ON COUNTABLY PERFECTLY MEAGER AND COUNTABLY PERFECTLY NULL SETS

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Abstract. We study a strengthening of the notion of a universally meager set and its dual counterpart that strengthens the notion of a universally null set.

We say that a subset A of a perfect Polish space X is countably perfectly meager (respectively, countably perfectly null) in X, if for every perfect Polish topology τ on X, giving the original Borel structure of X, A is covered by an F_σ-set F in X with the original Polish topology such that F is meager with respect to τ (respectively, for every finite, non-atomic, Borel measure μ on X, A is covered by an F_σ-set F in X with $\mu(F) = 0$).

We prove that if $2^{\aleph_0} \leq \aleph_2$, then there exists a universally meager set in $2^{\mathbb{N}}$ which is not countably perfectly meager in $2^{\mathbb{N}}$ (respectively, a universally null set in $2^{\mathbb{N}}$ which is not countably perfectly null in $2^{\mathbb{N}}$).

1. Introduction

We continue the study of countably perfectly meager sets undertaken by Pol and Zakrzewski [20]. We say (cf. [20]) that a subset A of a perfect Polish space X is *countably perfectly meager in* X ($A \in \mathbf{PM}_\sigma$), if for every sequence of perfect subsets $\{P_n : n \in \mathbb{N}\}$ of X, there exists an F_σ-set F in X such that $A \subseteq F$ and $F \cap P_n$ is meager in P_n for each n. Let us also recall that A is *universally meager* ($A \in \mathbf{UM}$), if for every Borel isomorphism f between X and any perfect Polish space Y the image of A under f is meager in Y (see [27], [28], [1], [2] and also [10], [11], [12], where this class was earlier studied by Grzegorek and denoted by $\overline{\mathbf{AFC}}$). By [2, Theorem 7] we have $\mathbf{PM}_\sigma \subseteq \mathbf{UM}$ and by [20, Theorem 1.1], this inclusion is consistently proper, namely it holds if there exists a universally meager set of cardinality $2^{\aleph_0}$, in particular, if CH is true.

In this note we prove (see Theorem 2.2) that $\mathbf{PM}_\sigma \neq \mathbf{UM}$ follows also from the assumption that $2^{\aleph_0} = \aleph_2$. Whether it is consistent that $\mathbf{PM}_\sigma = \mathbf{UM}$ remains an open problem (it is consistent that $\mathbf{UM} \subsetneq$

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PM but also that **UM** = **PM** (see [1]), where **PM** denotes the family of all perfectly meager subsets of X).

If I is a σ-ideal of subsets of X, i.e., it is hereditary, closed under taking countable unions and contains all singletons, then by I^* we denote the σ-ideal on X generated by the closed subsets of X which belong to I (cf. [23]).

If τ is a perfect Polish topology on X giving the original Borel structure of X, then by $\mathscr{M}(X,\tau)$ we denote the σ-ideal of meager sets with respect to τ. Let us note that $\mathscr{M}^*(X,\tau)$ consists of such $A \subseteq X$ that there exists an F_σ-set F in X (with the original Polish topology) with $A \subseteq F$ and $F \in \mathscr{M}(X,\tau)$. By [27, Theorem 2.1], A is universally meager in X if and only if A belongs to the intersection of all σ-ideals of the form $\mathscr{M}(X,\tau)$, whereas by [20, Proposition 4.6], A is countably perfectly meager in X if and only if A belongs to the intersection of all σ-ideals of the form $\mathscr{M}^*(X,\tau)$.

Universally meager sets may be seen as a category counterpart of *universally null* sets in X. Namely, if for a finite, non-atomic, Borel measure μ is on X (i.e., a countably additive measure $\mu : \mathbf{B}(X) \to [0,+\infty)$ defined on the σ-algebra $\mathbf{B}(X)$ of Borel subsets of X and vanishing on singletons of X), we denote by $\mathscr{N}(X,\mu)$ the σ-ideal of μ-null sets (i.e., sets of outer μ-measure zero), then the collection **UN** of universally null subsets of X is the intersection of all σ-ideals of the form $\mathscr{N}(X,\mu)$.

