**Logic homework #4**

**due November 5, 2019**

Recall that, given a satisfiable set $T \subseteq Form$, the *Lindenbaum–Tarski algebra* $B(T)$ consists of the set $\mathcal{A}$ (its universe) of all the equivalence classes of the relation of logical equivalence under $T$ (i.e., $\alpha \equiv_T \beta$ iff $T \models (\alpha \Leftrightarrow \beta)$) with distinguished elements $\mathbf{0}$ and $\mathbf{1}$, binary operations $+$, $\cdot$ and the unary operation $-$ defined as follows ($[\alpha]$ is the equivalence class of $\alpha$ under $\equiv_T$):

$$\mathbf{0} = [p \wedge \neg p], \mathbf{1} = [p \vee \neg p]$$

$$[\alpha] + [\beta] = [\alpha \vee \beta],$$

$$[\alpha] \cdot [\beta] = [\alpha \wedge \beta].$$

$$-[\alpha] = [\neg\alpha].$$

**Exercise 1.** Prove that the structure $B(T) = \langle \mathcal{A}, \mathbf{0}, \mathbf{1} +, \cdot, - \rangle$ is isomorphic to the structure $\mathbb{B} = \langle \mathcal{B}, \emptyset, X, \cup, \cap, ' \rangle$ ($'$ denotes the unary operation of taking the complement) for a certain field $\mathcal{B} \subseteq \mathcal{P}(X)$ of subsets of the set $X$ of all models $\nu : Var \to \{0, 1\}$ of $T$.

**Exercise 2.** Let $\mathcal{B}$ be a field of subsets of a non-empty set $X$.

Let $Var = \{p_A : A \in \mathcal{B}\}$ (where $p_A \neq p_B$ if $A \neq B$) and let $Form$ be the set of propositional formulas over the set of propositional variables $Var$. Let $\nu : Var \to \mathcal{B}$ be the identity valuation in $\mathcal{B}$, i.e., $\nu(p_A) = A$ for every $A \in \mathcal{B}$. Let $\bar{\nu} : Form \to \mathcal{B}$ be the unique extension of $\nu$ (defined in exercise 4 of Logic homework #3).

Let $T = \{\alpha \in Form : \bar{\nu}(\alpha) = X\}$. Prove that the structure $\mathbb{B} = \langle \mathcal{B}, \emptyset, X, \cup, \cap, ' \rangle$ ($'$ denotes the unary operation of taking the complement) is isomorphic to the Lindenbaum–Tarski algebra $B(T)$.

In Execises 3 and 4 (on the next page) let $\mathcal{A}$ be the universe of the Lindenbaum-Tarski algebra $\mathbb{A} = B(\emptyset)$ (so $\mathcal{A}$ consists of all the equivalence classes of the relation of logical equivalence). Define a binary relation $\leq$ on $\mathcal{A}$ as follows:

$$[\alpha] \leq [\beta] \quad \text{iff} \quad \text{the formula } (\alpha \Rightarrow \beta) \text{ is a tautology.}$$

**Exercise 3.** Prove that $\leq$ is a partial order on $\mathcal{A}$ with the following properties:

1. $\mathcal{A}$ has the smallest element $\mathbf{0}$ and the largest element $\mathbf{1}$,
2. $(\mathbb{A}, \leq)$ is a lattice, i.e., for every $\alpha, \beta \in Form$ there is the smallest upper bound and the greatest lower bound of the set $\{[\alpha]_\equiv, [\beta]_\equiv\}$ in $\mathcal{A}$.

A non-empty set $\mathcal{F} \subseteq \mathcal{A}$ is called a *filter in* $\mathbb{A}$ if it satisfies the following conditions:

1. $\mathbf{0} \notin \mathcal{F}$,
2. $[\alpha], [\beta] \in \mathcal{F}$ implies $[\alpha] \cdot [\beta] \in \mathcal{F}$,
3. $[\alpha] \in \mathcal{F}$ and $[\alpha] \leq [\beta]$ implies $[\beta] \in \mathcal{F}$.

If, moreover, for every $\alpha \in F$ either $[\alpha] \in \mathcal{F}$ or $-[\alpha] \in \mathcal{F}$, then $\mathcal{F}$ is called an *ultrafilter in* $\mathbb{A}$.

**Exercise 4.** Let $\mathcal{F} \subseteq \mathcal{A}$ and $S = \{\alpha \in Form : [\alpha] \in \mathcal{F}\}$. Prove that $\mathcal{F}$ is a filter in $\mathbb{A}$ if and only if $S$ is consistent (equivalently, $S$ is satisfiable) and is closed under logical consequences (i.e., for every $\alpha \in Form$, $S \models \alpha$ implies $\alpha \in S$). Moreover, $\mathcal{F}$ is an ultrafilter in $\mathbb{A}$ if and only if $S$ is consistent, closed under logical consequences and complete (i.e., for every $\alpha \in Form$, either $\alpha \in S$ or $\neg\alpha \in S$).