

By the continuity of Ψ at x_n, there exists an open subset $U_n \subseteq U_\tau$ such that $x_n \in U_n$ and $d_H(\Psi(x_n), \Psi(x)) < \delta/2$ for every $x \in U_n$. It follows that $\mathrm{dist}_d(K, \bigcup \Psi(U_n)) \geq \delta/2$, establishing (11).

Then, $a_n \in K$ and $d(a_n, x_n) < \min_{i<n} \mathrm{dist}_d(K, \bigcup \Psi(U_i))$, imply that $x_n \notin \bigcup_{i<n} \bigcup \Psi(U_i)$ and consequently $h(x_n) \notin \bigcup_{i<n} h(U_i)$.

Hence, by (8), $h(x_n) \notin \bigcup_{i<n} \mathrm{cl}_Y(h(B_i))$. So there is an open set $V \subseteq Y$ such that $h(x_n) \in V$ and $V \cap \bigcup_{i<n} \mathrm{cl}_Y(h(B_i)) = \emptyset$. Finally, we choose an open subset $B_n \subseteq U_n$ such that $x_n \in B_n$, and $\mathrm{cl}_Y(h(B_n)) \subseteq V$, thus establishing (8) and (9).

Having defined x_n, B_n and U_n satisfying (8) – (11), we let $D = \{x_n : n \in \mathbb{N}\}$ and $U_{\tau * n} = B_n$, for $n \in \mathbb{N}$. More precisely, we shrink each $U_{\tau * n}$ further, if necessary, so that all required conditions are satisfied. In particular, (6) follows from (10) and, by (6) and (5′), $K = \mathrm{cl}_X(\bigcup_{n\in\mathbb{N}} U_{\tau*n}) \setminus \bigcup_{n\in\mathbb{N}} \mathrm{cl}_X(U_{\tau*n})$, giving (6′).

□

The following immediate corollary to Theorem 5.1 implies that many important σ-ideals discussed in the literature have the 1-1 or constant property.

Corollary 5.2. *Let X be a compact metric space and let I be a σ-ideal on X generated by compact sets.*

If I is calibrated and $I \cap \mathscr{K}(X)$ is coanalytic then I has the 1-1 or constant property.

Remark 5.3. *Examples of σ-ideals satisfying the assumptions of Corollary 5.2 and hence having the 1-1 or constant property include (see* [5]*):*

- *the σ-ideal generated by closed sets of uniqueness in the group $\mathbb{T}$ of unit complex numbers,*
- *the σ-ideal generated by closed sets of extended uniqueness in $\mathbb{T}$,*
- *the σ-ideal generated by closed null-sets for a subadditive capacity on a compact, metric space (in particular: the σ-ideal generated by closed null-sets of the interval [0,1] or the Cantor group $2^{\mathbb{N}}$; the fact that this σ-ideal does not add Cohen reals has been proved earlier by Sabok* [13]*),*
- *the σ-ideal generated by closed σ-porous sets in a compact metric space,*
- *the σ-ideal generated by closed E-smooth subsets of a compact metric space X for a Borel non-smooth equivalence relation E on X.*

Some of the σ-ideals listed above have, moreover, the covering property, some of them not (see [5]). Let us note, however, that in Corollary 5.2 the assumption that I is coanalytic is needed only to justify the use of the Burgess-Hillard theorem (see the proof of Theorem 5.1) and in fact is not needed when the function f is continuous on a compact set not in I. In particular, the definability assumption can be dropped in the case of σ-ideals with the covering property which yields the following corollary. The second part of the assertion of this corollary follows readily from the Sabok-Zapletal dichotomy, by applying in a standard way to P_I forcing arguments related to not adding Cohen reals (see [18, Theorem 3.3.2]). However, arguments based on the proof of Theorem 5.1 provide a justification avoiding forcing methods.

Corollary 5.4. *Let X be a compact metric space and let I be a σ-ideal on X generated by compact sets.*

(1) *If I is calibrated then for every compact set $P \subseteq X$ not in I and every continuous function f from P into a Polish space Y with all fibers in I there is a G_δ-set $G \subseteq P$ not in I such that $f|G$ is 1-1.*

(2) *If I has the covering property then I has the 1-1 or constant property.*

Proof. To prove (1), one can follow closely the proof of Theorem 5.1, letting $B = P$ and $\mathcal{F}_n = I \cap \mathscr{K}(X)$ for each $n \in \mathbb{N}$. Note that under the present assumptions condition (2) from that proof is automatically fulfilled and we do not need additional definability assumptions about I.