The following definition of a measure analogue of countably perfectly meager sets was suggested by Taras Banakh. We say that A is *countably perfectly null in* X ($A \in \mathbf{PN}_\sigma$), if A belongs to the intersection of all σ-ideals of the form $\mathscr{N}^*(X,\mu)$. In other words, $A \in \mathbf{PN}_\sigma$ if for every finite, non-atomic, Borel measure μ on X, A is covered by an F_σ-set F in X with $\mu(F) = 0$. Let us note that if λ is the standard probability product measure on the Cantor space $2^{\mathbb{N}}$, then $\mathscr{N}^*(2^{\mathbb{N}},\lambda)$ is a well-known σ-ideal which is usually denoted by $\mathscr{E}$ (cf. [3]).

The name of the class $\mathbf{PN}_\sigma$ is further justified by the following observation.

Proposition 1.1. *A set $A \subseteq X$ is countably perfectly null in X if and only if for every sequence of perfect subsets $\{P_n : n \in \mathbb{N}\}$ of X with associated probability non-atomic Borel measures μ_n on P_n, there exists an F_σ-set F in X such that $A \subseteq F$ and $\mu_n(F \cap P_n) = 0$ for each n.*

Proof. If $A \in \mathbf{PN}_\sigma$ and for each n we have a perfect set P_n together with the respective measure μ_n on P_n, then it is enough to cover A by an F_σ-set F with $\mu(F) = 0$ for μ defined by

$$\mu(B) = \sum_n \frac{1}{2^n}\mu_n(B \cap P_n) \quad \text{for} \quad B \in \mathbf{B}(X).$$

For the other direction, given a finite, non-atomic, Borel measure μ on X let us note that the regularity of μ (cf. [16, 17.C]) implies the existence of (pairwise disjoint) perfect sets $\{P_n : n \in \mathbb{N}\}$ of positive μ-measure such that $\mu(X \setminus \bigcup_n P_n) = 0$. Then it suffices to cover A by an F_σ-set F with $\mu(F \cap P_n) = 0$ for each n.

□

Clearly, we have $\mathbf{PN}_\sigma \subseteq \mathbf{UN}$. One easily observes that we also have $\mathbf{PN}_\sigma \subseteq \mathbf{PM}_\sigma$.

Proposition 1.2. *Every countably perfectly null subset of X is countably perfectly meager.*

Proof. Let us assume that $A \in \mathbf{PN}_\sigma$ and let $\{P_n : n \in \mathbb{N}\}$ be a sequence of perfect subsets of X. For each n let μ_n be a Borel probability, non-atomic measure on P_n which assigns positive values to all non-empty, relatively open subsets of P_n (e.g., one may concentrate μ_n on a dense in P_n homeomorphic copy of the irrationals). Let F be an F_σ-set in X such that $A \subseteq F$ and $\mu_n(F \cap P_n) = 0$ for each n (cf. Proposition 1.1). Clearly, $F \cap P_n$ is meager in P_n for each n, so $A \in \mathbf{PM}_\sigma$. □

The inclusion $\mathbf{PN}_\sigma \subseteq \mathbf{PM}_\sigma$ is, at least consistently, proper. Indeed, if $A \subseteq 2^{\mathbb{N}}$ is a Sierpiński set with respect to the measure λ, then $A \in \mathbf{PM}_\sigma$ in $2^{\mathbb{N}}$ (cf. [20, Corollary 2.9 and Remark 2.11]) but A has positive outer measure λ.

An analogous argument shows the consistency of $\mathbf{PN}_\sigma \neq \mathbf{UN}$. Namely, if $A \subseteq 2^{\mathbb{N}}$ is a Luzin set in $2^{\mathbb{N}}$ (which exists e.g. under CH), then $A \in \mathbf{UN}$ (A has even strong measure zero, cf. [18]) but $A \notin \mathbf{PN}_\sigma$, A being non-meager in $2^{\mathbb{N}}$.

In this note we prove (see Theorem 3.2) that the inequality $\mathbf{PN}_\sigma \neq \mathbf{UN}$ follows also from the assumptions that either there exists a universally null set in $2^{\mathbb{N}}$ of cardinality $2^{\aleph_0}$ (then we actually have that even $\mathbf{UN} \setminus \mathbf{PM}_\sigma \neq \emptyset$; cf. Proposition 1.2) or $2^{\aleph_0} = \aleph_2$. Whether it is consistent that $\mathbf{PN}_\sigma = \mathbf{UN}$, remains an open problem.

