

COUNTABLY PERFECTLY MEAGER SETS

R. POL AND P. ZAKRZEWSKI

ABSTRACT. We study a strengthening of the notion of a perfectly meager set.

We say that a subset A of a perfect Polish space X is countably perfectly meager in X , if for every sequence of perfect subsets $\{P_n : n \in \mathbb{N}\}$ of X , there exists an F_σ -set F in X such that $A \subseteq F$ and $F \cap P_n$ is meager in P_n for each n .

We give various characterizations and examples of countably perfectly meager sets. We prove that not every universally meager set is countably perfectly meager correcting an earlier result of Bartoszyński.

1. INTRODUCTION

Let us recall that a subset A of a perfect Polish space X is *universally meager* ($A \in \mathbf{UM}$, see [24], [25], [2], [3]), if for every Borel isomorphism f between X and any perfect Polish space Y the image of A under f is meager in Y (this class of sets was earlier introduced and studied by Grzegorek [8], [9], [10] under the name of *absolutely of the first category* sets).

Let us also recall that A is *perfectly meager* ($A \in \mathbf{PM}$), if for all perfect subsets P of X , the set $A \cap P$ is meager in P . Clearly, $A \in \mathbf{PM}$ if and only if for every perfect subset P of X , there exists an F_σ -set F in X such that $A \subseteq F$ and $F \cap P$ is meager in P (cf. [3, Theorem 6]).

We shall say that A is *countably perfectly meager* ($A \in \mathbf{PM}_\sigma$), if for every sequence of perfect subsets $\{P_n : n \in \mathbb{N}\}$ of X , there exists an F_σ -set F in X such that $A \subseteq F$ and $F \cap P_n$ is meager in P_n for each n .

It follows directly from the definition that the class \mathbf{PM}_σ is, exactly as the other two classes, a σ -ideal of subsets of the underlying space X (shortly: a σ -ideal on X), i.e., it is hereditary, closed under taking countable unions and contains all singletons.

One readily checks that $\mathbf{UM} \subseteq \mathbf{PM}$ and it is consistent that $\mathbf{UM} \subsetneq \mathbf{PM}$ but also that $\mathbf{UM} = \mathbf{PM}$ (see [2]).

Bartoszyński [3, Theorem 7, (3) \Rightarrow (2) \Rightarrow (1)] proved that $\mathbf{PM}_\sigma \subseteq \mathbf{UM}$. Actually, it is in that paper where the property used by us to define the class \mathbf{PM}_σ first appeared (without any specific name)

Date: March 19, 2021.

2010 Mathematics Subject Classification. 03E20, 03E15, 54E52.

Key words and phrases. perfectly meager set, universally meager set, perfectly meager set in the transitive sense, λ' -set, σ -ideal.

and where it was claimed that this property characterizes universally meager sets in the Cantor space $2^{\mathbb{N}}$. Unfortunately, there is a flaw in the part of the argument showing the inclusion $\mathbf{UM} \subseteq \mathbf{PM}_{\sigma}$ (cf. [3, Theorem 7, (1) \Rightarrow (3)]).

In fact, the following theorem immediately implies that it is consistent (in particular, true under \mathbf{CH}) that there exists a universally meager subset of $2^{\mathbb{N}}$ which is not countably perfectly meager in $2^{\mathbb{N}}$.

Theorem 1.1. *Let T be a subset of $2^{\mathbb{N}}$ of cardinality 2^{\aleph_0} . There exist a set $H \subseteq T \times 2^{\mathbb{N}}$ intersecting each vertical section $\{t\} \times 2^{\mathbb{N}}$, $t \in T$, in a singleton and a homeomorphic copy E of H in $2^{\mathbb{N}}$ which is not a \mathbf{PM}_{σ} -set in $2^{\mathbb{N}}$. In particular, T is a continuous injective image of E .*

Under the notation from Theorem 1.1 it follows that if T is universally meager then H is universally meager as well (\mathbf{UM} being closed with respect to preimages under continuous injections (see [24]) and so is its homeomorphic copy E). A refinement of this argument also shows that, at least consistently, in contrast to both \mathbf{PM} and \mathbf{UM} the class \mathbf{PM}_{σ} is not closed with respect to homeomorphic images (see Theorem 3.1(3)). Consequently, unlike in the case of \mathbf{PM} and \mathbf{UM} , the statement that a subspace A of a Polish space is countably perfectly meager makes sense only if we specify a Polish space X in which it is embedded. We shall therefore speak of *countably perfectly meager sets in X* unless X is clear from the context.

In Section 2 we present various characterizations and some examples of countably perfectly meager sets in $2^{\mathbb{N}}$. These include (for the definitions see Section 2.2):

- sets perfectly meager in the transitive sense, in particular:
 - γ -sets,
 - strongly meager sets.
- sets with the Hurewicz property and no perfect subsets,
- λ' -sets.

Section 3 is largely devoted to a proof of Theorem 1.1. We derive from it various examples of subsets of $2^{\mathbb{N}}$ which are universally meager but not countably perfectly meager in $2^{\mathbb{N}}$.

In particular, if there is a λ -set in $2^{\mathbb{N}}$ of cardinality of the continuum, then there is also one which is not countably perfectly meager in $2^{\mathbb{N}}$ (see Theorem 3.1(2)).

Moreover, if there exists a λ' -set in $2^{\mathbb{N}}$ of cardinality of the continuum, then there is also one whose homeomorphic copy is not countably perfectly meager in $2^{\mathbb{N}}$ (see Theorem 3.1(3)). This strengthens the result of Sierpiński [21] that λ' -property is not invariant under homeomorphisms.

We end Section 3 with a proof based on one of the characterizations of Section 2 that the class \mathbf{PM}_{σ} is closed under products in the sense that

if A and B are \mathbf{PM}_σ -sets in perfect Polish spaces X and Y , respectively, then $A \times B$ is a \mathbf{PM}_σ -set in $X \times Y$ (see Theorem 3.2).

In Section 4 we gather some additional comments.

In Subsection 4.1 we present an example of a countably perfectly meager set in $2^{\mathbb{N}}$ which has neither the Hurewicz property nor λ' -property (see Example 4.1). We also give an example of a countably perfectly meager set in $2^{\mathbb{N}}$ which is not perfectly meager in the transitive sense (see Example 4.4).

Subsection 4.2 contains remarks on some σ -ideals related to the classes \mathbf{PM} and \mathbf{PM}_σ .

We thank the referee for the comments which improved the exposition of this paper.

2. CHARACTERIZATIONS AND EXAMPLES OF COUNTABLY PERFECTLY MEAGER SETS

2.1. Characterizations of countably perfectly meager sets.

In this subsection A is always a subset of a perfect (i.e., with no isolated points) Polish (i.e., a separable completely metrizable) topological space X and \mathbf{PM}_σ is the family of all subsets of X which are countably perfectly meager in X .

Let us recall that A is an s_0 -set if for every perfect (i.e., non-empty, closed and with no isolated points) set P there is a copy of the Cantor set $K \subseteq P$ with $K \cap A = \emptyset$. Clearly, every perfectly meager set has property s_0 .