To prove (2), note that given a Borel set $B \subseteq X$ not in I and a Borel function $f : B \to Y$ from B into a Polish space Y with all fibers in I, the covering property provides readily a compact set $P \subseteq B$ not in I such that the function $f|P$ is continuous. Then the conclusion follows from part (1).

□

Remark 5.5.

Our terminology concerning dimension theory follows [1].

(A) Given a compact metrizable space X, we denote by $\mathcal{F}_n(dim)$ the collection of closed at most n-dimensional subsets of X and let $I(dim)$ *be the σ-ideal of subsets of X that can be covered by countably many elements of $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n(dim)$.*

Each $\mathcal{F}_n(dim)$ is a G_δ-set in the hyperspace $\mathscr{K}(X)$ (see [4, §45, IV, Theorem 4]).

Moreover, if $F \in \mathscr{K}(X) \setminus I(dim)$ and $K_i \in \mathcal{F}_n(X)$, $i \in \mathbb{N}$, we have $dim(\bigcup_{i\in\mathbb{N}} K_i) \leq n$ by the sum theorem (see [1, 1.5.3]) and by the enlargement theorem (see [1, 1.5.11]), $\bigcup_{i\in\mathbb{N}} K_i$ is contained in a G_δ-set G in X with $dim(G) \leq n$. By the addition theorem (see [1, 1.5.10]), $dim(F \setminus G) = \infty$, and $F \setminus G$ being σ-compact, again using the sum theorem we infer that for each m there is a compact set $K \subseteq F \setminus G$ with $dim(K) \geq m$.

Therefore, *the σ-ideal $I(dim)$ has a coanalytic stratified calibration.*

However, *the σ-ideal $I(dim)$ in the Hilbert cube $[0,1]^{\mathbb{N}}$ is not calibrated.* Indeed, there is a compact set $F \subseteq [0,1]^{\mathbb{N}}$ not in $I(dim)$ containing a zero-dimensional G_δ-set H such that $F \setminus H \in I(dim)$ (see [1, Example 5.1.7]). Clearly, the calibration property fails for F.

(B) Let $dim_{\mathbb{Z}}$ be the cohomological dimension with respect to the ring of integers (see [1, page 75]) and let us consider, replacing in (A) the covering dimension dim by the cohomological dimension $dim_{\mathbb{Z}}$, the σ-ideal $I(dim_{\mathbb{Z}})$ associated with the cohomological dimension.

Then $I(dim_{\mathbb{Z}})$ *has a coanalytic stratified calibration.* A verification runs analogously to that in (A), but instead of using classical results, we have to appeal to a theorem of Dobrowolski and Rubin [2] that $\mathcal{F}_n(dim_{\mathbb{Z}})$ is a G_δ-set in the hyperspace, and to counterparts for $dim_{\mathbb{Z}}$ of the enlargement and addition theorems, established respectively by Rubin and Schapiro [12] and Rubin [11].

Theorem 5.1 combined with Remark 5.5 gives the following corollary which provides an answer to a question asked by Elekes during his seminar talk at the University of Warsaw in 2009.

Corollary 5.6. *Let X be a compact metrizable space and let I be the σ-ideal of subsets of X that can be covered by countably many finite-dimensional compact sets in X. Then I has the 1-1 or constant property.*

We shall end this section with a brief remark clarifying some relations between the 1-1 or constant property of I and certain forcing properties of P_I. As was already mentioned, these relations are well-known to experts on this topic.

Remark 5.7. *Let X be a Polish space. Let I be a σ-ideal on X generated by closed sets.*

If there is a Borel set $B \subseteq X$ not in I and a Borel function g from B into a Polish space Y without isolated points such that

(A) *the inverse image under g of every Borel subset $C \subseteq Y$ meager in Y is in I.*

then there is a Borel function $f : B \to \mathbb{N}^{\mathbb{N}}$ such that

(B) *all fibers of f are in I and there is no Borel set $A \subseteq B$ not in I such that $f|A$ is 1-1.*

Proof. Let $h : Y \to \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ be a Borel isomorphism between Y and $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that

$$\forall A \in BOR(Y)\ (A \in MGR(Y) \Leftrightarrow h(A) \in MGR(\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}})).$$

Let $f = proj \circ h \circ g$, where $proj$ is the projection onto the first coordinate. Then, by (A), all fibers of f are in I.

Assume now that A is a Borel subset of B and $f|A$ is 1-1. Then $proj|h(g(A))$ is 1-1 so the set $h(g(A)$, being analytic, is meager in $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$, by the Kuratowski-Ulam theorem. It follows, by (A), that $A \in I$.

□

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