Section 2 is devoted to the proof of Theorem 2.2 stating that if $2^{\aleph_0} = \aleph_2$, then there is a universally meager set in $2^{\mathbb{N}}$ which is not countably perfectly meager in $2^{\mathbb{N}}$.

In Section 3 we give some examples of countably perfectly null sets and prove Theorem 3.2 which shows the inequality $\mathbf{PN}_\sigma \neq \mathbf{UN}$ under the assumption that either there exists a universally null set in $2^{\mathbb{N}}$ of cardinality $2^{\aleph_0}$ (then we actually have that even $\mathbf{UN} \setminus \mathbf{PM}_\sigma \neq \emptyset$, cf. Proposition 1.2) or $2^{\aleph_0} = \aleph_2$.

In Section 4 we collect some remarks and open problems.

2. Universally meager not countably perfectly meager sets

Let us recall that the cardinal number $\mathfrak{b}$ is the minimal cardinality of a subset of $\mathbb{N}^{\mathbb{N}}$ which is unbounded in the ordering $\leq^*$ of eventual domination. Following [25, Definition 2.8], by a $\mathfrak{b}$-*scale* (in $\mathbb{N}^{\mathbb{N}}$) we mean a subset $B = \{f_\alpha : \alpha < \mathfrak{b}\}$ of $\mathbb{N}^{\mathbb{N}}$ with the following properties:

- $f_\alpha : \mathbb{N} \to \mathbb{N}$ is strictly increasing,
- $\alpha < \beta < \mathfrak{b}$ implies $f_\alpha <^* f_\beta$,
- for every $f \in \mathbb{N}^{\mathbb{N}}$ there is $\alpha < \mathfrak{b}$ with $f_\alpha \not\leq^* f$.

By identifying each f_α with the characteristic function of its range (or just its range, respectively), we obtain a homeomorphic copy A of B in $2^{\mathbb{N}}$ (respectively, in $\mathcal{P}(\mathbb{N})$ with the Cantor set topology) which we also call a $\mathfrak{b}$-*scale* in $2^{\mathbb{N}}$ (respectively, in $\mathcal{P}(\mathbb{N})$) (cf. [25]). It is well-known and easy to see that $\mathfrak{b}$-scales can be constructed in ZFC. They are also classical examples of sets which are both universally meager and universally null (cf. [19]).

Let us recall that given a subset A of a perfect Polish space X, by a γ–*cover* of A we mean a countable relatively open cover $\mathscr{U}$ of A which is infinite and such that for each $x \in A$ the set $\{U \in \mathscr{U} : x \notin U\}$ is finite. We say that A satisfies *property* $S_1(\Gamma, \Gamma)$ if for every sequence $(\mathscr{U}_n : n \in \mathbb{N})$ of γ-covers of A we can select for each n a set $V_n \in \mathscr{U}_n$ such that $\{V_n : n \in \mathbb{N}\}$ is a γ-cover of A (cf. [14], [25]). It is well-known (and due to Hurewicz [13]) that property $S_1(\Gamma, \Gamma)$ implies the Hurewicz property (for a definition of the Hurewicz property see Section 3).

If $\mathfrak{b} = \omega_1$, then there exists a $\mathfrak{b}$-scale $A = \{a_\alpha : \alpha < \mathfrak{b}\}$ in $\mathcal{P}(\mathbb{N})$ with the additional property that $\alpha < \beta < \mathfrak{b}$ implies that $a_\beta \setminus a_\alpha$ is finite (see [25, page 8]) and by a theorem of Scheepers [24] (see also [5, Theorem 123]), if A is such a $\mathfrak{b}$-scale in $\mathcal{P}(\mathbb{N})$, then $A \cup [\mathbb{N}]^{<\aleph_0}$ has property $S_1(\Gamma, \Gamma)$. The following observation is an easy corollary of this result. Let us recall that if κ is an infinite cardinal, then a set $A \subseteq X$ is κ-concentrated on a set $Q \subseteq X$, if $|A \setminus U| < \kappa$ for each open set U in X containing Q.