Theorem 2.1. *The following are equivalent:*

- (1) $A \in \mathbf{PM}_\sigma$.
- (2) For every sequence $\{K_n : n \in \mathbb{N}\}$ of copies of the Cantor set in X there is an F_σ -set F in X such that $A \subseteq F$ and $K_m \not\subseteq F$ for each $m \in \mathbb{N}$.
- (3) For every sequence $\{K_n : n \in \mathbb{N}\}$ of copies of the Cantor set in X there are closed sets F_n in X such that $A \subseteq \bigcup_n F_n$ and $K_m \not\subseteq F_n$ for each $m, n \in \mathbb{N}$.
- (4) For every sequence $\{K_n : n \in \mathbb{N}\}$ of pairwise disjoint copies of the Cantor set in X there are closed sets F_n in X such that $A \subseteq \bigcup_n F_n$ and $K_m \not\subseteq F_n$ for each $m, n \in \mathbb{N}$.
- (5) A is an s_0 -set and for every sequence $\{K_n : n \in \mathbb{N}\}$ of pairwise disjoint and disjoint from A copies of the Cantor set in X there are closed sets F_n in X such that $A \subseteq \bigcup_n F_n$ and $K_m \not\subseteq F_n$ for each $m, n \in \mathbb{N}$.

Proof. The implications (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) are obvious.

To prove that (4) \Rightarrow (5), it only suffices to show that if (4) holds, then A is an s_0 -set. To see this, let us fix a perfect set P and let

K_0, K_1, \dots be pairwise disjoint copies of the Cantor set in P such that each non-empty relatively open set in P contains some K_n . Now, by (4), $A \subseteq \bigcup_n F_n$ for some closed sets F_n in X such that no F_n covers any K_m . It follows that $F_n \cap P$ is nowhere dense in P for each n and so there exists a perfect set $K \subseteq P$ disjoint from $\bigcup_n F_n$. Then $K \cap A = \emptyset$ as well.

To prove that (5) \Rightarrow (1), we shall need the following simple observation.

Claim 2.2. *For every sequence of perfect sets $\{P_n : n \in \mathbb{N}\}$ in X , there is a sequence of pairwise disjoint Cantor sets $\{K_n : n \in \mathbb{N}\}$ with $K_n \subseteq P_n$ for each n .*

Indeed, first pick points $x_i \in P_i$ with $x_i \neq x_j$ for $i \neq j$, and then choose successively the Cantor sets K_n in P_n disjoint from K_i , for $i < n$, and $\{x_j : j > n\}$.

Now let us assume (5) and let $\{P_n : n \in \mathbb{N}\}$ be a sequence of perfect subsets of X .

For each n let $\{U_m^n : m \in \mathbb{N}\}$ be a basis of non-empty relatively open subsets of P_n and for each m let us pick a Cantor set $K_m^n \subseteq U_m^n$. By the claim and the fact that A is an s_0 -set we may assume that the sets K_m^n , $n, m \in \mathbb{N}$, are pairwise disjoint and disjoint from A .

By (5), there are closed sets F_i in X such that $A \subseteq \bigcup_i F_i$ and $K_m^n \not\subseteq F_i$ for each $m, n, i \in \mathbb{N}$. Letting $F = \bigcup_i F_i$ we easily conclude that $F \cap P_n$ is meager in P_n for each n . □

As a corollary let us formulate a characterization of countably perfectly meager sets stated and partially proved by Bartoszyński in [2].

Theorem 2.3. *The following are equivalent:*

- (1) $A \in \mathbf{PM}_\sigma$.
- (2) *For every sequence of countable dense-in-itself sets $\{A_n : n \in \mathbb{N}\}$ there are sets $B_n \subseteq A_n$ such that $\overline{A_n} = \overline{B_n}$ for each $n \in \mathbb{N}$ and $\bigcup_n B_n$ is a G_δ -set relative to $A \cup \bigcup_n A_n$.*

Proof. (1) \Rightarrow (2) was proved by Bartoszyński [2, Theorem 7, (3) \Rightarrow (2)].

(2) \Rightarrow (1). To prove that $A \in \mathbf{PM}_\sigma$, we shall check that A satisfies condition (5) of Theorem 2.1.

First note that $A \in \mathbf{PM}$ (for a proof see [2, Theorem 6, (2) \Rightarrow (1)]); in fact, Bartoszyński [2, Theorem 7, (2) \Rightarrow (1)] proved that $A \in \mathbf{UM}$ which implies that A is an s_0 -set.

Let $\{K_n : n \in \mathbb{N}\}$ be a sequence of (pairwise disjoint but this assumption is superfluous) copies of the Cantor set disjoint from A . We shall show that there is an F_σ -set F such that $A \subseteq F$ and $K_n \setminus F \neq \emptyset$ for each n . To this end, for each n let us choose a countable dense subset A_n of K_n . By (2), there are sets $B_n \subseteq A_n$ such that $\overline{B_n} = K_n$ for each

$n \in \mathbb{N}$ and a G_δ -subset G of X such that $(A \cup \bigcup_n A_n) \cap G = \bigcup_n B_n$. Let $F = X \setminus G$. Then $A \subseteq F$ and for each n we have $B_n \subseteq K_n \setminus F$, so F is as required. \square

Remark 2.4. *The characterization of \mathbf{PM}_σ -sets from Theorem 2.3 should be compared with the following characterization of \mathbf{PM} -sets by Bennett, Hosobuchi, and Lutzer [1]. (for a short proof see [2, Theorem 6]):*

The following are equivalent:

- (1) $A \in \mathbf{PM}$.
- (2) *For every countable dense-in-itself set A_0 there exists a set $B_0 \subseteq A$ such that $\overline{A_0} = \overline{B_0}$ and B_0 is a G_δ -set relative to $A \cup A_0$.*

Our last characterization of \mathbf{PM}_σ -sets is closely related to a characterization of \mathbf{UM} sets (cf. Remark 2.7).

Theorem 2.5. *The following are equivalent:*

- (1) $A \in \mathbf{PM}_\sigma$.
- (2) *For every continuous bijection $f : \mathbb{N}^\mathbb{N} \rightarrow X$ there are closed sets F_n in X such that $A \subseteq \bigcup_n F_n$ and $f^{-1}(F_n)$ is nowhere dense in $\mathbb{N}^\mathbb{N}$ for each $n \in \mathbb{N}$.*

Proof. (1) \Rightarrow (2). Since open sets in $\mathbb{N}^\mathbb{N}$ are mapped by f onto Borel sets, one can choose a sequence $\{B_n : n \in \mathbb{N}\}$ of Borel subsets of X such that $\{f^{-1}(B_n) : n \in \mathbb{N}\}$ is a basis of the topology of $\mathbb{N}^\mathbb{N}$. For each n let us pick a Cantor set $K_n \subseteq B_n$. Since $A \in \mathbf{PM}_\sigma$, there are closed sets F_n in X such that $A \subseteq \bigcup_n F_n$ and $K_m \not\subseteq F_n$ for each $m, n \in \mathbb{N}$. It readily follows that $f^{-1}(F_n)$ is nowhere dense in $\mathbb{N}^\mathbb{N}$ for each $n \in \mathbb{N}$.

