Logic homework #3

due October 29, 2019

Recall that a family $\mathcal{F}$ of subsets of a non-empty set X is called a *filter on X* if it satisfies the following properties:

- if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$ ($\mathcal{F}$ is upward closed),
- if $B_1 \in \mathcal{F}$ and $B_2 \in \mathcal{F}$, then $B_1 \cap B_2 \in \mathcal{F}$ ($\mathcal{F}$ is closed under finite intersections),
- $\emptyset \notin \mathcal{F}$ ($\mathcal{F}$ is a proper filter).

If, moreover,

- if $A \subseteq X$, then either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$,

then a filter $\mathcal{F}$ is called an *ultrafilter* on X.

Exercise 1. Prove that if a set X is finite, then:

(i) every filter on X has the form $\mathcal{F} = \{B \subseteq X : A \subseteq B\}$ for a certain $A \subseteq X$ (such a filter is called *the principal filter generated by A*),

(ii) every ultrafilter on X is a principal filter on X generated by a singleton.

Exercise 2. Prove that a filter on a set X is an ultrafilter on X iff it is a maximal (proper) filter on X.

Exercise 3. With the help of the Kuratowski-Zorn Lemma prove that every filter on a set X can be extended to an ultrafilter on X. As a consequence, show that on every infinite set X there exists a non-principal ultrafilter.

Exercise 4. Let $Form$ be the set of propositional formulas over a fixed set Var of propositional variables. Let $\mathcal{A}$ be a field (an algebra) of subsets of a non-empty set X. Let us call an arbitrary function $\nu : V \to \mathcal{A}$ a valuation in $\mathcal{A}$. By induction on formulas ν can be uniquely extended to a function $\bar{\nu} : Form \to \mathcal{A}$ such that for every $\alpha, \beta \in Form$:

1. $\bar{\nu}(\neg\alpha) = X \setminus \bar{\nu}(\alpha)$,
2. $\bar{\nu}(\alpha \vee \beta) = \bar{\nu}(\alpha) \cup \bar{\nu}(\beta)$,
3. $\bar{\nu}(\alpha \wedge \beta) = \bar{\nu}(\alpha) \cap \bar{\nu}(\beta)$,
4. $\bar{\nu}(\alpha \Rightarrow \beta) = (X \setminus \bar{\nu}(\alpha)) \cup \bar{\nu}(\beta)$,
5. $\bar{\nu}(\alpha \Leftrightarrow \beta) = X \setminus (\bar{\nu}(\alpha) \triangle \bar{\nu}(\beta))$.

Prove that if $\alpha \in Form$ is a tautology, then $\bar{\nu}(\alpha) = X$.