Lemma 2.1. *Assume that $\mathfrak{b} = \omega_1$. Let $A = \{a_\alpha : \alpha < \mathfrak{b}\}$ be a $\mathfrak{b}$-scale in $\mathcal{P}(\mathbb{N})$ with the additional property that $\alpha < \beta < \mathfrak{b}$ implies that $a_\beta \setminus a_\alpha$ is finite.*

For each n let $\mathscr{U}_n = \{U^n_k : k \in \mathbb{N}\}$ be an ascending (i.e., $U^n_k \subseteq U^n_{k+1}$) sequence of open sets in $\mathcal{P}(\mathbb{N})$ with $[\mathbb{N}]^{<\aleph_0} \subseteq \bigcup_k U^n_k$ but $[\mathbb{N}]^{<\aleph_0} \subseteq U^n_k$ for no k. Then we can select for each n a set $V_n = U^n_{k_n}$ such that $\{V_n : n \in \mathbb{N}\}$ is a γ-cover of $(A \cup [\mathbb{N}]^{<\aleph_0}) \setminus Y$ for a certain countable set $Y \subseteq A$.

Proof. The set A being a $\mathfrak{b}$-scale in $\mathcal{P}(\mathbb{N})$, is $\mathfrak{b}$-concentrated on $[\mathbb{N}]^{<\aleph_0}$ (see [25, Lemma 2.10]). Consequently, since $\mathfrak{b} = \omega_1$, there is $\xi < \omega_1$ such that if we let $A' = \{a_\alpha : \xi < \alpha < \mathfrak{b}\}$, then for each n we have $A' \cup [\mathbb{N}]^{<\aleph_0} \subseteq \bigcup_k U^n_k$ and by the properties of the sequence $\{U^n_k : k \in \mathbb{N}\}$, $\{(A' \cup [\mathbb{N}]^{<\aleph_0}) \cap U^n_k : k \in \mathbb{N}\}$ is a γ-cover of $A' \cup [\mathbb{N}]^{<\aleph_0}$. Since at the same time A' is still a $\mathfrak{b}$-scale in $\mathcal{P}(\mathbb{N})$ with the additional property above, Scheepers's theorem gives the desired conclusion. □

Let us recall that $\mathrm{non}(\mathscr{M})$ is the smallest cardinality of a non-meager subset of $2^{\mathbb{N}}$. It is well-knows that if τ is a perfect Polish topology on a Polish space X, then $\mathrm{non}(\mathscr{M})$ is the smallest cardinality of a subset of X not in $\mathscr{M}(X,\tau)$. We denote by $\mathbb{Q}$ the copy of the rationals in $2^{\mathbb{N}}$ consisting of all eventually zero binary sequences.

Now we are ready to prove the main result of this section (cf. the proof of [26, Theorem 4]).

Theorem 2.2. *If $2^{\aleph_0} \leq \aleph_2$, then there is a universally meager set in $2^{\mathbb{N}}$ which is not countably perfectly meager in $2^{\mathbb{N}}$.*

Proof. If $2^{\aleph_0} = \aleph_1$, then the result follows from [20, Theorem 1.1], so from now on let us assume that $2^{\aleph_0} = \aleph_2$.

We shall split the argument into three cases.

Case (A): $\mathrm{non}(\mathscr{M}) = \aleph_2$.

Then, by a result of Grzegorek (see [11, Theorem 1]), there exists a universally meager set in $2^{\mathbb{N}}$ of cardinality $\aleph_2 = 2^{\aleph_0}$ and the result follows from [20, Theorem 1.1].

Case (B): $\mathfrak{b} = \aleph_2$.

This case is already covered by the previous one, since it is well-known that $\mathfrak{b} \leq \mathrm{non}(\mathscr{M})$.

Case (C): $\mathrm{non}(\mathscr{M}) = \mathfrak{b} = \aleph_1$.

Let C and D be disjoint copies of the Cantor set in $2^{\mathbb{N}}$ such that

(1) the operation $+$ of addition is a homeomorphism between $C \times D$ and $C + D$ (cf. [21]).

Let us fix a homeomorphism $h : 2^{\mathbb{N}} \to C$.

Let $A = \{a_\alpha : \alpha < \mathfrak{b}\}$ be a $\mathfrak{b}$-scale in $\mathcal{P}(\mathbb{N})$ with the additional property that $\alpha < \beta < \mathfrak{b}$ implies that $a_\beta \setminus a_\alpha$ is finite (cf. the paragraph preceding Lemma 2.1) and let us identify it with its homeomorphic copy in $2^{\mathbb{N}}$.

Let $X = A \cup \mathbb{Q}$ and $\tilde{X} = h(X)$. Since X is universally meager, so is $\tilde{X}$.