(2) \Rightarrow (1). To prove that $A \in \mathbf{PM}_\sigma$, we shall check condition (4) of Theorem 2.1.

Let $\{K_n : n \in \mathbb{N}\}$ be a sequence of pairwise disjoint Cantor sets in X .

A key observation is the following fact.

Claim 2.6. *There is a continuous bijection $f : \mathbb{N}^\mathbb{N} \rightarrow X$ such that $f^{-1}(K_n)$ is open in $\mathbb{N}^\mathbb{N}$ for each $n \in \mathbb{N}$.*

To prove the claim, let τ be the (perfect Polish) topology of X and let us first extend τ to the topology τ' whose basic open sets are elements of τ and relatively open subsets of K_n 's. More precisely, let τ_n be the topology generated by $\tau \cup \{K_n\}$ and then let τ' be the topology generated by $\bigcup_n \tau_n$. This topology is Polish (cf. [12, 13.A]) and it is easy to check that it is also perfect. It follows (cf. [12, 7.15]) that there is a bijection $f : \mathbb{N}^\mathbb{N} \rightarrow X$ which is continuous in the sense of τ' hence also in the sense of τ .

Having proved the claim we may now apply the assumption about A to find closed sets F_n in X such that $A \subseteq \bigcup_n F_n$ and $f^{-1}(F_n)$ is

nowhere dense in $\mathbb{N}^{\mathbb{N}}$ for each $n \in \mathbb{N}$. But $f^{-1}(K_m)$ being open in $\mathbb{N}^{\mathbb{N}}$ we conclude that $K_m \not\subseteq F_n$ for each $m, n \in \mathbb{N}$. This shows that A satisfies condition (4) of Theorem 2.1 completing the proof. \square

Remark 2.7. *The characterization of \mathbf{PM}_{σ} -sets from Theorem 2.5 should be compared with the following characterization of \mathbf{UM} -sets (cf. [25, Theorem 2.4]):*

The following are equivalent:

- (1) $A \in \mathbf{UM}$.
- (2) *For every continuous bijection $f : \mathbb{N}^{\mathbb{N}} \rightarrow X$ there are sets F_n in X such that $A \subseteq \bigcup_n F_n$ and $f^{-1}(F_n)$ is closed and nowhere dense in $\mathbb{N}^{\mathbb{N}}$ for each $n \in \mathbb{N}$.*

In particular, in view of Theorem 2.5, this gives another proof of the inclusion $\mathbf{PM}_{\sigma} \subseteq \mathbf{UM}$.

2.2. Examples of countably perfectly meager sets. Let us recall that a subset A of $2^{\mathbb{N}}$ is *perfectly meager in the transitive sense* ($A \in \mathbf{AFC}'$, cf. [15], [16] and [23]) if for every perfect subset P of $2^{\mathbb{N}}$, there exists an F_{σ} -set F in X such that $A \subseteq F$ and $F \cap (P + t)$ is meager in $P + t$ for each $t \in 2^{\mathbb{N}}$ or, equivalently (cf. [16, Lemma 6]), if for every sequence $\{K_n : n \in \mathbb{N}\}$ of copies of the Cantor set in $2^{\mathbb{N}}$ there are closed sets F_n in $2^{\mathbb{N}}$ such that $A \subseteq \bigcup_n F_n$ and $K_m + t \not\subseteq F_n$ for each $m, n \in \mathbb{N}$ and $t \in 2^{\mathbb{N}}$. Combining this with Theorem 2.1 we get the following result which somewhat strengthens the fact that $\mathbf{AFC}' \subseteq \mathbf{UM}$ established by Nowik and Weiss [16, Theorem 2] by a similar argument.

Theorem 2.8. *Every subset of $2^{\mathbb{N}}$ which is perfectly meager in the transitive sense is countably perfectly meager in $2^{\mathbb{N}}$.*

As a corollary we obtain a list of some classical classes of sets which being perfectly meager in the transitive sense are countably perfectly meager as well.

Corollary 2.9. *The following collections of subsets of $2^{\mathbb{N}}$ are countably perfectly meager in $2^{\mathbb{N}}$:*

- (1) *meager-additive sets,*
- (2) *γ -sets,*
- (3) *strongly meager sets,*
- (4) *Sierpiński sets.*

Proof. (1). See [26, Proposition 6.6]

(2). See [15]. This also follows from (1), since by [17, Proposition 3.7], every γ -set is meager-additive.

(3). See [15, Theorem 9].

(4). This follows from (3), since Pawlikowski [18] proved that every Sierpiński set is strongly meager. \square

The following result gives more examples of universally meager sets which are countably perfectly meager as well.

Let us recall that given a perfect Polish space X a set $A \subseteq X$

- has the *Hurewicz property*, if every continuous image of A in $\mathbb{N}^{\mathbb{N}}$ is bounded in the ordering \leq^* of eventual domination,
- is a λ' -set in X if every countable set $D \subseteq X$ is relatively G_δ in $A \cup D$.

The cardinal number \mathfrak{b} is the minimal cardinality of a subset of $\mathbb{N}^{\mathbb{N}}$ which is unbounded in the ordering \leq^* .

Proposition 2.10. *The following collections of sets are countably perfectly meager in the respective spaces:*

- (1) subsets of a perfect Polish space X with the Hurewicz property and no perfect subsets; in particular, subsets of X of cardinality less than \mathfrak{b} ,
- (2) λ' -subsets in a perfect Polish space X , in particular:
 - (a) sets in $\mathbb{N}^{\mathbb{N}}$ of the form $\{f_\alpha : \alpha < \mathfrak{b}\}$ where
 - $\alpha < \beta < \mathfrak{b}$ implies $f_\alpha <^* f_\beta$,
 - for every $f \in \mathbb{N}^{\mathbb{N}}$ there is $\alpha < \mathfrak{b}$ with $f_\alpha \not\leq^* f$.
 - (b) Hausdorff (ω_1, ω_1^*) -gaps in $\mathcal{P}(\mathbb{N})$.

Proof. (1). This was actually shown in Proposition 2.3 of Zakrzewski [24].

(2). Let A be a λ' -set in X . To prove that $A \in \mathbf{PM}_\sigma$, we shall check condition (5) of Theorem 2.1. Clearly, A is an s_0 -set. For a sequence $\{K_n : n \in \mathbb{N}\}$ of pairwise disjoint and disjoint from A copies of the Cantor set in X for each n let us pick a point $d_n \in K_n$ and let $D = \{d_n : n \in \mathbb{N}\}$. Then, A being a λ' -set, there is an F_σ set F in X such that $A \subseteq F$ and $F \cap D = \emptyset$, so d_n witnesses that $K_n \not\subseteq F$ for any $n \in \mathbb{N}$.

Sets described in (a) and (b) are classical examples of λ' sets due to Rothberger and Hausdorff (see [14]).

□

Remark 2.11. *An easier way of proving that every Sierpiński set in $2^{\mathbb{N}}$ is \mathbf{PM}_σ (cf. Corollary 2.9) is to combine Proposition 2.10(1) with Theorem 7 of Fremlin and Miller [7] which states that every Sierpiński set has the Hurewicz property.*

Likewise, another way of proving that every γ -set in $2^{\mathbb{N}}$ is in \mathbf{PM}_σ (cf. Corollary 2.9) is to combine Proposition 2.10(1) with Theorem 2 of Galvin and Miller [6] which states that every γ -set has the Hurewicz property.

On the other hand, the set described in Proposition 2.10(2)(a), is λ' in $\mathbb{N}^{\mathbb{N}}$ but does not have the Hurewicz property as an unbounded subset of $\mathbb{N}^{\mathbb{N}}$. Likewise, not every subset of $2^{\mathbb{N}}$ with the Hurewicz property

and no perfect subsets (cf. Proposition 2.10(1)) is a λ' -set in $2^{\mathbb{N}}$, see Example 4.1 in the comment section below.

3. \mathbf{PM}_σ VERSUS \mathbf{UM}

Theorem 1.1, which we are now going to prove, reveals an essential difference between the classes \mathbf{UM} and \mathbf{PM}_σ .

Proof of Theorem 1.1. Let C_0, C_1, \dots be pairwise disjoint meager Cantor sets in $2^{\mathbb{N}}$ such that:

- (1) each non-empty open set in $2^{\mathbb{N}}$ contains some C_n .

Let $P = 2^{\mathbb{N}} \setminus \bigcup_n C_n$.

We shall justify the theorem in three steps (A), (B) and (C).

(A) We claim that there exists a set $H \subseteq T \times P$ intersecting each vertical section $\{t\} \times P$, $t \in T$, in a singleton, such that each F_σ -set in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ containing H contains also $\{t\} \times V$ for some $t \in T$ and a non-empty open set V in $2^{\mathbb{N}}$.

Indeed, let $\{F_t : t \in T\}$ be a parametrization on T of all F_σ -sets in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$. For each $t \in T$, we pick $(t, \varphi(t)) \in (\{t\} \times P) \setminus F_t$, whenever this is possible, and we let $\varphi(t)$ be an arbitrary fixed element of P , otherwise.

Let us check that the graph $H = \{(t, \varphi(t)) : t \in T\}$ has the required property.

Let F be an F_σ -set in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ containing H , and let $t \in T$ be such that $F = F_t$. Then $(t, \varphi(t)) \in F_t$, hence F_t contains $\{t\} \times P$. Consequently, P being a dense G_δ -set in $2^{\mathbb{N}}$, the Baire category theorem provides a non-empty open set V in $2^{\mathbb{N}}$ with $\{t\} \times V \subseteq F_t$, completing the proof of the claim.

(B) For any $s \in 2^{<\mathbb{N}}$ let $N_s = \{x \in 2^{\mathbb{N}} : s \subseteq x\}$ be the standard basic open set in $2^{\mathbb{N}}$ determined by s .

Let \sim be the equivalence relation on $2^{\mathbb{N}} \times 2^{\mathbb{N}}$, whose equivalence classes are given by:

$$[(x, y)]_\sim = \begin{cases} N_{x|n} \times \{y\}, & \text{if } y \in C_n, \\ \{(x, y)\}, & \text{if } y \in P \end{cases}$$

Let $\pi(x, y) = [(x, y)]_\sim$ be the quotient map onto the quotient space $K = (2^{\mathbb{N}} \times 2^{\mathbb{N}}) / \sim$ (whose topology consists of sets $U \subseteq K$ such that $\pi^{-1}(U)$ is open in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$).

Claim. The space K is homeomorphic to $2^{\mathbb{N}}$.

The equivalence classes of \sim form an upper-semicontinuous decomposition of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ (i.e., the saturation of every closed set in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ is closed). It follows that the decomposition space $K = 2^{\mathbb{N}} \times 2^{\mathbb{N}} / \sim$ is metrizable (cf. [5, Theorem 4.2.13]). Moreover, K is compact, zero-dimensional and has no isolated points, and hence the claim follows.

However, for reader's convenience, we shall provide a direct argument to that effect, avoiding the metrization theorem.

Let \mathcal{B} consist of clopen subsets of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ of the form $N_s \times N_t$, where $s, t \in 2^{<\mathbb{N}}$ and $N_t \cap C_k = \emptyset$ for each $k < \text{length}(s)$. We shall show that $\{\pi(B) : B \in \mathcal{B}\}$ is a countable basis for K consisting of clopen sets.

First, let us note that each set from \mathcal{B} is saturated, i.e., is the union of equivalence classes. Indeed, if $(x, y) \in N_s \times N_t$, $N_t \cap \bigcup_{k < n} C_k = \emptyset$ and $n = \text{length}(s)$, then either $y \in P$ and then $[(x, y)]_{\sim} = \{(x, y)\}$ or $y \in C_m$ for some $m \geq n$ which implies that $[(x, y)]_{\sim} = N_{x|m} \times \{y\} \subseteq N_s \times N_t$.

It follows that for each $B \in \mathcal{B}$, $\pi(B)$ and $\pi((2^{\mathbb{N}} \times 2^{\mathbb{N}}) \setminus B)$ are disjoint open sets in K , hence $\pi(B)$ is clopen in K .

Next, let us fix an open set W in K and let $c = \pi(x, y) \in W$. Then, since $\pi^{-1}(W)$ is open in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ and $(x, y) \in \pi^{-1}(W)$, we have $N_{x|n} \times N_{y|m} \subseteq \pi^{-1}(W)$ for some n and m . Moreover, if $y \in \bigcup_k C_k$, then we additionally assume that n is the unique k for which $y \in C_k$ (let us note that in this case $(x, y) \in N_{x|n} \times \{y\} \subseteq \pi^{-1}(W)$). In any case, $y \notin \bigcup_{k < n} C_k$ and $\bigcup_{k < n} C_k$ being closed, there is large enough $m' \geq m$ for which $N_{y|m'} \cap \bigcup_{k < n} C_k = \emptyset$. Then $B = N_{x|n} \times N_{y|m'} \in \mathcal{B}$ and $\pi(B)$ is a neighbourhood of c contained in W .

We have checked that $\{\pi(B) : B \in \mathcal{B}\}$ is a countable basis for K consisting of clopen sets. Clearly, no $\pi(B)$ is a singleton, hence K has no isolated points.

Finally, the equivalence classes of \sim are closed in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$, hence the singletons of K are closed.

It follows that the space K , being T_1 and having a basis consisting of clopen sets, is also Hausdorff and it is compact as a continuous image of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$. Consequently, being a compact, Hausdorff, second countable, zero-dimensional topological space without isolated points, K is homeomorphic to $2^{\mathbb{N}}$, which completes the proof of the claim.

Let us also note that the sets

$$(2) P_s = \pi(N_s \times C_n),$$

where $s \in 2^{<\mathbb{N}}$ and $n = \text{length}(s)$, are perfect subsets of K .