Let us fix a set $M \subseteq D$ of cardinality $\mathrm{non}(\mathscr{M}) = \aleph_1$ such that

(2) M is relatively non-meager in D.

Since $|\tilde{X}| = \aleph_1$, we can fix a surjection $m : \tilde{X} \to M$ onto M and let $H = \{(x, m(x)) : x \in \tilde{X}\} \subseteq C \times D$ be the graph of m. Let us note that since $\tilde{X}$ is the injective continuous image of H under the projection onto the first axis and $\tilde{X}$ is universally meager, so is H.

Finally, let $Z = \{x+m(x) : x \in \tilde{X}\}$. Clearly, Z is universally meager as the image of H under the homeomorphism $+$ between $C \times D$ and $C + D$ (cf. (1)).

We shall show that

(3) Z is not a $\mathbf{PM}_\sigma$-set in $2^{\mathbb{N}}$

and this will end the proof of the theorem.

To that end, let $\tilde{\mathbb{Q}} = h(\mathbb{Q}) = \{q_n : n \in \mathbb{N}\}$ and let us suppose, towards a contradiction, that there are closed sets F_n in $2^{\mathbb{N}}$ such that $Z \subseteq \bigcup_n F_n$ and F_n is relatively nowhere dense in $q_k + D$ or equivalently, $(q_k + F_n) \cap D$ is relatively nowhere dense in D for each n and k.

Let $\{I_n : n \in \mathbb{N}\}$ be an enumeration with infinitely many repetitions of the elements of a countable basis $\mathscr{B}$ of D.

Let us fix an arbitrary i and let $F = F_i$.

As the set F is compact, for each n we can define by induction on k an ascending sequence $\{U^n_k : k \in \mathbb{N}\}$ of open sets in C with $\{q_i : i < k\} \subseteq U^n_k \cap \tilde{\mathbb{Q}} \neq \tilde{\mathbb{Q}}$ for every k together with a sequence $\{D^n_k : k \in \mathbb{N}\}$ of non-empty, relatively clopen sets in D such that

(4) $D^n_{k+1} \subseteq D^n_k \subseteq I_n$ and $cl_D((U^n_k + F) \cap D) \cap D^n_k = \emptyset$ for every k.

Now, since $\tilde{X}$ and $\tilde{\mathbb{Q}}$ are the respective images of X and $\mathbb{Q}$ under the homeomorphism h, and $\mathscr{U}_n = \{U^n_k : k \in \mathbb{N}\}$ is an ascending sequence of open sets in C with $\tilde{\mathbb{Q}} \subseteq \bigcup_k U^n_k$ but $\tilde{\mathbb{Q}} \subseteq U^n_k$ for no k, Lemma 2.1 enables us to select for each n a set $V_n = U^n_{k_n}$ such that

(5) $\{V_n : n \in \mathbb{N}\}$ is a γ-cover of $\tilde{X} \setminus Y$ for a certain countable set $Y \subseteq \tilde{X}$.

We will show that

(6) $((\tilde{X} \setminus Y) + F) \cap D$ is meager in D.

To see this, for each m let $K_m = \bigcap_{n \geq m} cl_D((V_n + F) \cap D)$ and let us note that K_m is a closed relatively nowhere dense subset of D. Indeed, any open set from $\mathscr{B}$ is of the form I_n for some $n \geq m$ and $I_n \not\subseteq K_m$ by (4).

Moreover, we have $((\tilde{X} \setminus Y) + F) \cap D \subseteq \bigcup_m K_m$. Indeed, if $c \in \tilde{X} \setminus Y$, then there is m such that $c \in V_n$ for every $n \geq m$ (cf. (5)). Consequently, $(c + F) \cap D \subseteq \bigcap_{n \geq m}((V_n + F) \cap D) \subseteq K_m$, completing the proof of (6).

Let us summarize: for each i we have found a countable set $Y_i \subseteq \tilde{X}$ such that $((\tilde{X} \setminus Y_i) + F_i) \cap D$ is meager in D.

Consequently, letting $\tilde{Y} = \bigcup_i Y_i$ we get a countable subset of C such that $((\tilde{X} \setminus \tilde{Y}) + \bigcup_n F_n) \cap D$ is meager in D.

But since $Z \subseteq \bigcup_n F_n$, we conclude that

(7) $((\tilde{X} \setminus \tilde{Y}) + Z) \cap D$ is meager in D.

On the other hand, $M \setminus m(\tilde{Y}) \subseteq (\tilde{X} \setminus \tilde{Y}) + Z$. Indeed, if $m \in M \setminus m(\tilde{Y})$, then $m = m(x)$ for some $x \in \tilde{X} \setminus \tilde{Y}$ and then $m = (x + (x + m(x))) \in x + Z$. This implies that $((\tilde{X} \setminus \tilde{Y}) + Z) \cap D$ is not meager in D (cf. (2)) contradicting (7) and thus completing the proof of (3).

□

Let us note that under CH we have $\mathrm{non}(\mathscr{M}) = \mathfrak{b} = \aleph_1$ and Case (C) of the proof above establishes the consistency of $\mathbf{PM}_\sigma \neq \mathbf{UM}$ in the way which avoids the use of [20, Theorem 1.1].

3. Countably perfectly null sets

Let us recall that given a perfect Polish space X a set $A \subseteq X$ has *the Hurewicz property*, if for each sequence $\mathscr{U}_1, \mathscr{U}_2, \ldots$ of open covers of A, there are finite subfamilies $\mathcal{F}_n \subseteq \mathscr{U}_n$ such that $A \subseteq \bigcup_n \bigcap_{m \geq n} (\bigcup \mathcal{F}_m)$. If A is a zero-dimensional subspace of X, then by a result of Hurewicz (cf. [13] and [22]) this is equivalent to the statement that every continuous image of A in $\mathbb{N}^\mathbb{N}$ is bounded in the ordering $\leq^*$ of eventual domination.

The smallest cardinality of a subset of $2^\mathbb{N}$ which is nonmeasurable with respect to the standard probability product measure λ on $2^\mathbb{N}$ is denoted by $\mathrm{non}(\mathcal{N})$. It is well-known that if μ is a non-zero, finite, non-atomic, Borel measure on X, then $\mathrm{non}(\mathcal{N})$ is the smallest cardinality of a subset of X not in $\mathcal{N}(X, \mu)$.

Let us also recall that by $\mathbb{Q}$ we denote the copy of the rationals in $2^\mathbb{N}$ consisting of all eventually zero binary sequences.

The following result provides examples of universally null countably perfectly meager sets which are countably perfectly null as well.

Proposition 3.1. *The following collections of sets are countably perfectly null in the respective perfect Polish spaces:*

(1) universally null sets with the Hurewicz property in any perfect Polish space X,
(2) any sets of cardinality less than $\min(\mathit{non}(\mathcal{N}), \mathfrak{b})$ in any perfect Polish space X,
(3) γ-sets in any perfect Polish space X,
(4) $\mathfrak{b}$-scales in $2^\mathbb{N}$,
(5) Hausdorff (ω_1, ω_1^*)-gaps in $\mathcal{P}(\mathbb{N})$.

Proof. (1) Let $A \subseteq X$ be a universally null set with the Hurewicz property and let μ be a non-zero, finite, non-atomic Borel measure on X. Since $A \in \mathbf{UN}$, there is a G_δ-set G in X such that $A \subseteq G$ and $\mu(G) = 0$. Now, since A has the Hurewicz property, there is an F_σ set F in X such that $A \subseteq F \subseteq G$ (cf. [14, Theorem 5.7]). Consequently, $\mu(F) = 0$ which shows that $A \in \mathbf{PN}_\sigma$.

Statements (2) – (4) can be derived from (1) as follows.

(2) Sets of cardinality less than $\mathrm{non}(\mathcal{N})$ are universally null and sets of cardinality less than $\mathfrak{b}$ have the Hurewicz property.

(3) γ-sets are universally null (as they actually have Rothberger's property C'', cf. [8]) and they have the Hurewicz property, by [7, Theorem 2].

(4) Let us assume that A is a $\mathfrak{b}$-scale in $2^{\mathbb{N}}$. Let $B = A \cup \mathbb{Q}$. Then B is a universally null set with the Hurewicz property (see e.g., [20, Example 4.1 and Remark 4.2]), so $B \in \mathbf{PN}_\sigma$ in $2^{\mathbb{N}}$. Consequently, $A \in \mathbf{PN}_\sigma$ in $2^{\mathbb{N}}$.