(C) Finally, let $E = \pi(H)$ (cf. (A)). Clearly, E is a homeomorphic copy of H in K and T is the injective image of E under the continuous function $\text{proj}_1 \circ \pi^{-1}|_E$, where proj_1 is the projection of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ onto the first axis.

We shall show that

$$(3) E \text{ is not a } \mathbf{PM}_{\sigma}\text{-set in } K.$$

To that end, let us consider an F_{σ} -set F^* in K such that $E \subseteq F^*$. Then

$$(4) F = \pi^{-1}(F^*)$$

is an F_σ -set in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ containing H , so there are $t \in T$ and a non-empty open set V in $2^{\mathbb{N}}$ such $\{t\} \times V \subseteq F$, cf. (A).

Let us fix $C_n \subseteq V$ (cf. (1)) and let $s = t|n$ be the unique sequence in 2^n such that $t \in N_s$. We have $\{t\} \times C_n \subseteq F$ and let us notice that $\pi(\{t\} \times C_n) = P_s$ (cf. (2)).

Consequently, $P_s \subseteq F^*$, cf. (4).

It follows that any F_σ -set in K containing E also contains some P_s , which confirms (3), completing the proof of the theorem. \square

As a corollary we have the following result which shows that, at least consistency-wise, the classes **UM** and **PM** $_\sigma$ are different (part (1)).

Its part (2) strengthens the result of Nowik, Scheepers and Weiss [15] that assuming the Continuum Hypothesis there is a λ -set in $2^{\mathbb{N}}$ which is not perfectly meager in the transitive sense (A a λ -set if every countable set $D \subseteq A$ is relatively G_δ in A).

Part (3) strengthens the result of Sierpiński [21] that (assuming the continuum hypothesis) λ' -property is not invariant under homeomorphisms.

Theorem 3.1.

- (1) *If there exists a universally meager set in $2^{\mathbb{N}}$ of cardinality of the continuum, then there is also one which is not countably perfectly meager.*
- (2) *If there exists a λ -set in $2^{\mathbb{N}}$ of cardinality of the continuum, then there is also one which is not countably perfectly meager.*
- (3) *If there exists a λ' -set in $2^{\mathbb{N}}$ of cardinality of the continuum, then there is also one whose homeomorphic copy in $2^{\mathbb{N}}$ is not countably perfectly meager. In particular, the class **PM** $_\sigma$ is not closed with respect to homeomorphic images.*

In particular, assuming the continuum hypothesis there is a λ' -set in $2^{\mathbb{N}}$ whose homeomorphic copy in $2^{\mathbb{N}}$ is not countably perfectly meager.

Proof. We keep the notation from the proof of Theorem 1.1. For a set $T \subseteq 2^{\mathbb{N}}$ of cardinality 2^{\aleph_0} let $H \subseteq T \times P$ and $E \subseteq 2^{\mathbb{N}}$ satisfy the assertions of Theorem 1.1.

(1) and (2). It suffices to notice that if T is either universally meager or a λ -set, then so is E , respectively (cf. [24]). But $E \notin \mathbf{PM}_\sigma$.

(3). It can be readily checked that if T is a λ' -set in $2^{\mathbb{N}}$, then H is λ' -set in $2^{\mathbb{N}} \times P$. Since P can be homeomorphically embedded as a G_δ -set in $2^{\mathbb{N}}$, we may identify H with a λ' -set in $2^{\mathbb{N}}$. But then E is a homeomorphic copy of H in $2^{\mathbb{N}}$ which is not **PM** $_\sigma$ in $2^{\mathbb{N}}$. \square

We close this section by showing that like **UM** (see [24]) (but, at least consistency-wise, unlike **PM**, see [19]), the class **PM** $_\sigma$ is closed under products.

Theorem 3.2. *The product of two countably perfectly meager sets is countably perfectly meager in the sense that if A and B are \mathbf{PM}_σ -sets in perfect Polish spaces X and Y , respectively, then $A \times B$ is a \mathbf{PM}_σ -set in $X \times Y$.*

Proof. Let A and B be \mathbf{PM}_σ -sets in perfect Polish spaces X and Y , respectively.

To prove that $A \times B$ is a \mathbf{PM}_σ -set in $X \times Y$, we shall check condition (3) of Theorem 2.1.

Let $\{K_n : n \in \mathbb{N}\}$ be a sequence of Cantor sets in $Z = X \times Y$ and for each n let L_n and M_n be the images of K_n under projections of Z onto X and Y , respectively.

Let $S = \{n \in \mathbb{N} : L_n \text{ is uncountable}\}$ and for each $n \in S$ let us pick a Cantor set $L'_n \subseteq L_n$. Likewise, let $T = \{n \in \mathbb{N} : M_n \text{ is uncountable}\}$ and for each $n \in T$ let us pick a Cantor set $M'_n \subseteq M_n$.

The sets A and B being countably perfectly meager, there are sequences $\{F_i^A : i \in \mathbb{N}\}$ and $\{F_j^B : j \in \mathbb{N}\}$ of closed sets in X and Y , respectively, such that $A \subseteq \bigcup_i F_i^A$, $B \subseteq \bigcup_j F_j^B$, $L'_m \not\subseteq F_i^A$ and $M'_n \not\subseteq F_j^B$ whenever $m \in S$, $n \in T$ and $n, m, j, i \in \mathbb{N}$.

Let $F_{i,j} = F_i^A \times F_j^B$ for $i, j \in \mathbb{N}$.

Clearly, $A \times B \subseteq \bigcup_{i,j} F_{i,j}$ and we claim that for any n, i, j we have $K_n \not\subseteq F_{i,j}$.

Indeed, let us notice that $S \cup T = \mathbb{N}$ since for each n we have $K_n \subseteq L_n \times M_n$. It follows that either $L'_n \not\subseteq F_i^A$ (if $n \in S$) or $M'_n \not\subseteq F_j^B$ (if $n \in T$), so in either case $K_n \not\subseteq F_{i,j}$. This completes the proof. \square

4. COMMENTS

4.1. \mathbf{PM}_σ versus λ' , Hurewicz property and \mathbf{AFC}' . The relationship of classes of subsets of $2^\mathbb{N}$ with the Hurewicz property (and no perfect subsets, cf. Proposition 2.10(1)), or of λ' -sets (cf. Proposition 2.10(2)), or of \mathbf{AFC}' -sets (cf. Theorem 2.9) to an apparently larger class of \mathbf{PM}_σ -sets seems particularly close.

E.g., the characterizations of \mathbf{PM}_σ -sets given in Theorem 2.1 somewhat resemble the following characterization of sets with the Hurewicz property, obtained by Just, Miller, Scheepers and Szeptycki [11, Theorem 5.7]: $A \subseteq 2^\mathbb{N}$ has the Hurewicz property if and only if for every sequence $\{K_n : n \in \mathbb{N}\}$ of copies of the Cantor set in $2^\mathbb{N}$ disjoint from A there are closed sets F_n in $2^\mathbb{N}$ such that $A \subseteq \bigcup_n F_n$ and $K_m \cap F_n = \emptyset$ for each $m, n \in \mathbb{N}$. In particular, if $A \subseteq 2^\mathbb{N}$ has either the Hurewicz property or λ' -property, then for every countable set $D \subseteq 2^\mathbb{N}$ disjoint from A there is an F_σ -set F in $2^\mathbb{N}$ such that $A \subseteq F$ and $F \cap D = \emptyset$.

The following example, based on a result of Bartoszyński and Shelah [4] and classical ideas of Rothberger (cf. [14]), shows that there exists (in ZFC) a countably perfectly meager set in $2^\mathbb{N}$ of cardinality \mathfrak{b} which lacks the latter property and thus has neither the Hurewicz nor λ'

property. It also shows that the Hurewicz and λ' properties are not the same.

Example 4.1. *Inductively, one easily constructs a subset $\{f_\alpha : \alpha < \mathfrak{b}\}$ of $\mathbb{N}^{\mathbb{N}}$ with the following properties (cf. Proposition 2.10(2(a))):*

- f_α is strictly increasing,
- $\alpha < \beta < \mathfrak{b}$ implies $f_\alpha <^* f_\beta$,
- for every $f \in \mathbb{N}^{\mathbb{N}}$ there is $\alpha < \mathfrak{b}$ with $f_\alpha \not\leq^* f$.

By identifying each f_α with the characteristic function of its range, we obtain a homeomorphic copy A of $\{f_\alpha : \alpha < \mathfrak{b}\}$ in $2^{\mathbb{N}}$.

Let $B = A \cup \mathbb{Q}$, where \mathbb{Q} consists of all eventually zero binary sequences. Then:

- $\{f_\alpha : \alpha < \mathfrak{b}\}$, being unbounded in $\mathbb{N}^{\mathbb{N}}$, does not have the Hurewicz property,
- $\{f_\alpha : \alpha < \mathfrak{b}\}$ is a λ' -set in $\mathbb{N}^{\mathbb{N}}$ (cf. Proposition 2.10(2)(a)),
- B has the Hurewicz property and has no perfect subsets (cf. [4, Theorem 1], [22, Theorem 2.12] and Remark 4.2 below) so, by Proposition 2.10(1), B is countably perfectly meager in $2^{\mathbb{N}}$,
- A is countably perfectly meager in $2^{\mathbb{N}}$ as a subset of B ,
- A does not have the Hurewicz property as the homeomorphic image of $\{f_\alpha : \alpha < \mathfrak{b}\}$,
- if F is any F_σ -set in $2^{\mathbb{N}}$ such that $A \subseteq F$, then $F \cap \mathbb{Q} \neq \emptyset$ (since otherwise F viewed as a subset of $\mathbb{N}^{\mathbb{N}}$ is bounded, whereas A is unbounded). In particular, neither A nor B are λ' -sets in $2^{\mathbb{N}}$.

Remark 4.2. *It seems useful to indicate another proof of the fact that the set B described in Example 4.1 has the Hurewicz property. In fact, in the argument below it is enough to assume that $\{f_\alpha : \alpha < \mathfrak{b}\}$ is any well-ordered by eventual domination and unbounded subset of $\mathbb{N}^{\mathbb{N}}$, $h : \mathbb{N}^{\mathbb{N}} \rightarrow P$ is a homeomorphism onto the set P of irrationals in $I = [0, 1]$, $\mathbb{Q} = I \setminus P$, $A = \{h(f_\alpha) : \alpha < \mathfrak{b}\}$ and $B = A \cup \mathbb{Q}$.*

Let us recall that the Hurewicz property of $H \subseteq I$ is equivalent to the following covering property (this is the original Hurewicz's definition, cf. [22]):

for each sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of open in I covers of H , there are finite subfamilies $\mathcal{F}_n \subseteq \mathcal{U}_n$ such that $H \subseteq \bigcup_n \bigcap_{m \geq n} (\bigcup \mathcal{F}_m)$.

Proof. Let $\mathcal{U}_1, \mathcal{U}_2, \dots$ be open in I covers of B and let $G = \bigcap_n (\bigcup \mathcal{U}_n)$. Since $I \setminus G = P \setminus G$ is σ -compact and so is $S = h^{-1}(P \setminus G)$, there is $f \in \mathbb{N}^{\mathbb{N}}$ such that $g \leq^* f$ for any $g \in S$. Let us pick $\alpha < \mathfrak{b}$ so that $f_\alpha \not\leq^* f$. Then $T = \{g \in \mathbb{N}^{\mathbb{N}} : f_\alpha \leq^* g\}$ is an F_σ -set in $\mathbb{N}^{\mathbb{N}}$ disjoint from S and $f_\beta \in T$, whenever $\beta \geq \alpha$.

Let $L = h(T) \cup \mathbb{Q}$. Then $|B \setminus L| < \mathfrak{b}$, as $B \setminus L \subseteq \{h(f_\xi) : \xi < \alpha\}$. Consequently, $B \setminus L$ has the Hurewicz property and since $B \setminus L \subseteq G$, there are finite collections $\mathcal{F}'_n \subseteq \mathcal{U}_n$ such that $B \setminus L \subseteq \bigcup_n \bigcap_{m \geq n} (\bigcup \mathcal{F}'_m)$.

Let us notice that the set L is σ -compact in I so it also has the Hurewicz property and since $L \subseteq G$, we can pick finite $\mathcal{F}_n'' \subseteq \mathcal{U}_n$ such that $L \subseteq \bigcup_n \bigcap_{m \geq n} (\bigcup \mathcal{F}_m')$.

Then, letting $\mathcal{F}_n = \mathcal{F}_n' \cup \mathcal{F}_n''$, we get finite collections $\mathcal{F}_n \subseteq \mathcal{U}_n$ such that $B \subseteq \bigcup_n \bigcap_{m \geq n} (\bigcup \mathcal{F}_m)$. This shows that B has the Hurewicz property. \square

The fact that **AFC'**-subsets of $2^{\mathbb{N}}$ are closely related to **PM** $_{\sigma}$ -sets in $2^{\mathbb{N}}$ is revealed by the following characterization of Bartoszyński (private communication). We are grateful to Tomek Bartoszyński for allowing us to include his result in our paper.

Proposition 4.3 (T. Bartoszyński). *For a set $A \subseteq 2^{\mathbb{N}}$ the following are equivalent:*

- (1) $A \in \mathbf{PM}_{\sigma}$.
- (2) for every perfect subset P of $2^{\mathbb{N}}$, there exists an F_{σ} -set F in $2^{\mathbb{N}}$ such that $A \subseteq F$ and $F \cap (P + q)$ is meager in $P + q$ for every $q \in \mathbb{Q}$, where \mathbb{Q} consists of all eventually zero binary sequences.

Proof. Implication (1) \Rightarrow (2) follows directly from the definition of countably perfectly meager sets.

To prove that (2) \Rightarrow (1), we shall appeal to condition (2) of Theorem 2.1. So let $\{K_n : n \in \mathbb{N}\}$ be a sequence of Cantor sets in $2^{\mathbb{N}}$. For each n one can pick a non-empty relatively open compact set L_n in K_n (an intersection of K_n with a basic neighborhood in $2^{\mathbb{N}}$) and $q_n \in \mathbb{Q}$ such that all points in $T_n = L_n + q_n$ have n first coordinates zero, and the sets T_n are pairwise disjoint. Then the union P of $\{0\}$ and the sets T_n is perfect and $(K_n + q_n) \cap P$ has relatively non-empty interior in P for each $n \in \mathbb{N}$.