(5). This may actually be established by a classical argument showing that the Hausdorff gap is universally null, which we sketch here for the sake of completeness. Following the proof of [15, Lemma 20.5], let $\langle\langle a_\alpha : \alpha < \omega_1\rangle, \langle b_\alpha : \alpha < \omega_1\rangle\rangle$ be a Hausdorff gap, $F_\alpha = \{c \in \mathcal{P}(\mathbb{N}) : a_\alpha \subseteq^* c \subseteq^* b_\alpha\}$ for $\alpha < \omega_1$ and let μ be a non-zero, finite, non-atomic Borel measure on $\mathcal{P}(\mathbb{N})$. Then F_α's are F_σ-sets in $\mathcal{P}(\mathbb{N})$ and for a sufficiently large ξ we have $\mu(F_\xi) = 0$ (see [15, the proof of Lemma 20.5]). Letting

$$F = F_\xi \cup \{a_\alpha : \alpha < \xi\} \cup \{b_\alpha : \alpha < \xi\},$$

we get an F_σ-set with $\{a_\alpha : \alpha < \omega_1\} \cup \{b_\alpha : \alpha < \omega_1\} \subseteq F$ and $\mu(F) = 0$ which shows that $\{a_\alpha : \alpha < \omega_1\} \cup \{b_\alpha : \alpha < \omega_1\} \in \mathbf{PN}_\sigma$ in $\mathcal{P}(\mathbb{N})$.

□

The main result of this section is a measure counterpart of [20, Theorem 1.1] and Theorem 2.2.

Theorem 3.2. *If either*

(a) there exists a universally null set in $2^{\mathbb{N}}$ *of cardinality* $2^{\aleph_0}$

or

(b) $2^{\aleph_0} \le \aleph_2$*,*

then there is a universally null set in $2^{\mathbb{N}}$ *which is not countably perfectly null in* $2^{\mathbb{N}}$*.*

Proof. (a) Let T be a universally null set in $2^{\mathbb{N}}$ of cardinality $2^{\aleph_0}$.

By Proposition 1.2, it suffices to show that there is also one which is not countably perfectly meager.

Let us recall that by [20, Theorem 1.1], there exist a set $H \subseteq T \times 2^{\mathbb{N}}$ intersecting each vertical section $\{t\} \times 2^{\mathbb{N}}$, $t \in T$, in a singleton and a homeomorphic copy E of H in $2^{\mathbb{N}}$ which is not a $\mathbf{PM}_\sigma$-set in $2^{\mathbb{N}}$. Now, since T is universally null, so is E as a preimage of T under a continuous injective function.

(b) If $2^{\aleph_0} = \aleph_1$, then any Luzin set in $2^{\mathbb{N}}$ provides an example of a non-meager, universally null set.

From now on let us assume that $2^{\aleph_0} = \aleph_2$.

Following closely the scheme of proof of the Theorem 2.2, we split the argument into three cases.

Case (A): $\mathrm{non}(\mathcal{N}) = \aleph_2$.

Then, by a theorem of Grzegorek (see [9]), there exists a universally null set in $2^{\mathbb{N}}$ of cardinality $\aleph_2 = 2^{\aleph_0}$ and the result follows from part (a).

Case (B): $\mathfrak{b} = \aleph_2$.

In this case any $\mathfrak{b}$-scale in $2^{\mathbb{N}}$ is a universally null set of cardinality $\mathfrak{b} = 2^{\aleph_0}$ and the result again follows from part (a).

Case (C): $\mathrm{non}(\mathcal{N}) = \mathfrak{b} = \aleph_1$.

As in the proof of Theorem 2.2, we fix copies $C,\ D$ of the Cantor set in $2^{\mathbb{N}}$ such that

(1) the operation $+$ of addition is a homeomorphism between $C \times D$ and $C + D$ (cf. [21]),

a homeomorphism $h : 2^{\mathbb{N}} \to C$, a $\mathfrak{b}$-scale X in $2^{\mathbb{N}}$ and we let $\tilde{X} = h(X)$. Since X is universally null, so is $\tilde{X}$.

We also fix a homeomorphism $g : 2^{\mathbb{N}} \to D$ and we define a Borel measure μ on $2^{\mathbb{N}}$ by letting

$$\mu(B) = \lambda(g^{-1}(B \cap D)), \quad \text{for} \quad B \in \mathbf{B}(2^{\mathbb{N}}).$$

Then we fix a set $M \subseteq D$ of cardinality $\mathrm{non}(\mathcal{N}) = \aleph_1$ with

(2) $\mu^*(M) > 0$,

we let $m : \tilde{X} \to M$ be a surjection onto M and we put $H = \{(x, m(x)) : x \in \tilde{X}\}$. Since $\tilde{X}$ is the injective continuous image of H under the projection onto the first axis and $\tilde{X}$ is universally null, so is H.