Using (2), we fix an F_{σ} -set F in $2^{\mathbb{N}}$ such that $A \subseteq F$ and $(F + q) \cap P$ is meager in P for every $q \in \mathbb{Q}$.

But now it is clear that $K_n \subseteq F$ for no $n \in \mathbb{N}$, since otherwise we would have $(K_n + q_n) \cap P \subseteq (F + q_n) \cap P$, contradicting the choice of F . \square

The following example is a slight modification of a remarkable construction of Reclaw [20] and shows that, at least consistently, there exists a countably perfectly meager in $2^{\mathbb{N}}$, in fact a λ' -set in $2^{\mathbb{N}}$, which is not perfectly meager in the transitive sense (a similar construction was also used by Weiss [23, Theorem 3] in his proof that the existence of a universally meager set in $2^{\mathbb{N}}$ of cardinality of the continuum implies that there is also one which is not perfectly meager in the transitive sense). This is also yet another (cf. Theorem 3.1) strengthening of the result of Nowik, Scheepers and Weiss [15] that under the Continuum Hypothesis there is a λ -set in $2^{\mathbb{N}}$ which is not perfectly meager in the transitive sense.

Example 4.4. *Let us assume that there exists a λ' -set in $2^{\mathbb{N}}$ of cardinality of the continuum.*

Let C, D be disjoint copies of the Cantor set in $2^{\mathbb{N}}$ such that

- (1) *the operation $+$ of addition is a homeomorphism between $C \times D$ and $C + D$ (cf. [20]).*

Let $T \subseteq C$ be a λ' -set in C of cardinality of the continuum.

Arguing as in part (A) of the proof of Theorem 1.1, we obtain a set $H \subseteq T \times D$ intersecting each vertical section $\{t\} \times D$, $t \in T$, in a singleton, such that

- (2) *each F_σ -set in $C \times D$ containing H contains also $\{t\} \times D$ for some $t \in T$.*

Now, T being a λ' -set in C , one readily checks that H is a λ' -set in $C \times D$ (cf. the proof of Theorem 3.1(3)). It follows that, cf. (1), if we let $Y = +(H)$, then Y is a λ' -set in $C + D$ and hence also in $2^{\mathbb{N}}$.

On the other hand, Y is not perfectly meager in the transitive sense. Indeed, if F is an arbitrary F_σ -set in $2^{\mathbb{N}}$ with $Y \subseteq F$ and we let $E = (+)^{-1}(F \cap (C + D))$, then E is an F_σ -set in $C \times D$ containing H so it also contains, cf. (2), $\{t\} \times D$ for some $t \in T$. Consequently, $t + D \subseteq F$, completing the proof.

4.2. Remarks on related σ -ideals. In this section \mathcal{F} always denotes a countable (possibly finite but non-empty) collection of perfect sets in a perfect Polish space X .

If I is a σ -ideal of subsets of X , then by I^* we denote the σ -ideal generated by the closed subsets of X which belong to I .

4.2.1. The σ -ideals $MGR(\mathcal{F})$. Following Kechris and Solecki [13] let us put

$$MGR(\mathcal{F}) = \{A \subseteq X : A \cap P \text{ is meager in } P \text{ for every } P \in \mathcal{F}\}.$$

Let us note that $MGR(\mathcal{F})$ is a σ -ideal on X generated by Borel, in fact $F_{\sigma\delta}$ -sets, and fulfills the c.c.c. Moreover, the quotient Boolean algebra $Bor(X)/(MGR(\mathcal{F}) \cap Bor(X))$ is isomorphic to the Cohen algebra, cf. [3]. Clearly, the intersection of all σ -ideals of the form $MGR(\mathcal{F})$ is precisely the σ -ideal **PM**.

On the other hand, by [24, Theorem 2.1], the σ -ideal **UM** is the intersection of all σ -ideals I on X such that the quotient Boolean algebra $Bor(X)/(I \cap Bor(X))$ is isomorphic to the Cohen algebra. Any such I is precisely of the form $\mathcal{M}(X, \tau)$, by which we denote the σ -ideal consisting of meager sets with respect to a perfect Polish topology τ on X giving the original Borel structure of X .

4.2.2. The σ -ideals $MGR^*(\mathcal{F})$. By the definition (see the beginning of Section 4), the σ -ideal $MGR^*(\mathcal{F})$ consists of such sets $A \subseteq X$ that there exists an F_σ -set F in X with $A \subseteq F$ and such that $F \cap P$ is meager in P for every $P \in \mathcal{F}$. It is the σ -ideal $MGR^*(\mathcal{F})$ (not the σ -ideal

$MGR(\mathcal{F})$ erroneously employed in [3, Theorem 7, (1) \Rightarrow (3)]) that is relevant to the definition of countably perfectly meager sets. Indeed, the intersection of all σ -ideals of the form $MGR^*(\mathcal{F})$ is precisely the σ -ideal \mathbf{PM}_σ .

It turns out that not all of the σ -ideals of the form $MGR^*(\mathcal{F})$ fulfill the c.c.c. Let us elaborate on this a little further with the help of a theory developed by Kechris and Solecki [13].

Proposition 4.5.

- (1) *If every non-empty open set U in X contains a nowhere dense set $P \in \mathcal{F}$, then the σ -ideal $MGR^*(\mathcal{F})$ does not fulfill the c.c.c.*
- (2) *The intersection of all σ -ideals in X generated by closed sets which fulfill the c.c.c. is precisely the σ -ideal \mathbf{PM} .*
- (3) *The intersection of all σ -ideals of the form $MGR^*(\mathcal{F})$ which fulfill the c.c.c. is precisely the σ -ideal \mathbf{PM} .*
- (4) *The intersection of all σ -ideals of the form $MGR^*(\mathcal{F})$ which do not fulfill the c.c.c. is precisely the σ -ideal \mathbf{PM}_σ .*

Proof. (1). This immediately follows from [13, Lemma 9].

(2) and (3). First, let $A \in \mathbf{PM}$ and let I be a σ -ideal in X which is generated by closed sets and fulfills the c.c.c. Then, by [13, Theorem 3], I is of the form $MGR(\mathcal{F})$ for a countable family of perfect subsets of X . Consequently, $A \in I$.

For the other direction, assume that $A \in MGR^*(\mathcal{F})$ for every \mathcal{F} such that the σ -ideal $MGR^*(\mathcal{F})$ fulfills the c.c.c. Suppose that P is an arbitrary perfect subset of X and let $\mathcal{F} = \{P\}$. Then we have $MGR^*(\mathcal{F}) = MGR(\mathcal{F})$, so the σ -ideal $MGR^*(\mathcal{F})$ fulfills the c.c.c. Consequently, $A \in MGR^*(\mathcal{F})$ which just means that $A \cap P$ is meager in P , completing the proof that $A \in \mathbf{PM}$.