Finally, let $Z = \{x + m(x) :\ x \in \tilde{X}\}$. Clearly, Z is universally null as the image of H under the homeomorphism $+$ between $C \times D$ and $C + D$ (cf. (1)).

We shall show that on the other hand

(3) Z is not a $\mathbf{PN}_\sigma$-set in $2^{\mathbb{N}}$,

thus completing the proof of the theorem.

To that end, let $\tilde{\mathbb{Q}} = h(\mathbb{Q}) = \{q_n : n \in \mathbb{N}\}$ and let us suppose, towards a contradiction, that there are closed μ-null sets F_n in $2^{\mathbb{N}}$ such that $Z \subseteq \bigcup_n F_n$ and $\mu(q_k + F_n) = 0$ for each n and k (cf. Proposition 1.1.)

Let us fix an arbitrary $\varepsilon > 0$.

For each n, F_n being compact and μ-null, there is an open set U_n in C such that $\tilde{\mathbb{Q}} \subseteq U_n$ and

(4) $\mu(U_n + F_n) < \frac{\varepsilon}{2^{n+1}}$.

Now, X being a $\mathfrak{b}$-scale in $2^{\mathbb{N}}$, is $\mathfrak{b}$-concentrated on $\mathbb{Q}$ (see [25, Lemma 2.10]). Consequently, $\tilde{X}$ is $\mathfrak{b}$-concentrated on $\tilde{\mathbb{Q}}$ which, taking into account that $\mathfrak{b} = \aleph_1$, implies that for each n there is a countable set $Y_n \subseteq \tilde{X}$ such that $\tilde{X} \setminus Y_n \subseteq U_n$. It follows (cf. (4)) that $\mu^*((\tilde{X} \setminus Y_n) + F_n) < \frac{\varepsilon}{2^{n+1}}$ which implies that, letting $F = \bigcup_n F_n$ and $\tilde{Y} = \bigcup_n Y_n$, we have $\mu^*((\tilde{X} \setminus \tilde{Y}) + F) < \varepsilon$. But since $Z \subseteq F$ and the choice of ε was arbitrary, we conclude that

(5) $\mu^*((\tilde{X} \setminus \tilde{Y}) + Z) = 0$.

On the other hand, exactly as in the proof of Theorem 2.2, we have $M \setminus m(\tilde{Y}) \subseteq (\tilde{X} \setminus \tilde{Y}) + Z$ which, $\tilde{Y}$ being countable, implies that $\mu^*((\tilde{X} \setminus \tilde{Y}) + Z) > 0$ (cf. (2)), contradicting (5) and thus completing the proof of (3).

□

4. Remarks and open problems

The results of Sections 2 and 3 motivate the following questions. The first two are directly related to Theorems 2.2 and 3.2, respectively.

Problem 1. Is $\mathbf{PM}_\sigma = \mathbf{UM}$ consistent?

Problem 2. Is $\mathbf{PN}_\sigma = \mathbf{UN}$ consistent?

Let us note that we consistently have $\mathbf{PM}_\sigma \subseteq \mathbf{UN}$ since in the model obtained by adding $\aleph_2$ Cohen reals to a model of GCH we have $\mathbf{UM} \subseteq \mathbf{UN}$ (see Corazza [6, Theorem 0.6(b)] and Miller [17]; by a theorem of Bartoszyński and Shelah, cf [4, Theorem 3], it is consistently true that even all perfectly meager sets are universally null). By the fact that $\mathbf{PN}_\sigma \subseteq \mathbf{PM}_\sigma$ (see Proposition 1.2), the dual statement that $\mathbf{PN}_\sigma \subseteq \mathbf{UM}$ is just true but the following question remains open.

Problem 3. Is $\mathbf{PN}_\sigma = \mathbf{PM}_\sigma$ consistent?

Finally, in view of the inclusion $\mathbf{PN}_\sigma \subseteq \mathbf{PM}_\sigma \cap \mathbf{UN}$ one may ask

Problem 4. Is $\mathbf{PN}_\sigma = \mathbf{PM}_\sigma \cap \mathbf{UN}$ true/consistent?

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