(4). Let us assume that $A \in MGR^*(\mathcal{F})$ for every \mathcal{F} such that the σ -ideal $MGR^*(\mathcal{F})$ does not fulfill the c.c.c. To prove that $A \in \mathbf{PM}_\sigma$, let \mathcal{F} be an arbitrary countable family of perfect subsets of X . By extending \mathcal{F} , if necessary, we may assume that every non-empty open set U in X contains a nowhere dense set $P \in \mathcal{F}$. By part (1), the σ -ideal $MGR^*(\mathcal{F})$ does not fulfill the c.c.c. Consequently, $A \in MGR^*(\mathcal{F})$ and we are done. □

4.2.3. *The σ -ideals $\mathcal{M}^*(X, \tau)$.* In this subsection τ always denotes a perfect Polish topology on X giving the original Borel structure of X . Recall that $\mathcal{M}(X, \tau)$ is the σ -ideal of meager sets with respect to τ and $\mathcal{M}^*(X, \tau)$ consists of such $A \subseteq X$ that there exists an F_σ -set F in X (with the original Polish topology) with $A \subseteq F$ and $F \in \mathcal{M}(X, \tau)$.

It turns out that we can characterize perfectly meager and countably perfectly meager sets with the help of the σ -ideals $\mathcal{M}^*(X, \tau)$ in an analogous way to their characterizations in terms of the σ -ideals $MGR^*(\mathcal{F})$ (cf. 4.5).

Proposition 4.6.

- (1) *The intersection of all σ -ideals of the form $\mathcal{M}^*(X, \tau)$ which fulfill the c.c.c. is precisely the σ -ideal \mathbf{PM} .*
- (2) *The intersection of all σ -ideals of the form $\mathcal{M}^*(X, \tau)$ is precisely the σ -ideal \mathbf{PM}_σ . Moreover, for a set A to be countably perfectly meager it is enough to belong to all σ -ideals of the form $\mathcal{M}^*(X, \tau)$ which do not fulfill the c.c.c.*

Proof. (1). This follows directly from points (2) and (3) of Proposition 4.5, since every σ -ideal of the form $MGR^*(\mathcal{F})$ is also of the form $\mathcal{M}^*(X, \tau)$ which in turn is generated by closed sets.

(2). Let us assume that $A \in \mathbf{PM}_\sigma$ and let $\{U_n : n \in \mathbb{N}\}$ be a basis of a perfect Polish topology τ on X such that every U_n is an (uncountable) Borel set in X . For each n let us pick a Cantor set $K_n \subseteq U_n$. We complete the argument exactly as in the proof of implication (1) \Rightarrow (2) in Theorem 2.5.

For the other direction we again use the fact that every σ -ideal of the form $MGR^*(\mathcal{F})$ is also of the form $\mathcal{M}^*(X, \tau)$ and apply point (4) of Proposition 4.5.

□

REFERENCES

1. H. R. Bennett, M. Hosobuchi and D. J. Lutzer, *A note on perfect generalized ordered spaces*, Rocky Mountain J. Math. **10(3)** (1999), 1195–1207.
2. T. Bartoszyński, *On perfectly meager sets*, Proc. Amer. Math. Soc. **130(4)** (2002), 1189–1195.
3. T. Bartoszyński, *Remarks on small sets of reals*, Proc. Amer. Math. Soc. **131(2)** (2003), 625–630.
4. T. Bartoszyński, S. Shelah, *Continuous images of sets of reals*, Topology Appl. **116** (2001), 243–253.
5. R. Engelking, *General Topology*, Sigma Series in Pure Mathematics, Heldermann Verlag, 1989.
6. F. Galvin, A. W. Miller, *γ -sets and other singular sets of real numbers*, Topology Appl. **17** (1984), 145–155.
7. D. H. Fremlin, A. W. Miller, *On some properties of Hurewicz, Menger and Rothberger*, Fund. Math. **129** (1988), 17–33.
8. E. Grzegorek, *Solution to a problem of Banach on σ -fields without continuous measures*, Bull. Acad. Pol. Sci. **28** (1980), 7–10.
9. E. Grzegorek, *Always of the first category sets*, Rend. Circ. Mat. Palermo, II. Ser. Suppl. **6** (1984), 139–147.
10. E. Grzegorek, *Always of the first category sets. II*, Rend. Circ. Mat. Palermo, II. Ser. Suppl. **10** (1985), 43–48.
11. W. Just, A.W. Miller, M. Scheepers, P.J. Szeptycki, *The combinatorics of open covers (II)*, Topology Appl. **73** (1996), 241–266.
12. A. S. Kechris, *Classical descriptive set theory*, Graduate Texts in Math. 156, Springer-Verlag, 1995.
13. A. S. Kechris, S. Solecki, *Approximation of analytic by Borel sets and definable countable chain conditions*, Israel J. Math. **89** (1995), 343–356.

14. A. W. Miller, *Special subsets of the real line* in *Handbook of set-theoretic topology*, North-Holland 1984, 201–233.
15. A. Nowik, M. Scheepers, T. Weiss, *The algebraic sum of sets of real numbers with strong measure zero sets*, *J. Symbolic Logic* **63**(1) (1998), 301–324.
16. A. Nowik, T. Weiss, *Not every Q -set is perfectly meager in the transitive sense*, *Proc. Amer. Math. Soc.* **128**(10) (2000), 3017–3024.
17. A. Nowik, T. Weiss, *On the Ramseyan properties of some special subsets of 2^ω and their algebraic sums*, *J. Symbolic Logic* **67**(2) (2002), 547–556.
18. J. Pawlikowski, *Every Sierpiński set is strongly meager*, *Arch. Math. Logic* **35** (1996), 281–285.
19. I. Reclaw, *Products of perfectly meagre sets*, *Proc. Amer. Math. Soc.* **112**(4) (1991), 1029–1031.
20. I. Reclaw, *Some additive properties of special subsets of reals*, *Colloquium Mathematicum* **62**(2) (1991), 221–226.
21. W. Sierpiński, *Sur la non-invariance topologique de la propriété λ'* , *Fund. Math.* **33** (1945), 264–268.
22. B. Tsaban, *Menger's and Hurewicz's Problems: Solutions from "The Book" and refinements*, *Contemp. Math.* **533** (2011), 211–226.
23. T. Weiss, *On perfectly meager sets in the transitive sense*, *Proc. Amer. Math. Soc.* **130**(2) (2001), 591–594.
24. P. Zakrzewski, *Universally Meager Sets*, *Proc. Amer. Math. Soc.* **129**(6) (2001), 1793–1798.
25. P. Zakrzewski, *Universally Meager Sets, II*, *Topology Appl.* **155** (2008), 1445–1449.
26. O. Zindulka, *Meager-additive sets in topological groups*, (2018), <https://arxiv.org/abs/1806.06674>.

INSTITUTE OF MATHEMATICS, UNIVERSITY OF WARSAW, UL. BANACHA 2,
02-097 WARSAW, POLAND

E-mail address: pol@mimuw.edu.pl, piotrzak@mimuw.edu.pl